JOURNAL OF MATHEMATICAL NANOSCIENCE

JMNS

Vol. 1, No. 1, 2011, 51-57

Hosoya index and Fibonacci numbers

SAEID ALIKHANI

Department of Mathematics, Yazd University, 89195-741, Yazd, Iran
alikhani@yazduni.ac.ir

ABSTRACT. Let $G=(V,E)$ be a simple graph. The Hosoya index $Z(G)$ of G is defined as the total number of edge independent sets of G. Fibonacci numbers are terms of the sequence defined in a quite simple recursive fashion. In this paper, we investigate the relationships between Hosoya index and Fibonacci numbers. Also we consider Fibonacci cubes and study some of its parameters which is related to Fibonacci numbers.

Keywords: Hosoya index; Fibonacci number; Fibonacci cube.

1. INTRODUCTION

Let $G=(V,E)$ be a simple graph of order n and size m. An r-matching of G is a set of r edges of G which no two of them have common vertex. The maximum number of edges in a matching of a graph G is called the matching number of G and denoted by $\alpha'(G)$.

The Hosoya index $Z(G)$ of a graph G is defined as the total number of its matchings [6]. If $m(G,k)$ denotes the number of its k-matchings, then

$$Z(G)=\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} m(G,k).$$

The Hosoya index has been studied intensively in the literature [1, 4,9, 13,15,16].

For $v \in V(G)$, we denote by $G-v$ the graph obtained from G by deleting the vertex v together with their incident edges. For $e \in E(G)$, we denote by $G-e$ the graph obtained from G by removing the edge e. Let $deg(v)$ denotes the vertex degree of v. We denote by P_n, S_n and C_n the path, the star and the cycle on n vertices, respectively.

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The *corona* of two graphs G_1 and G_2, as defined by Frucht and Harary in [3], is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added. The *join* of two graphs G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1)$ and $v \in V(G_2)\}$.

Fibonacci numbers are terms of the sequence defined in a quite simple recursive fashion. We define Fibonacci numbers: $F_0 = 0, F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

In section 2 we consider graphs with speci ic constructions and study their Hosoya index. In Section 3 we consider Fibonacci cubes and study some of its parameters which is related to Fibonacci numbers.

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2. HOSOYA INDEX OF CERTAIN GRAPHS

In this section we compute the Hosoya index for some certain graphs. We state the following theorem:

Theorem 1. ([5])

1. Let $e = uv$ be an edge of a graph G. Then
$$Z(G) = Z(G-e) + Z(G - \{u,v\}).$$
2. Let v be a vertex of a graph G. Then
$$Z(G) = Z(G-v) + \sum_{uv} Z(G-uv),$$
where the summation extends over all vertices adjacent to v.

3. If $G_1, G_2, \ldots, G_k$ are connected components of G, then $Z(G) = \prod_{i=1}^{k} Z(G_i)$.

Here we consider some kind of graphs and obtain their Hosoya indices.

Let P_{m+1} be a path with vertices labeled by $y_0, y_1, \ldots, y_m$, for $m \geq 0$ and let G be any graph. Denote by $G_{v_0}(m)$ (or simply $G(m)$, if there is no likelihood of confusion) a graph obtained from G by identifying the vertex v_0 of G with an end vertex y_0 of P_{m+1} (see Figure 1). For example, if G is a path P_2, then $G(m) = P_2(m)$ is the path P_{m+2}.

Let P_m be a path with vertices labeled $y_1, \ldots, y_m$ and let a, b be two specific vertices of a graph G (note that may be $a = b$). Denote by $G'_{a,b}(m)$ (or simply $G'(m)$, if there is no likelihood of confusion) a graph obtained from G and P_m by identifying verices a and y_1 and also b and y_m, See Figure 1. Through-out our discussion these two vertices a and b are fixed.

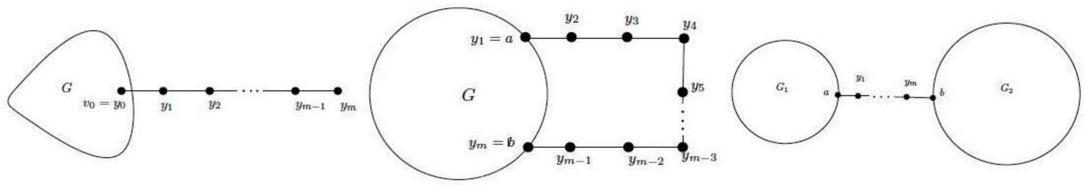

Figure 1. Graphs $G(m)$,$G'(m)$ and $G_1(m)G_2$, respectively.

