



Research Paper

Characterization of some finite groups by prime order element graph

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Abstract. For a finite group G , the prime order element graph of G , is defined as follows: the vertex set is the elements of G and two distinct vertices a and b are adjacent if and only if the order of ab is prime. This graph is denoted by $\Gamma(G)$. In this paper, we prove that the alternating groups A_4 and A_5 , all groups of order pq , where p and q are prime numbers, and cyclic groups \mathbb{Z}_p of prime order are uniquely characterized by the prime order element graph.

Keywords. finite group, prime order element graph, characterization, alternating group.

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1 Introduction

Many graphs have been constructed where the vertex set is the elements of a group G , and edges describe structural properties of G . An extensive survey of such graphs is provided in [1]. The motivation behind defining these graphs, is to get the most information about a group G from its corresponding graph. Extending this idea, a natural question is: does the count of elements of prime order in G reveal algebraic information about the group, or even uniquely characterize G ? For groups of small order, uniqueness can often be established. However, as the number of such elements rises, it is difficult to classify such groups. Consequently, the prime order element graph is introduced in [4] and it is defined for finite groups as follows:

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Definition 1.1 ([4]). For a finite group G , the prime order element graph of G which is denoted by $\Gamma(G)$ is defined as follows: the elements of the group G are the vertex set and two distinct vertices a, b are adjacent if the order of the element ab is prime.

This graph is undirected, because in any finite group G , the orders of the elements ab and ba are equal. The prime order element graph of a group G is a special form of addition Cayley graphs [2]. Let S be a subset of the finite group G , the addition Cayley graph $\text{Cay}_G^+(S)$ is the undirected simple graph, where the elements of G are vertices of graph and two vertices a and b are adjacent if and only if $a + b \in S$. For the prime order element graph $\Gamma(G)$, the set S specifically consists of all elements in G of prime order, and G is assumed to be finite. Fundamental properties of $\Gamma(G)$, such as its completeness, connectivity, and vertex degrees, were investigated in [4]. In this article, we prove that the alternating groups A_4 and A_5 , all groups of order pq , where p and q are prime numbers, and cyclic groups \mathbb{Z}_p of prime order are uniquely characterized by the prime order element graph. Throughout this paper, G denotes a finite group, and any unexplained notation are defined in [3, 6].

If G is a finite group, then according to Theorem 2.1 in [4], the degree of the identity element e in $\Gamma(G)$ is maximum. Let $a \in G$, the degree of the element a in $\Gamma(G)$ is maximum if and only if the order of a^2 in G is not prime. In addition, if the order of a^2 is prime, then the degree of a in $\Gamma(G)$ equals to $\text{deg}(e) - 1$. As a consequence of these results, we get that $\Gamma(G)$ contains a universal vertex (a vertex adjacent to every other vertex) if and only if every non-identity element of G has prime order.

Definition 1.2. If a finite group G has elements of orders o_1, o_2, \dots, o_k , then we set:

$$\text{nse}(G) = \{n_{o_1}, n_{o_2}, \dots, n_{o_k}\},$$

where n_{o_i} denotes the number of elements in G with order o_i .

2 Results and Discussion

Remark 2.1. Let G be a finite group in which $\Gamma(G) = \Gamma(\mathbb{Z}_p)$, where p is a prime number. Therefore, $\Gamma(G)$ has p vertices and so $|G| = p$. The only group of order p is \mathbb{Z}_p . Thus $G \cong \mathbb{Z}_p$, i.e. \mathbb{Z}_p is characterized by its prime order element graph.

Theorem 2.2. If $\Gamma(G) = \Gamma(A_4)$, and G is a finite group, then $G \cong A_4$.

Proof. In A_4 , the orders of elements are 1, 2, or 3. The group A_4 contains one element of order 1 (the identity element), three elements of order 2 and eight elements of order 3.

(1) Identity element (e) is adjacent to all 11 non-identity elements, since $o(e \cdot g) = o(g)$. Thus, $\text{deg}(e) = 11$.

(2) Elements of order 2: For $x \in A_4$ with $o(x) = 2$, $x^2 = e$, so $o(x^2) = 1$, is not prime. Therefore, by Theorem 2.1 in [4], $\text{deg}(x) = \text{deg}(e) = 11$.

