Journal of Discrete Mathematics and Its Applications 7 (3) (2022) 119–131

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Edge version of some degree based topological descriptors of graphs

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Academic Editor: Saeid Alikhani

Abstract. In this paper, we study the edge version of some degree based topological indices such as general sum-connectivity index, Randic index, inverse sum indeg index, symmetric division deg index, augmenting Zagreb index and harmonic polynomial for joint graphs and certain graph operations.

Keywords: topological index, line graph, joint graph
Mathematics Subject Classification (2010): 05C09.

1 Introduction

A topological index is a mathematical measure which correlates to the chemical structures of any simple finite graph. They are invariant under the graph isomorphism. They play an important role in the study of $QSAR/QSPR$. There are numerous topological descriptors that have some applications in theoretical chemistry. Among these topological descriptors the degree-based topological indices are of great importance. The first degree-based topological indices that were defined by Gutman and Trinajstić in [7] 1972, are the first and second Zagreb indices. These indices were originally defined as follows:

$$M_1(G) = \sum_{u \in V(G)} \Big(d_G(u)\Big)^2 = \sum_{uv \in E(G)} \Big(d_G(u) + d_G(v)\Big) \text{ and } M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v),$$

where $d_G(u)$ is the degree of a vertex u in G.

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Received 1 September 2022; Revised 10 September 2022; Accepted 22 September 2022
First Publish Date: 1 October 2022

In a graph G, if the corresponding edges share a vertex in G , the *line graph* $L(G)$ of a graph G is considered as a graph with vertices of the edges in G, and it possesses two adjacent vertices. Similarly, the degree of an edge $e \in E(G)$ is represented by the number of its adjacent vertices in $V(L(G))$. In this paper, we study the edge version of some degree based topological indices such as general sum-connectivity index, Randic index, inverse sum indeg index, symmetric division deg index, augmenting Zagreb index and harmonic polynomial for joint graphs and certain graph operations.

2 Preliminaries

2.1 Edge version of atom-bond connectivity index

The atom-bond connectivity index was proposed by Estrada et al. [1] which is defined as $ABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_G(u)+d_G(v)-2}{d_G(u)\times d_G(v)}}$. For more details about ABC index, see [2–5]. Referring to the end vertex degree $d_G(e)$ and $d_G(f)$ of edges e and f in a line graph of G, Farahani [6] proposed the edge version of atom-bond connectivity index which is defined as ${}_eABC(G) = \sum_{uv \in E(G)} \sqrt{\frac{d_{L(G)}(e)+d_{L(G)}(f)-2}{d_{L(G)}(e)\times d_{L(G)}(f)}}$, where $d_{L(G)}(e)$ is the degree of the edge e in the line graph $L(G)$.

2.2 Edge version of sum-connectivity index

The *sum-connectivity index* was proposed by Zhou and Trinajstić [8] in 2009, which is defined as the sum over all the edges of the graph of the terms $(d_G(u)+d_G(v))^{\frac{-1}{2}}$. This concept was extended to the general sum-connectivity index in 2010 [9], which is defined as follows: $\chi_\alpha(G) = \sum\limits_{uv \in E(G)} (d_G(u)+d_G(v))^\alpha$. The edge version of sum-connectivity index expressed as ${}_e\chi_\alpha(G) = \sum\limits_{e,f \in E(L(G))} (d_{L(G)}(e)+d_{L(G)}(f))^\alpha$, where α is a real number. The sum-connectivity index correlate well with the π-electron energy of benzenoid hydrocarbons [17].

2.3 Edge version of Randic index

The *Randic index* which is defined as the multiple over all the edges of the graph of the terms $(d_G(u)d_G(v))^{\frac{-1}{2}}$. The edge version of Randic index expressed as

$$ {}_eR_\alpha(G) = \sum_{e,f \in E(L(G))} (d_{L(G)}(e)d_{L(G)}(f))^\alpha. $$

2.4 Edge version of Harmonic index

Zhang [15, 16] introduced the harmonic index in 2012 which is defined as

$$ H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(x)+d_G(y)}. $$

The edge version of harmonic index is defined as

$$_eH(G) = \sum_{ef \in E(L(G))} \frac{2}{d_{L(G)}(e) + d_{L(G)}(f)}.$$

2.5 Edge version of inverse sum indeg and symmetric division deg indices

The edge version of inverse sum indeg index is defined as

$$_eISI(G) = \sum_{ef \in E(L(G)} \frac{d_{L(G)}(e) d_{L(G)}(f)}{d_{L(G)}(e) + d_{L(G)}(f)}.$$

where $d_{L(G)}(e)$ is the degree of the edge e in $L(G)$ and the edge version of symmetric division deg index is defined as $_eSDD(G) = \sum_{ef \in E(L(G)} \frac{d_{L(G)}(e)^2 + d_{L(G)}(f)^2}{d_{L(G)}(e) d_{L(G)}(f)}$.