By Theorem 1, we have the following theorem for Hosoya indices of graphs $G(m)$,$G'(m)$ and $G_1(m)G_2$:

Theorem 2 .

1. The Hosoya index of graph $G(m)$ satisfy

$$Z(G(m))=Z(G(m-1))+Z(G(m-2)).$$

2. The Hosoya index of graph $G_1(m)G_2$ satisfy

$$Z(G_1(m)G_2)=Z(G_1(1))Z(G_2(m-1))+Z(G_1)Z(G_2(m-2)).$$

3. The Hosoya index of graph $G'(m)$ satisfy

$$Z(G'(m))=Z(G(1)(m-1))+Z(G(m-2)).$$

As a consequence of Theorem 2, we have the following corollary:

Corollary 3.

1. Let P_n be a path with n vertices. Then for every $n \geq 2$, $Z(P_n)=F_{n+1}$, with $Z(P_0)=0$, $Z(P_1)=1$.
2. Let C_n be a cycle of order n, then

$$Z(C_n)=F_{n-1}+F_{n+1}.$$

Proof.

1. Using Theorem2 (i) for $G=K_1$ we have the result.
2. It suf ices to use Theorem2 (iii) for $G=K_2$.

Let $L_{n,k}(s,t)$ be the set of all unicyclic graphs as shown in Figure 2. We assume that u_0 and v_0 are adjacent in $L_{n,k}$ and n is the order of graph, i.e. $s+t+k=n$. Obviously these graphs are of the form $C_k(t)(s)$.

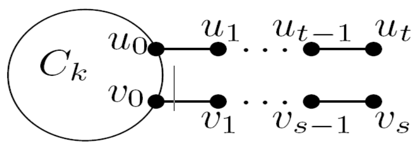

Figure 2. Graph $L_{n,k}(s,t)$.

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We have the following theorem for Hosoya indices of graphs $L_{n,k}(s,t)$.

Theorem 4.

$$Z(L_{n,k}(s,t)) = F_{n+1} + F_{k-1}.F_{s+1}.F_{n-s-k+1}.$$

Proof. By Theorem 1 we have

$$\begin{aligned} Z(L_{n,k}(s,t)) &= Z(L_{n,k}(s,t) - v_0v_1) + Z(L_{n,k}(s,t) - v_0 - v_1) \\ &= Z(P_s)Z(L_{n-s,k}) + Z(P_{s-1})Z(P_{n-s-1}) \\ &= F_{s+1}(F_{n-s+1} + F_{k-1}F_{n-s-k+1}) + F_sF_{n-s} \\ &= F_{s+1}F_{n-s+1} + F_sF_{n-s} + F_{k-1}.F_{s+1}F_{n-s-k+1} \\ &= F_{n+1} + F_{k-1}.F_{s+1}.F_{n-s-k+1}. \end{aligned}$$

Here we consider the corona of P_n and C_n with K_1. We denote $P_n \circ K_1$ and $C_n \circ K_1$ simply by P_n^* and C_n^*, respectively. See Figure 3. We denote the graph obtained from G_n^* by deleting the vertex labeled $2n$ as $G_n^* - \{2n\}$. We have the following theorem:

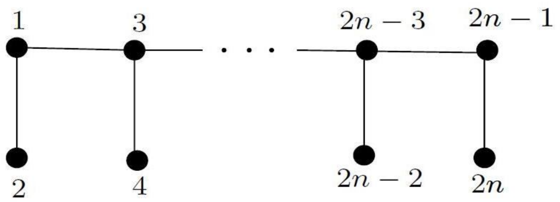

Figure 3. Labeled centipede P_n^*.

Theorem 5.

1. For every $n \geq 3$, $Z(P_n^*) = 2Z(P_{n-1}^*) + Z(P_{n-2}^*)$, $Z(P_1^*) = F_3 = 2$, $Z(P_2^*) = F_5 = 5$.
2. For every $n \geq 3$, $Z(C_n^*) = 2Z(P_{n-1}^*) + 2Z(P_{n-2}^*)$, $Z(P_1^*) = F_3 = 2$, $Z(P_2^*) = F_5 = 5$.

Proof.

1. By Theorem 2 (i) we have $Z(P_n^*) = Z(P_n^* - \{2n\}) + Z(P_{n-1}^*)$. Also for graph $P_n^* - \{2n\}$ we have $Z(P_n^* - \{2n\}) = Z(P_{n-1}^* + Z(P_{n-2}^*)$. By these two equations we have $Z(P_n^*) = 2Z(P_{n-1}^*) + Z(P_{n-2}^*)$.

