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Energy of graphs

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Abstract. Let $G = (V, E)$ be a simple graph of order n and m edges. The energy of a graph G, denoted by $\mathcal{E}(G)$, is defined by Ivan Gutman as the sum of the absolute values of all eigenvalues of G. In the past decade, interest in graph energy has increased and many different versions have been introduced. The aim of this paper is to present the graph energy of normal Cayley graphs in terms of their character tables.

Keywords: energy of graph, Laplacian energy, characteristic polynomial
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1 Introduction

The energy $\mathcal{E}(G)$ of a graph G is equal to the sum of the absolute values of the graph eigenvalues, namely the sum of the eigenvalues of the adjacency matrix $A(G)$ of G. The origin of this concept comes from the π-electron energy in the Hückel molecular orbital model, but has also gained purely mathematical interest. In the past decade many kinds of energy have been introduced. In 2006, Gutman and Zhou defined the Laplacian energy of a graph as the sum of the absolute deviations of the eigenvalues of its Laplacian matrix [15]. The signless Laplacian, the distance, the incidence and many other versions of energy associated with a graph were defined, see [23]. In 2010, Cavers, Fallat and Kirkland first studied the Normalized Laplacian energy of a graph known as the Randic energy related to the Randic

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index [3].

Let $G=(V,E)$ be a simple graph on n vertices and m edges with vertex set $V(G)=\{v_1,v_2,\cdots,v_n\}$. If vertices v_i and v_j are adjacent, we denote that by $v_iv_j\in E(G)$. The adjacency matrix $A(G)$ of G is defined as follows: the entry $a_{ij}=1$ if $v_iv_j\in E(G)$ and 0 otherwise.

The characteristic polynomial $\chi(X)$ of graph X with adjacency matrix A is defined as $\chi(X)=det(xI-A)$. It is a monic polynomial of degree n. The roots of the charachteristic polynomial are eigenvalues of X and form the spectrum of X. Since all considered graphs are undirected, the adjacency matrix A is symmetric. Consequently, all eigenvalues are real.

Suppose $\lambda_1\geq\lambda_2\geq\lambda_{n-1}\geq\lambda_n$ denote the eigenvalues of $A(G)$. Some well-known results concerning the energy of graph are as follows:

$$\sum_{i=1}^{n}\lambda_i=0, \tag{1}$$

$$\sum_{i=1}^{n}\lambda_i^2=2m, \tag{2}$$

$$det(A)=\Pi_{i=1}^{n}\lambda_i. \tag{3}$$

The energy of the graph G is defined as

$$\mathcal{E}(G)=\sum_{i=1}^{n}|\lambda_i|,$$

where λ_i, $i=1,2,\cdots,n$ are the eigenvalues of graph G, see [1–7, 12–14, 18] as well as [8–11].

2 Definitions and preliminaries

A general linear group $GL(\mathcal{V})$ of vector space $\mathcal{V}$ is the set of all $\mathcal{A}\in End(\mathcal{V})$ where $\mathcal{A}$ is invertible. A representation of a group Γ is a homomorphism $\alpha:\Gamma\to GL(\mathcal{V})$ and the degree of α is equal to the dimension of $\mathcal{V}$. A trivial representation is a homomorphism $\alpha:\Gamma\to\mathbb{C}^*$ where $\alpha(g)=1$ for all $g\in\Gamma$. Let $\varphi:\Gamma\to GL(\mathcal{V})$ be a representation with $\varphi(g)=\varphi_g$, the character $\chi_\varphi:\Gamma\to\mathbb{C}$ of φ is defined as $\chi_\varphi(g)=tr(\varphi_g)$. An irreducible character is the character of an irreducible representation and the character χ is linear, if $\chi(1)=1$. We denote the set of all irreducible characters of G by $Irr(\Gamma)$, see [17] for more details on character theory.

Let Γ be a finite group with symmetric subset S. The symmetric subset S is a normal if $g^{-1}Sg=S$, for all $g\in\Gamma$. The Cayley graph $X=Cay(\Gamma,S)$ has the elemnts of Γ as its vertices and two vertices x and y are adjacent if and only if $yx^{-1}\in S$.

Theorem 2.1. *Let Γ be a finite group with a normal symmetric subset S. Let A be the adjacency matrix of the graph $X=Cay(\Gamma,S)$. Then the eigenvalues of A are given by $[\lambda_\varphi]^{\varphi(1)^2}$, where*

$$\lambda_\varphi=\frac{1}{\varphi(1)}\sum_{s\in S}\varphi(s)$$

and $\varphi\in Irr(\Gamma)$.