2.6 Edge version of Augumenting Zagreb index

Furtula et al [12] modify the ABC index and named as Augumenting Zagreb index. The correlating ability among several topological indices possess by AZI. The edge version of AZI is defined as $_eAZI(G) = \sum_{ef \in E(L(G)} \left(\frac{d_{L(G)}(e) d_{L(G)}(f)}{d_{L(G)}(e) + d_{L(G)}(f) - 2} \right)^3$.

2.7 Edge version of Harmonic polynomial

The harmonic polynomial is defined in [11] as $H(G,x) = \sum_{uv \in E(G)} 2x^{d_G(u) + d_G(v) - 1}$. Note that $\int_0^1 H(G,x) dx = H(G)$. The edge version of harmonic polynomial is defined as $_eH(G,x) = \sum_{ef \in E(L(G)} 2x^{d_{L(G)}(e) + d_{L(G)}(f) - 1}$.

3 Main results

Joint structure and graph operation of basic molecular structures are frequently found in the new chemical compounds, nanomaterials and drugs in the fields of chemical and pharmaceutical engineering. The phenomenon provides us some hints on the significance and feasibility of the research on the chemical and pharmacological properties of these molecular structures. In this section, the edge version of some topological indices for joint graphs and certain graph operations are determined.

Theorem 3.1. *Let $G = P_n + C_m$ be the join graph for $n \geq 4, m \geq 3$, see Figure 1. Then*
(i) $_e\chi_\alpha(P_n + C_m) = (n + m - 7)(4)^\alpha + 3^\alpha + 3(5)^\alpha + 3(6)^\alpha$.
(ii) $_eR_\alpha(P_n + C_m) = (n + m - 7)(4)^\alpha + 2^\alpha + 3(6)^\alpha + 3(9)^\alpha$.

Proof. (i) Apply induction method, one can see that this line graph possesses $n + m$ edges. Let $d_{L(G)}(e)$ and $d_{L(G)}(f)$ be the degree of edge of e. From the structure of the graph $P_n + C_m$, we consider the following.

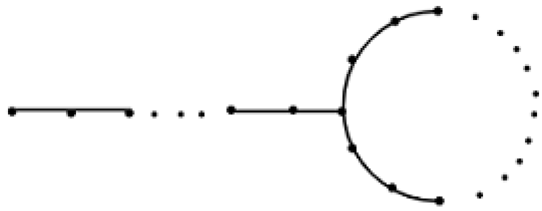

Figure 1. The graph $P_n + C_m$.

Table 1. Degree of vertices in $L(G)$.

$(d_{L(G)}(e),d_{L(G)}(f))$, where $ef \in E(L(G))$	Number of edges
(1,2)	1
(2,2)	$n+m-7$
(2,3)	3
(3,3)	3

From the definition of ${}_e\chi_\alpha$ and Table 1.1, we have

$$\begin{aligned}{}_e\chi_\alpha(G) &= \sum_{e,f\in E(L(G))} (d_{L(G)}(e) + d_{L(G)}(f))^\alpha \\ &= (n+m-7)(4)^\alpha + 3^\alpha + 3(5)^\alpha + 3(6)^\alpha.\end{aligned}$$

A similar argument of (i), we obtain the result of (ii).

Corollary 3.2. *Let $G = P_n + C_m$ be the join graph. Then*

$$ {}_e\chi_\alpha(G) = \begin{cases} 4(n+m+2), & \textit{if} \quad \alpha = 1; \\ \frac{1}{60}(15n+15m+71), & \textit{if} \quad \alpha = -1; \\ 16(n+m+5), & \textit{if} \quad \alpha = 2; \\ 2(n+m-7)+\sqrt{3}+3\sqrt{5}+3\sqrt{6}, & \textit{if} \quad \alpha = \frac{1}{2}; \\ \frac{n+m-7}{2}+\frac{3}{\sqrt{5}}+\frac{3}{\sqrt{6}}+\frac{1}{\sqrt{3}}+, & \textit{if} \quad \alpha = \frac{-1}{2}. \end{cases}$$

Proof: Putting $\alpha = 1$ in Theorem 2.1, we get

$$\begin{aligned}{}_e\chi(G) &= 3 + (n+m-1)4 + 15 + 18 \\ &= 4(n+m+2).\end{aligned}$$

Putting $\alpha = -1$ in Theorem 2.1, we get

$$\begin{aligned}{}_e\chi_{-1}(G) &= (n+m-7)(4)^{-1} + 3^{-1} + 3(5)^{-1} + 3(6)^{-1} \\ &= \frac{1}{60}(15n+15m+71).\end{aligned}$$