2. By Theorem 2 (i) we have $Z(C_n^*) = Z(P_n^*) + Z(P_{n-2}^*)$ Now we have the result by part (i).

Now we shall extend previous result to graphs of the form $P_n \circ \overline{K_i}$ and $C_n \circ \overline{K_i}$, where $i \geq 1$.

Theorem 6.

1. For every $n \geq 3$, $Z(P_n \circ \overline{K_i}) = (i+1)Z(P_{n-1} \circ \overline{K_i}) + Z(P_{n-2} \circ \overline{K_i})$.

2. For every $n \geq 3$, $Z(C_n \circ \overline{K_i}) = (i+1)Z(P_{n-1} \circ \overline{K_i}) + 2Z(P_{n-2} \circ \overline{K_i})$.

3. On the parameters of Fibonacci cubes

As we have seen, the Hosoya index of some graphs are relates to Fibonacci numbers. In this section we shall consider Fibonacci cube and study the relationship between its parameters with Fibonacci numbers.

Let $B = \{0,1\}$ and for $n \geq 1$ set $B_n = \{b_1 b_2 \ldots b_n \mid b_i \in B, 1 \leq i \leq n\}$. The n-cube Q_n is the graph defined on the vertex set B_n, vertices $b_1 b_2 \ldots b_n$ and $b'_1 b'_2 \ldots b'_n$ being adjacent if $b_i \neq b'_i$ holds for exactly one $i \in \{1, \ldots, n\}$. Clearly, $|V(Q_n)| = 2^n$. To obtain additional graphs (or networks) with similar properties as hypercubes, but on vertex sets whose order is not a power of two, Hsu [7] (see also [8]) introduced Fibonacci cubes as follows.

For $n \geq l$ let $F_n = \{b_1 b_2 \ldots b_n \in B_n \mid b_i b_{i+1} = 0, 1 \leq i \leq n-1\}$. The Fibonacci cube Γ_n, $n \geq 1$, has F_n as the vertex set, two vertices being adjacent if they differ in exactly one coordinate. In other words, Γ_n is the graph obtained from Q_n by removing all vertices that contain at least two consecutive 1s. The Fibonacci cube Γ_5 is shown in Figure 4. It is easy to see that $|V(\Gamma_n)| = F_{n+2}$. For more information on Fibonacci cubes refer to [12].

Fibonacci cubes were introduced as interconnection networks [7] and later studied from many aspects, see the recent survey [11]. From our point of view it is important that Fibonacci cubes also play a role in mathematical chemistry. Fibonacci cubes are precisely the resonance graphs of fibonacenes which in turn form an important class of hexagonal chains [10].

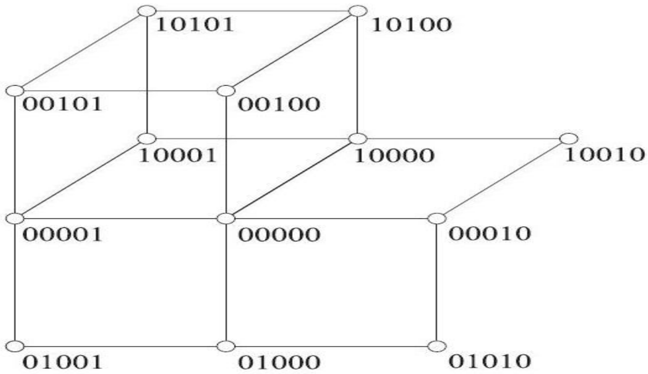

Figure 4. 5-dimensional Fibonacci cube Γ_5.

A less direct representation of Fibonacci cubes appeared in theoretical chemistry. The resonance graph or the Z-transformation graph of a hexagonal graph

H has perfect matchings as vertices, two vertices being adjacent if they differ on exactly one 6-cycle and on this cycle their symmetric difference is the whole cycle; see [17] and references therein, as well as [18] for a generalization of this concept to all plane bipartite graphs. A fibonacene is a hexagonal chain in which no three hexagons are linearly attached. These concepts lead to the following representation of Fibonacci cubes:

Theorem 7. ([10]) *Let G be a fibonacene with n hexagons. Then the resonance graph of G is isomorphic to Γ_n.*

An independent set of a graph G is a set of vertices where no two vertices are adjacent. The independence number is the size of a maximum independent set in the graph and denoted by α.

Lemma 8 ([2]) Γ_n has a Hamiltonian path for any $n \geq 0$.

The following theorem give us the independence number of Γ_n.

Theorem 9. For any $n \geq 0$, $\alpha(\Gamma_n) = \lceil \frac{F_{n+2}}{2} \rceil$.