Example 2.2. *[19] Consider the dihedral group D_8 with the following presentation*

$$D_8 = \langle a, b : a^4 = b^2 = 1, \ b^{-1}ab = a^{-1} \rangle.$$

The character table of dihedral group D_8 is reported in Table 1. Let $S = \{a, a^{-1}\}$, then by using Theorem 2. 1, all eigenvalues of D_8 are:

$$\lambda_{\chi_1} = \lambda_{\chi_3} = 2, \lambda_{\chi_2} = \lambda_{\chi_4} = -2 \ \ and \ \ \lambda_{\chi_5} = 0.$$

In other words, suppose $S = \{a, a^{-1}\}$, then the spectrum of $X = Cay(D_8, S)$ is as follows:

$$Spec(X) = \{[-2]^2, \ [0], \ [2]^2\}.$$

Table 1. The character table of group D_8.

$\mathcal{M}(D_8)$	1	g_1	g_2	g_3	g_4
χ_1	1	1	1	1	1
χ_2	1	−1	1	1	−1
χ_3	1	1	−1	1	−1
χ_4	1	−1	−1	1	1
ψ_5	2	0	0	−2	0

3 Main results

Suppose Γ is a group with irreducble characters $Irr(\Gamma)$ and a normal symmetric subset S. Then the energy of Cayley graph $X = Cay(\Gamma, S)$ is

$$\mathcal{E}(X) = \sum_{\varphi \in Irr(\Gamma)} \varphi(1) | \sum_{s \in S} \varphi(s)|.$$

Indeed, if Γ is abelian group then it is a well-known fact that all characters are linear and so

$$\mathcal{E}(X) = \sum_{\varphi \in Irr(\Gamma)} | \sum_{s \in S} \varphi(s)|.$$

Corollary 3.1. *Suppose Γ is a finite group with exactly n conjugacy classes $\{x_1^\Gamma, \ldots, x_n^\Gamma\}$ and $X = Cay(\Gamma, S)$. Then*

$$\mathcal{E}(X) = \sum_{\varphi \in Irr(\Gamma)} \varphi(1) \sum_{i=1}^{n} |x_i^\Gamma| \times | \sum_{s \in x_i^\Gamma} \varphi(s)|.$$

Theorem 3.2. *Suppose Γ is a finite group with exactly n conjugacy classes and S, S' are normal subsets of $\Gamma \setminus \{e\}$ such that $S' = (\Gamma \setminus S) \cup \{e\}$. Moreover, we assume that $\Gamma = Cay(\Gamma, S)$, $\Gamma' = Cay(\Gamma, S')$. Set $Spec(\Gamma) = \{\lambda_1, \cdots, \lambda_n\}$ and $Spec(\Gamma') = \{\beta_1, \cdots, \beta_n\}$. Then $\beta_i = -\lambda_i - 1, 1 \leq i \leq n$.*

Proof. It is clear that S' is a symmetric and generating subset of G. So,

$$\begin{aligned}\beta_i &= \frac{1}{\chi_i(e)}\sum_{s'\in S'}\chi_i(s') \\ &= \frac{1}{\chi_i(e)}\Big(\sum_{s'\in\Gamma}\chi_i(s') - \sum_{s\in S\cup\{e\}}\chi_i(s)\Big) \\ &= \frac{1}{\chi_i(e)}\sum_{s'\in\Gamma}\chi_i(s') - \frac{1}{\chi_i(e)}\sum_{s\in S}\chi_i(s) - \frac{1}{\chi_i(e)}\chi_i(e) \\ &= -\lambda_i - 1,\end{aligned}$$

proving the result. □ □

Theorem 3.3. *Suppose $\Gamma = \{g_1, g_2, \cdots, g_n\}$ and F is a subset of $\{1, 2, \cdots, n\}$ such that for all $j \in F$, the order of g_j is a power of a prime p_j. Define $S = \bigcup_{j\in F} g_j^\Gamma$ and $X = Cay(\Gamma, S)$. We also assume that for each $\chi \in Irr(\Gamma)$, $\chi(g_j)$ is an integer. Then $Spec(X) = \{\lambda_1, \cdots, \lambda_n\}$, where $\lambda_i = \sum_{j\in F} |g_j^\Gamma|\Big(1 + \frac{k_j p_j}{\chi_i(e)}\Big)$, for some integer k_j.*

Proof. Suppose χ_l, $1 \le l \le |Irr(\Gamma)|$, is an arbitrary irreducible character of Γ. Then by [17, Corollary 22.27], $\chi_l(g_j) \equiv \chi_l(e)\ (mod\ p_j)$. Thus, $\chi_l(g_j) - \chi_l(e) = p_j k_j$, for some integer k_j. This implies that $\frac{\chi_l(g_j)}{\chi_l(e)} = 1 + \frac{p_j k_j}{\chi_l(e)}$ and so,

$$\lambda_l = \sum_{s\in S}\frac{\chi_l(s)}{\chi_l(e)} = \sum_{j\in F}\sum_{s\in g_j^G}\frac{\chi_l(s)}{\chi_l(e)} = \sum_{j\in F}\sum_{s\in g_j^\Gamma}\Big(1 + \frac{p_j k_j}{\chi_l(e)}\Big) = \sum_{j\in F}|g_j^\Gamma|\Big(1 + \frac{p_j k_j}{\chi_l(e)}\Big).$$