Putting $\alpha = 2$ in Theorem 2.1, we get

$$\begin{aligned} {}_e\chi_2(G) &= (n+m-7)(4)^2 + 3^2 + 3(5)^2 + 3(6)^2 \\ &= 16(n+m+5). \end{aligned}$$

Putting $\alpha = \frac{1}{2}$ in Theorem 2.1, we get

$$\begin{aligned} {}_e\chi_{\frac{1}{2}}(G) &= (n+m-7)(4)^{\frac{1}{2}} + 3^{\frac{1}{2}} + 3(5)^{\frac{1}{2}} + 3(6)^{\frac{1}{2}} \\ &= 2(n+m-7) + \sqrt{3} + 3\sqrt{5} + 3\sqrt{6}. \end{aligned}$$

Putting $\alpha = \frac{-1}{2}$ in Theorem 2.1, we get

$$\begin{aligned} {}_e\chi_{\frac{-1}{2}}(G) &= (n+m-7)(4)^{\frac{-1}{2}} + 3^{\frac{-1}{2}} + 3(5)^{\frac{-1}{2}} + 3(6)^{\frac{-1}{2}} \\ &= \frac{1}{\sqrt{3}} + \frac{n+m-7}{2} + \frac{3}{\sqrt{5}} + \frac{3}{\sqrt{6}}. \end{aligned}$$

Corollary 3.3. *Let $G = P_n + C_m$ be the join graph. Then*

$${}_eR_\alpha(G) = \begin{cases} 4n+4m+19, & if \quad \alpha = 1; \\ \frac{1}{12}(3n+3m-5), & if \quad \alpha = -1; \\ 16n+16m+243, & if \quad \alpha = 2; \\ 2(n+m) + \sqrt{2} + 3\sqrt{6} - 5, & if \quad \alpha = \frac{1}{2}; \\ \frac{n+m-5}{2} + \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{6}}, & if \quad \alpha = \frac{-1}{2}. \end{cases}$$

Using Table 1.1 and definitions of the edge version of ISI SDD AZI and ${}_eH(G,x)$, we obtain the following theorem.

Theorem 3.4. *Let $G = P_n + C_m$ be the join graph. Then*
(i) ${}_eSDD(G) = 2(n+m) - 1$.
(ii) ${}_eISI(G) = n + m + \frac{53}{30}$.
(iii) ${}_eAZI(G) = \frac{1}{16}(128n + 128m - 141)$.
(iv) ${}_eH(G,x) = 2x^2(1 + x(n+m-7)) + 6x^4(1+x)$.

Theorem 3.5. *Let $G = P_n + S_m$ be the join graph for $n, m \geq 4$, see Figure 2. Then*
(i) ${}_e\chi_\alpha(G) = 3^\alpha + (n-4)4^\alpha + (2+m)^\alpha + \frac{(m-1)(m-2)}{2}(2(m-1))^\alpha + (m-1)(2m-1)^\alpha$.
(ii) ${}_eR_\alpha(G) = 2^\alpha + 4^\alpha(n-4) + (2m)^\alpha + \frac{(m-1)(m-2)}{2}((m-1)^2)^\alpha + (m-1)(m^2-m)^\alpha$.

Proof. Apply the induction method, one can see that this line graph possesses

$$\frac{2(m+2) + (m-1)(m-2)}{2}$$

edges. Let $d_{L(G)}(e)$ be the degree of edge of e. From the structure of the graph $P_n + S_m$, we consider the following edge partitions.

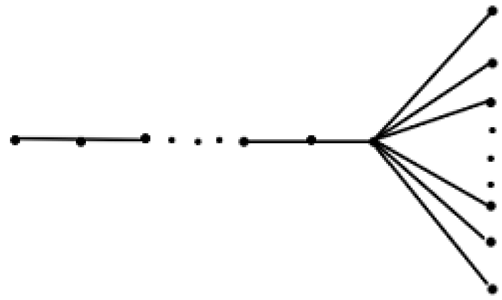

Figure 2. The graph $P_n + S_m$.

Table 2. Degree of vertices in $L(G)$.

$(d_{L(G)}(e),d_{L(G)}(f))$, where $ef \in E(L(G))$	Number of edges
(1,2)	1
(2,2)	$n-4$
(2,m)	1
$(m-1,m-1)$	$\frac{(m-1)(m-2)}{2}$
$(m-1,m)$	$m-1$

From the definition of ${}_e\chi_\alpha$ and Table 1.2, we obtain

$$\begin{aligned} {}_e\chi_\alpha(G) &= \sum_{e,f\in E(L(G))} (d_{L(G)}(e) + d_{L(G)}(f))^\alpha \\ &= 3^\alpha + (n-4)4^\alpha + (2+m)^\alpha + \frac{(m-1)(m-2)}{2}(2(m-1))^\alpha \\ &+ (m-1)(2m-1)^\alpha. \end{aligned}$$

A similar argument of (i), we get (ii).