Proof. Since Γ_n has a Hamiltonian path, $\alpha(\Gamma_n) \leq \lceil \frac{F_{n+2}}{2} \rceil$. On the other hand, if X and Y are the bipartition of Γ_n, then $\alpha(\Gamma_n) \geq \lceil \frac{F_{n+2}}{2} \rceil$. Therefore we have the result.

The following theorem is about the number of edges and Wiener index of Fibonacci cube.

Theorem 10. ([11])

1. $|E(\Gamma_n)| = \frac{nF_{n+1} + 2(n+1)F_n}{5}$.
2. For any $n \geq 0$, $W(\Gamma_n) = \sum_{i=1}^{n} F_i F_{i+1} F_{n-i+1} F_{n-i+2}$.

A set $S \subseteq V$ is a dominating set if every vertex in V\S is adjacent to at least one vertex in S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G. Pike and Zhou [14] proved the following lower bound for domination number of Fibonacci cubes.

Theorem 11. For any $n \geq 4$, $\gamma(F_n) \geq \lceil \frac{F_{n+2} - 3}{n-2} \rceil$.

Ordered Hosoya polynomial is the counting polynomial of the distances among ordered pairs of vertices:

$$\overline{H}(G,x) = \sum_{(u,v) \in V(G) \times V(G)} x^{d(u,v)}.$$

Theorem 12. ([11]) *The generating function of the sequence of ordered Hosoya polynomials of Γ_n is*

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$$f(x,z)=\sum_{n\geq 0}\overline{H}(\Gamma_n,x)z^n=\frac{1+z+xz(1-z)}{1-z-z^2+xz(-1-z+z^2)}$$

REFERENCES

1. O. Chan, I. Gutman, T.K. Lam and R, Merris, Algebraic connections between topological indices, J. Chem. Inform. Comput. Sci. 38 (1998), 62-65.
2. B. Cong, S. Zheng and S. Sharma, On simulations of linear arrays, rings and 2nd meshes on Fibonacci cube networks. In Proceedings of the 7th International Parallel Processing Symphosium, (1993), 747-751.
3. R. Frucht and F. Harary, On the corona of two graphs, Aequationes Math, 4 (1970), 322-324.
4. I. Gutman, On the Hosoya index of very large molecules, MATCH Commum. Math. Comput. Chem. 23 (1988), 95–103.
5. I. Gutman and O. E. Polansky, Mathematical Concepts in Organic Chemistry, Springer, Berlin, (1986).
6. H. Hosoya, Topological index: a newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, Bull. Chem. Soc. Jpn. 44 (1971), 2332-2339.
7. W.-J. Hsu. Fibonacci cubes—a new interconnection technology. IEEE Trans. Parallel Distrib. Syst., 4(1) (1993), 3-12.
8. W.-J. Hsu, C. V. Page, and J.-S. Liu. Fibonacci cubes—a class of self-similar graphs. Fibonacci Quart., 31(1) (1993), 65–72.
9. H. Hua, Hosoya index of unicyclic graphs with prescribed pendent vertices, J. Math. Chem. 43 (2008), 831–844.
10. S. Klavzar and P. Zigert, Fibonacci cubes are the resonance graphs of Fibonaccenes, Fibonacci Quart. 43 (2005), 269-276.
11. S. Klavzar and M. Mollard, Wiener index and Hosoya polynomial of Fibonacci and Lucas cubes, avvailable at http://hal.archives-ouvertes.fr/hal-00624188/fr/
12. S. Klavzar and M. Mollard. Wiener index and Hosoya polynomial of Fibonacci and Lucas cubes. To appear in MATCH Commun. Math. Comput. Chem., (2012).
13. S. Li and Z. Zhu, The number of independent sets in unicyclic graphs with a given diameter, Discrete Appl. Math. 157 (2009), 1387–1395.
14. D. A. Pike and Y. Zou. The domination number of Fibonacci cubes. To appear in J. Combin. Math. Combin. Comput.
15. S. Wagner, Extemal trees with respect to Hosoya index and Merrifield-Simmons index, MATCH Commun. Math. Comput. Chem. 57 (2007), 221-233.
16. K. Xu, On the Hosoya index and the Merrifield-Simmons index of graphs with a given clique number, Appl. Math. Lett. 23 (2010), 395–398.
17. H. Zhang, P. C. B. Lam, and W. C. Shiu. Resonance graphs and a binary coding for the 1-factors of benzenoid systems. SIAM J. Discrete Math., 22(3) (2008), 971–984.
18. H. Zhang and F. Zhang. Plane elementary bipartite graphs. Discrete Appl. Math., 105(1-3) (2000), 291–311.