Corollary 3.4. *Suppose Γ is a finite group, S is a normal subset of Γ such that all elements of S are involutions and $X = Cay(\Gamma, S)$. If F is a representative set for $\Gamma-$conjugacy classes of S then the following statements hold:*

1. *if Γ is abelian then all $X-$eigenvlues are odd or all of them are even,*
2. *if Γ is a simple group then $\lambda_\chi = \sum_{g\in F}|g^\Gamma|(1 + \frac{4k_g}{\chi(e)})$.*

The energy of Cayley graphs of cyclic, dicyclic, semi-dihedral, Suzuki, Ree groups are computed in [16].

Example 3.5. *Let A_5 be the alternating group on five symbols. In [19], it is proved that the conjugacy classes of alternating group A_5 are*

$$()^\Gamma, (1,2)(3,4)^\Gamma, (1,2,3)^\Gamma, (1,2,3,4,5)^\Gamma, (1,2,3,5,4)^\Gamma,$$

where $a = (1,2,3,4,5)$ and $b = (2,3)(4,5)$ are the generators of this group. Let $S = \{a, a^{-1}, b\}$. Then the Cayley graph $X = Cay(A_5, S)$ is as depicted in Figure 1. The spectrum of this graph is

$$3^1, 2.7^3, 2.3^5, 1.8^3, 1.6^4, 1^9, 0.6^5, -0.1^3, -0.4^4, -1.3^4, -1.4^3, -1.6^5, -2^4, -2.57^4, -2.6^3.$$

Hence, $\mathcal{E}(Cay(A_5, S)) \cong 91.78.$

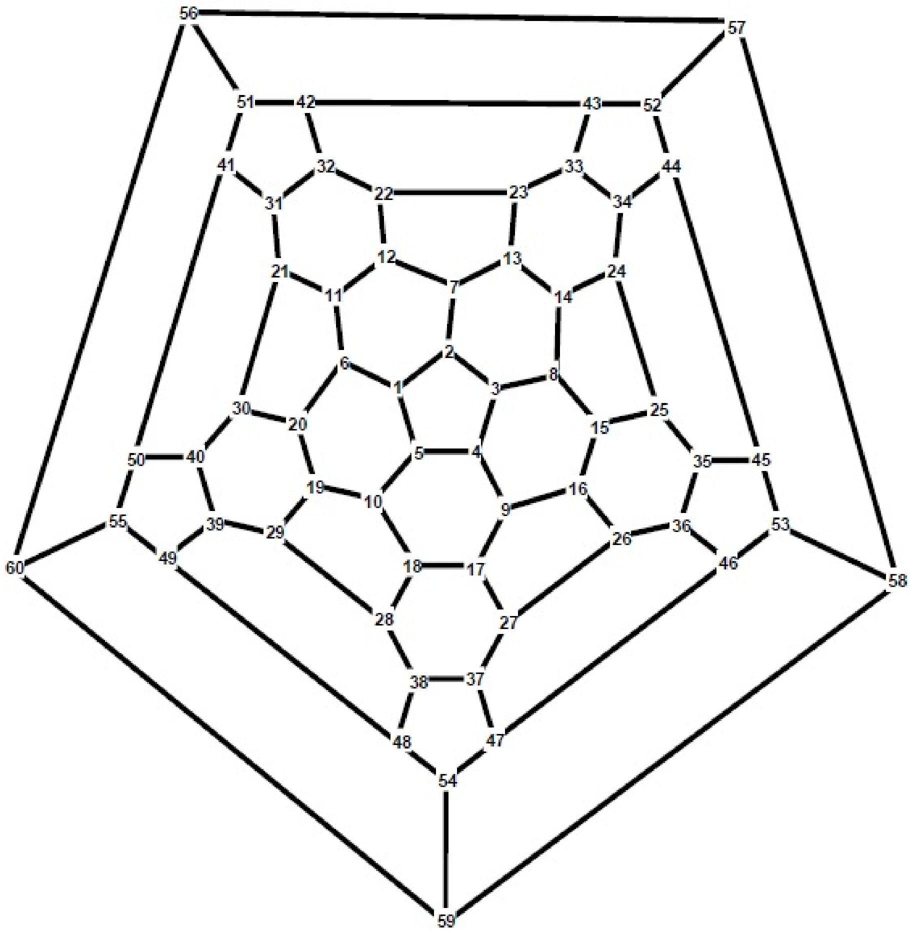

Figure 1. Cayley graph $X = Cay(A_5, S)$, where $S = \{(1,2,3,4,5), (1,5,4,3,2), (2,3)(4,5)\}$.

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