Corollary 3.6. *Let $G = P_n + S_m$ be the join graph. Table 1.2, we obtain*

$${}_e\chi_\alpha(G) = \begin{cases} m^3 - 2m^2 + 3m + 4n - 8, & \text{if} \quad \alpha = 1; \\ n + \frac{1}{2+m} + \frac{(m-1)(m-2)}{4(m-1)} + \frac{m-1}{2m-1}, & \text{if} \quad \alpha = -1; \\ 2m^4 - 6m^3 + 25m^2 + 13m + 16n - 40, & \text{if} \quad \alpha = 2; \\ \sqrt{m+2} + \frac{m(m-1)}{2}\sqrt{(2m-1)} + \sqrt{3} + 2(n-4), & \text{if} \quad \alpha = \frac{1}{2}; \\ \frac{1}{\sqrt{2+m}} + \frac{(m-1)(m-2)}{2\sqrt{(2(m-1))}} + \frac{(m-1)}{\sqrt{(2m-1)}} + \frac{1}{\sqrt{3}} + \frac{(n-4)}{2}, & \text{if} \quad \alpha = \frac{-1}{2}. \end{cases}$$

Proof. Putting $\alpha = 1$ in Theorem 2.7, we get

$$\begin{aligned} {}_e\chi(G) &= 3 + (n-4)4 + (2+m) + \frac{(m-1)(m-2)}{2}(2(m-1)) + (m-1)(2m-1) \\ &= m^3 - 2m^2 + 3m + 4n - 8. \end{aligned}$$

Putting $\alpha = -1$ in Theorem 2.7, we get

$$ {}_e\chi_{-1}(G) = 3^{-1} + (n-4)4^{-1} + (2+m)^{-1} + \frac{(m-1)(m-2)}{2}(2(m-1))^{-1} + (m-1)(2m-1)^{-1} = n + \frac{1}{2+m} + \frac{(m-1)(m-2)}{4(m-1)} + \frac{m-1}{2m-1}. $$

Putting $\alpha = 2$ in Theorem 2.7, we get

$$ {}_e\chi_2(G) = 3^2 + (n-4)4^2 + (2+m)^2 + \frac{(m-1)(m-2)}{2}(2(m-1))^2 + (m-1)(2m-1)^2 = 2m^4 - 6m^3 + 25m^2 + 13m + 16n - 40. $$

Putting $\alpha = \frac{1}{2}$ in Theorem 2.7, we get

$$ {}_e\chi_{\frac{1}{2}}(G) = 3^2 + (n-4)4^{\frac{1}{2}} + (2+m)^{\frac{1}{2}} + \frac{(m-1)(m-2)}{2}(2(m-1))^{\frac{1}{2}} + (m-1)(2m-1)^{\frac{1}{2}} = \sqrt{3} + 2(n-4) + \sqrt{2+m} + \frac{(m-1)(m-2)}{2}\sqrt{(2(m-1))} + (m-1)\sqrt{(2m-1)}. $$

Putting $\alpha = \frac{-1}{2}$ in Theorem 2.7, we get

$$ {}_e\chi_{\frac{-1}{2}}(G) = 3^2 + (n-4)4^{\frac{-1}{2}} + (2+m)^{\frac{-1}{2}} + \frac{(m-1)(m-2)}{2}(2(m-1))^{\frac{-1}{2}} + (m-1)(2m-1)^{\frac{-1}{2}} = \frac{1}{\sqrt{3}} + \frac{(n-4)}{2} + \frac{1}{\sqrt{2+m}} + \frac{(m-1)(m-2)}{2\sqrt{(2(m-1))}} + \frac{(m-1)}{\sqrt{(2m-1)}}. $$

Corollary 3.7. *Let $G = P_n + S_m$ be the join graph. Then*

$$ {}_eR_\alpha(G) = \begin{cases} \frac{1}{2}(m^4 - 3m^3 - 5m^2 - m + 8n + 26), & \text{if } \alpha = 1; \\ \frac{(m-2)}{2(m-1)} + \frac{(m-1)}{(m^2-m)} + \frac{1}{2m} + \frac{n-4}{4} + \frac{1}{2}, & \text{if } \alpha = -1; \\ \frac{(m-1)(m-2)}{2}(m^2-2m+1)^2 + (m-1)(m^4+m^2-2m^3) + 4m^2 + 4 \\ +16(n-4), & \text{if } \alpha = 2; \\ \frac{m^3-4m+5m}{2} + \sqrt{2m} + 2n + \sqrt{2} - 9 + (m-1)\sqrt{m^2-m}, & \text{if } \alpha = \frac{1}{2}; \\ \frac{m-1}{\sqrt{m^2-m}} + \frac{1}{\sqrt{2m}} + \frac{n+m-6}{2} + \frac{1}{\sqrt{2}}, & \text{if } \alpha = \frac{-1}{2}. \end{cases} $$

Proof: Putting $\alpha = 1$ in Theorem 2.9, we get

$$ {}_eR(G) = 2 + 4(n-4) + (2m) + \frac{(m-1)(m-2)}{2}((m-1)^2) + (m-1)(m^2-m) = \frac{1}{2}(m^4 - 3m^3 - 5m^2 - m + 8n + 26). $$

Putting $\alpha = -1$ in Theorem 2.9, we get

$$ {}_eR_{-1}(G) = 2^{-1} + 4^{-1}(n-4) + (2m)^{-1} + \frac{(m-1)(m-2)}{2}((m-1)^2)^{-1} + (m-1)(m^2-m)^{-1} = \frac{1}{2} + \frac{n-4}{4} + \frac{1}{2m} + \frac{(m-1)(m-2)}{2(m-1)^2} + \frac{(m-1)}{m^2-m}. $$

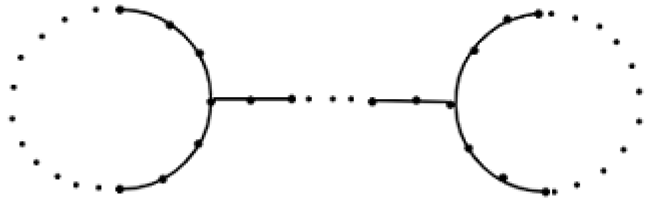

Figure 3. The graph $C_m + P_n + C_m$.

Putting $\alpha = 2$ in Theorem 2.9, we get

$$\begin{aligned} {}_eR_2(G) &= 2^2 + 4^2(n-4) + (2m)^2 + \frac{(m-1)(m-2)}{2}((m-1)^2)^2 + (m-1)(m^2-m)^2 \\ &= \frac{(m-1)(m-2)}{2}(m^2-2m+1)^2 + (m-1)(m^4 = m^2 - 2m^3) + 4m^2 + 4 + 16(n-4). \end{aligned}$$

Putting $\alpha = \frac{1}{2}$ in Theorem 2.9, we get

$$\begin{aligned} {}_eR_{\frac{1}{2}}(G) &= 2^{\frac{1}{2}} + 4^{\frac{1}{2}}(n-4) + (2m)^{\frac{1}{2}} + \frac{(m-1)(m-2)}{2}((m-1)^2)^{\frac{1}{2}} + (m-1)(m^2-m)^{\frac{1}{2}} \\ &= \frac{m^3 - 4m + 5m}{2} + \sqrt{2m} + 2n + \sqrt{2} - 9 + (m-1)\sqrt{m^2-m}. \end{aligned}$$

Putting $\alpha = \frac{-1}{2}$ in Theorem 2.9, we get

$$\begin{aligned} {}_eR_{\frac{-1}{2}}(G) &= 2^{\frac{-1}{2}} + 4^{\frac{-1}{2}}(n-4) + (2m)^{\frac{-1}{2}} + \frac{(m-1)(m-2)}{2}((m-1)^2)^{\frac{-1}{2}} + (m-1)(m^2-m)^{\frac{-1}{2}} \\ &= \frac{1}{\sqrt{2}} + \frac{n-4}{2} + \frac{1}{\sqrt{2m}} + \frac{(m-1)(m-2)}{2(m-1)} + \frac{m-1}{\sqrt{m^2-m}}. \end{aligned}$$

Using Table 1.2 and definitions of the edge version of ISI, SDD, AZI and $H(G,x)$, we obtain the following theorem.

Theorem 3.8. *Let $G = P_n + S_m$ be the join graph. Then*
(i) ${}_eSDD(G) = \frac{6n-19}{3} + \frac{m^2+4}{4m} + \frac{m^4-5m^3+9m^2-7m+2}{2(m-1)} + \frac{2m^3-4m^2+3m-1}{2m-1}$,
(ii) ${}_eISI(G) = \frac{(m-1)^2(m-2)}{4} + \frac{m^3-2m^2+m}{2m-1} + \frac{2m}{2+m} + \frac{3n-10}{3}$,
(iii) ${}_eAZI(G) = 8n - 16 + \frac{(m-1)^7(m-2)}{16m^3} + \frac{(m-1)^4m^3}{(m^2-m-2)^3}$.
(iv) ${}_eH(G,x) = (n-4)2x^3 + 2x^2 + 2x^{(m-1)} + (m-1)(m-2)x^{2m-1} + (m-1)2x^{2m}$.

Theorem 3.9. *Let $G = C_m + P_n + C_m$ be the join graph for $n \geq 4, m \geq 3$, see Figure 3. Then*
(i) *The edge version of general sum connectivity index is* ${}_e\chi_\alpha(G) = (n+2m-10)(4)^\alpha + 6(5)^\alpha + 6(6)^\alpha$.
(ii) *The edge version of Randic index is* ${}_eR_\alpha(C_m + P_n + C_m) = 2^\alpha + 4^\alpha + 6(6)^\alpha + 6(9)^\alpha$.

Proof. Apply the induction method, one can see that this line graph possesses $n+2m+2$ edges. Let $d_{L(G)}(e)$ and $d_{L(G)}(f)$ be the degree of edge of e. From the structure of the graph $C_m+P_n+C_m$, we consider the following degree cases

Table 1.3. Degree of vertices in $L(G)$.

$(d_{L(G)}(e), d_{L(G)}(f))$, where $ef \in E(L(G))$	Number of edges
(2,2)	$n+m-10$
(2,3)	6
(3,3)	6

From the definition of ${}_e\chi_\alpha$ and Table 1.3, we have

$$\begin{aligned} {}_e\chi_\alpha(G) &= \sum_{e,f\in E(L(G))} (d_{L(G)}(e)+d_{L(G)}(f))^\alpha \\ &= (n+2m-10)(4)^\alpha + 6(5)^\alpha + 6(6)^\alpha. \end{aligned}$$

A similar argument of (i), we get (ii).

Corollary 3.10. *Let $G = C_m+P_n+C_m$ be the join graph. Then*

$${}_e\chi_\alpha(G) = \begin{cases} 4n-8m+26, & \text{if} \quad \alpha=1; \\ \frac{5n+10m-6}{20}, & \text{if} \quad \alpha=-1; \\ 16n+32m+206, & \text{if} \quad \alpha=2; \\ (2n+2m-10)+6\sqrt{5}+6\sqrt{6}, & \text{if} \quad \alpha=\frac{1}{2}; \\ \frac{n+2m-10}{2}+\frac{6}{\sqrt{5}}+\frac{6}{\sqrt{6}}, & \text{if} \quad \alpha=\frac{-1}{2}. \end{cases}$$

Corollary 3.11. *Let $G = C_m+P_n+C_m$ be the join graph. Then*

$${}_eR_\alpha(G) = \begin{cases} 2(2n+4m+5), & \text{if} \quad \alpha=1; \\ \frac{1}{12}(3n+6m-10), & \text{if} \quad \alpha=-1; \\ 16n+32m+542, & \text{if} \quad \alpha=2; \\ 2n+4m-2+2\sqrt{6}, & \text{if} \quad \alpha=\frac{1}{2}; \\ \frac{n+2m-6+2\sqrt{6}}{2}, & \text{if} \quad \alpha=\frac{-1}{2}. \end{cases}$$

Using Table 1.3 and definitions of the edge version of the indices ISI, SDD, AZI and $H(G,x)$, we obtain the following theorem.

Theorem 3.12. *Let $G = C_m+P_n+C_m$ be the join graph. Then*
(i) ${}_eSDD(G) = 2n+4m+5.$
(ii) ${}_eISI(G) = n+2m+\frac{31}{5}.$
(iii) ${}_eAZI(G) = 2n+22m.$
(iv) ${}_eH(G,x) = (n+2m)2x^3 - 4x^3(2x^2+2x+5).$

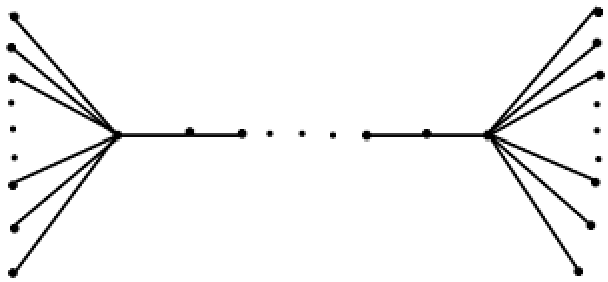

Figure 4. The graph $S_m + P_n + S_m$.

Theorem 3.13. *Let $G = S_m + P_n + S_m$ be the join graph for $n, m \geq 4$, see Figure 4. Then*
(i) ${}_e\chi_\alpha(G) = (n-4)4^\alpha + 2(2+m)^\alpha + (m-1)(m-2)(2(m-1))^\alpha + 2(m-1)(2m-1)^\alpha$.
(ii) ${}_eR_\alpha(G) = 4^\alpha(n-4) + 2(2m)^\alpha + (m-1)(m-2)((m-1)^2)^\alpha + 2(m-1)(m^2-m)^\alpha$.

Proof. Apply induction method, one can see that this line graph possesses $m(m-1) + n - 2$ edges. If $d_{L(G)}(e)$ and $d_{L(G)}(f)$ are the degree of edge of e. From the structure of the graph $S_m + P_n + S_m$, we consider the following degree cases.

Table 1.4. Degree of vertices in $L(G)$.

$(d_{L(G)}(e), d_{L(G)}(f))$, where $ef \in E(L(G))$	Number of edges
(2,2)	$n-4$
$(2,m)$	2
$(m-1, m-1)$	$(m-1)(m-2)$
$(m, m-1)$	$2(m-1)$

From the definition of ${}_e\chi_\alpha$, and Table 1.4, we have

$$\begin{aligned} {}_e\chi_\alpha(G) &= \sum_{e,f \in E(L(G))} (d_{L(G)}(e) + d_{L(G)}(f))^\alpha \\ &= (n-4)4^\alpha + 2(2+m)^\alpha + (m-1)(m-2)(2(m-1))^\alpha \\ &\quad + 2(m-1)(2m-1)^\alpha. \end{aligned}$$

A similar argument of (i) we get the result of (ii).

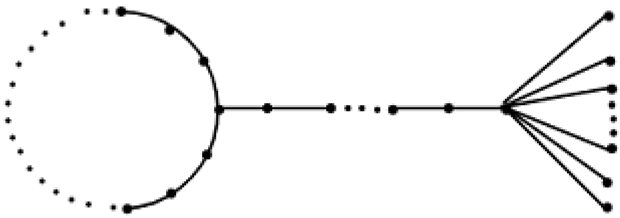

Figure 5. The graph $C_m + P_n + S_r$.

Corollary 3.14. *Let $G = S_m + P_n + S_m$ be the join graph. Then*

$$_e\chi_\alpha(G) = \begin{cases} 2(2n + m^3 - 2m^2 + 3m - 7), & \text{if} \quad \alpha = 1; \\ \frac{(n-4)+2(m-2)}{4} + \frac{2}{2+m} + \frac{2(m-1)}{2m-1}, & \text{if} \quad \alpha = -1; \\ 2m^4 - 2m^3 + 30m^2 + 4m - 74, & \text{if} \quad \alpha = 2; \\ (n-4)2 + 2\sqrt{2+m} + (m-1)(m-2)\sqrt{2(m-1)} & \\ +2(m-1)\sqrt{2m-1}, & \text{if} \quad \alpha = \frac{1}{2}; \\ \frac{n-4}{2} + \frac{2}{\sqrt{2+m}} + \frac{(m-1)(m-2)}{2\sqrt{(m-1)}} + \frac{2(m-1)}{\sqrt{2m-1}}, & \text{if} \quad \alpha = \frac{-1}{2}. \end{cases}$$

Corollary 3.15. *Let $G = S_m + P_n + S_m$ be the join graph. Then*

$$_eR_\alpha(G) = \begin{cases} m^4 - 3m^3 + 5m^2 - m - 14, & \text{if} \quad \alpha = 1; \\ \frac{n-4}{4} + \frac{m^2+m-3}{m(m-1)}, & \text{if} \quad \alpha = -1; \\ 16n + 2m^5 + 3m^4 - 6m^3 + 12m^2 - 4m - 63, & \text{if} \quad \alpha = 2; \\ (n-4)2 + 2\sqrt{2m} + (m-1)^2(m-2) + \sqrt{2(m-1)}m^2 - m, & \text{if} \quad \alpha = \frac{1}{2}; \\ \frac{2(m-1)}{m^2-m} + (m-2) + \frac{2}{\sqrt{2m}} + \frac{(n-4)}{4}, & \text{if} \quad \alpha = \frac{-1}{2}. \end{cases}$$

Using Table 1.4 and definitions of the edge version of ISI, SDD, AZI and $H(G,x)$, we obtain the following theorem.

Theorem 3.16. *Let $G = S_m + P_n + S_m$ be the join graph. Then*
(i) $_eSDD(G) = 2(n-4) + \frac{3m^4-9m^3+16m^2-10m+1}{m(n-1)}$.
(ii) $_eISI(G) = n - 4 + \frac{4m}{m+2} + \frac{m^3-4m^2+5m-2}{2} + \frac{m^3-m^2}{2m-1}$.
(iii) $_eAZI(G) = 8n + (m-1)^4\left(\frac{(m-2)(m-1)^2}{2m-4} + \frac{2m^3}{2m-3}\right) - 16$.
(iv) $_eH(G,x) = 2x^3(n-4) + 4x^{m+1} + (m-1)(m-2)2x^{2m-1} + 2(m-1)2x^{2m-2}$.

Theorem 3.17. *Let $G = C_m + P_n + S_r$ be the join graph for $n, r \geq 4, m \geq 3$, see Figure 5. Then*
(i) $_e\chi_\alpha(G) = (n+m-7)4^\alpha + 3(5)^\alpha + r + 1^\alpha + 3(6)^\alpha + \frac{(r-1)(r-2)}{2}(2(r-1))^\alpha + (r-1)(2r-1)^\alpha$.
(ii) $_eR_\alpha(S_m + P_n + S_r) = (n+m-7)4^\alpha + 3(6)^\alpha + (2(r-1))^\alpha + 3(9)^\alpha + \frac{(r-1)(r-2)}{2}(r-1)^\alpha + (r-1)(r(r-1))^\alpha$.

Proof. Apply induction method, one can see that this line graph possesses $\frac{2(n+m)+(r-1)r}{2}$ edges. Let $d_{L(G)}(e)$ and $d_{L(G)}(f)$ are the degree of edge of e. From the structure of the graph $C_m + P_n + S_r$, we consider the following degree cases.

Table 1.5. Degree of vertices in $L(G)$.

$(d_{L(G)}(e), d_{L(G)}(f))$, where $ef \in E(L(G))$	Number of edges
(2,2)	$n+m$-7
$(r-1, r-1)$	$\frac{(r-1)(r-2)}{2}$
(2,3)	3
$(2, r-1)$	1
(3,3)	3
$(r-1, r)$	$r-1$

From the definitions of ${}_e\chi_\alpha$, ${}_eR_\alpha$, and Table 1.4, we obtain the required results.

Corollary 3.18. *Let $G = C_m + P_n + S_r$ be the join graph. Then*

$$
{}_e\chi_\alpha(G) = \begin{cases} 4(n+m) + r^3 - 2r^2 + 2r + 4, & \text{if} \quad \alpha = 1; \\ \frac{n+m-7}{4} + \frac{r^2+3r+2}{4(r+1)} + \frac{(r-1)}{2r-1} + \frac{11}{10}, & \text{if} \quad \alpha = -1; \\ (n+m-7)16 + 2r^4 - 6r^3 + 11r^2 - 7r + 187, & \text{if} \quad \alpha = 2; \\ (n+m-7)2 + 3\sqrt{5} + \sqrt{r+1} + 3\sqrt{6} + \frac{(r-1)n}{2}\sqrt{2(r-1)}, & \text{if} \quad \alpha = \frac{1}{2}; \\ \frac{n+m-7}{2} + \frac{3}{\sqrt{5}} + \frac{1}{\sqrt{r+1}} + \frac{3}{\sqrt{6}} + \frac{(r-1)(r-2)}{2\sqrt{2(n-1)}} + \frac{(r-1)}{\sqrt{2r-1}}, & \text{if} \quad \alpha = \frac{-1}{2}. \end{cases}
$$

Corollary 3.19. *Let $G = C_m + P_n + S_r$ be the join graph. Then*

$$
{}_eR_\alpha(G) = \begin{cases} 4(n+m) + \frac{3r^3-7r^2+9r+29}{2}, & \text{if} \quad \alpha = 1; \\ \frac{(r-1)(r-2)+1}{2(r-1)} + \frac{r-1}{r(r-1)} + \frac{n+m-7}{4} + \frac{5}{6}, & \text{if} \quad \alpha = -1; \\ (n+m)16 + 4(r^2-2r) + r^2(r-1)^3 + \frac{(r-1)^3(r-2)}{2} + 243, & \text{if} \quad \alpha = 2; \\ (n+m-7)2 + \frac{(r-1)^2(r-2)}{2} + (r-1)\sqrt{r(r-1)} \\ + \sqrt{2(r-1)} + 3\sqrt{6} + 9, & \text{if} \quad \alpha = \frac{1}{2}; \\ \frac{(n+m-7)}{2} + \frac{(r-1)(r-2)}{2\sqrt{r-1}} + \frac{(r-1)}{\sqrt{r(r-1)}} + \frac{1}{\sqrt{2(r-1)}} + \frac{\sqrt{6}+3}{\sqrt{6}}, & \text{if} \quad \alpha = \frac{-1}{2}. \end{cases}
$$

Corollary 3.20. *Let $G = C_m + P_n + S_r$ be the join graph. Then*
(i) ${}_eSDD(G) = (n+m)2 + \frac{53}{2} + \frac{4+(r-1)^2}{2(r-1)} + (r-1)(r-2) + \frac{(r-1)^3+r^2}{r(r-1)}$.
(ii) ${}_eISI(G) = n+m + \frac{11}{10} + \frac{2(r-1)}{2+(r-1)} + \frac{(r-1)^2(r-2)}{4} + \frac{2r-1}{r}$.
(iii) ${}_eAZI(G) = (n+m)8 + \frac{139}{64} + \frac{8(r-2)^3}{(r-1)^3} + \frac{(r-1)^6}{16(r-2)} + \frac{r^3(r-1)^4}{(2r-3)^3}$.
(iv) ${}_eH(G,x) = (n+m-7)2x^3 + 6x^4(1+x) + 2x^r + (r-1)(r-2)x^{(2r-3)} + 2(r-1)x^{2(r-1)}$.

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Citation: K. Pattabiraman, T. Suganya, Edge version of some degree based topological descriptors of graphs, J. Disc. Math. Appl. 7(3) (2022) 119–131.

https://doi.org/10.22061/JDMA.2022.721