(3) Elements of order 3: For $y \in A_4$ with $o(y) = 3$, $o(y^2) = 3$, which is prime. Thus by Theorem 2.1 in [4], $\text{deg}(y) = \text{deg}(e) - 1 = 10$. Hence $\Gamma(A_4)$ is connected, because every

non-identity element is adjacent to e and elements of distinct orders are connected via e . Therefore, $\Gamma(A_4)$ has 4 vertices of degree 11 (identity and 3 involutions), 8 vertices of degree 10 (elements of orders 3).

We know that $\Gamma(G) = \Gamma(A_4)$, so G has 4 vertices of degree 11 and 8 vertices of degree 10. By Theorem 2.1 in [4], $\deg(e) = 11$ and the order of every non-identity element in G is prime. If $x \in G$ is an element whose order is not 2, then x is non-adjacent to its inverse. Because $o(xx^{-1}) = 1$, which is not prime. Therefore, the 3 vertices of degree 11 (except e), must satisfy $x = x^{-1}$, i.e., $o(x) = 2$. Thus in G , we have 3 elements of order 2. Since $|G| = |A_4| = 12$, and the order of every non-identity element of G is prime, so the other eight elements of G have order 3. Hence $nse(G) = \{1, 3, 8\} = nse(A_4)$. Since by [5], the only group of order 12 with $nse(G) = \{1, 3, 8\}$, is A_4 . Thus $G \cong A_4$. \square

Theorem 2.3. *If $\Gamma(G) = \Gamma(A_5)$, and G is a finite group, then $G \cong A_5$.*

Proof. In A_5 , the orders of elements are 1, 2, 3, or 5. The group A_5 contains one element of order 1 (the identity element), fifteen elements of order 2, twenty elements of order 3, and twenty-four elements of order 5.

(1) Identity element (e) is adjacent to all 59 non-identity elements, since $o(e \cdot g) = o(g)$. Thus, $\deg(e) = 59$.

(2) Elements of order 2: For $x \in A_5$ with $o(x) = 2$, $x^2 = e$, so $o(x^2) = 1$, is not prime. Therefore, by Theorem 2.1 in [4], $\deg(x) = \deg(e) = 59$.

(3) Elements of order 3 or 5: For $y \in A_5$ with $o(y) = 3$ or 5 , $o(y^2) = 3$ or 5 , which is prime. Thus by Theorem 2.1 in [4], $\deg(y) = \deg(e) - 1 = 58$. Hence $\Gamma(A_5)$ is connected, because every non-identity element is adjacent to e and elements of distinct orders are connected via e .

Therefore $\Gamma(A_5)$ has 16 vertices of degree 59 (identity and 15 involutions), 44 vertices of degree 58 (elements of orders 3 or 5).

We know that $\Gamma(G) = \Gamma(A_5)$, so G has 16 vertices of degree 59 and 44 vertices of degree 58. By Theorem 2.1 in [4], $\deg(e) = 59$ and the order of every non-identity element in G is prime. If $x \in G$ is an element whose order is not 2, then x is non-adjacent to the inverse of x . Therefore, the 15 vertices of degree 59 (except e), must satisfy $x = x^{-1}$, i.e., $o(x) = 2$. Thus in G , we have 15 vertices of order 2.

By Sylow theorems, for a finite group G of order $p^a m$ ($p \nmid m$), the number of Sylow p -subgroups n_p satisfies $n_p \equiv 1 \pmod p$ and $n_p \mid m$. Now suppose $n_5 = 1$. Then G has a unique normal Sylow 5-subgroup P_5 . A Sylow 3-subgroup P_3 acts on P_5 by conjugation, implying $P_3 P_5 \leq G$ and $|P_3 P_5| = 15$. However, such a subgroup would be cyclic. Thus G contains elements of order 15, which is a contradiction. Thus, $n_5 \neq 1$. By Sylow's theorems, $n_5 = 6$. We know that the only finite group of order 60 with non-normal Sylow 5-subgroup is A_5 . Hence, $G \cong A_5$. \square

Theorem 2.4. *Let S be a group of order pq , where p and q are prime numbers. Then S is uniquely characterized by its prime order element graph $\Gamma(S)$.*

Proof. Let G be a finite group with $\Gamma(G) = \Gamma(S)$. Since $\Gamma(G)$ and $\Gamma(S)$ have $|S| = pq$ vertices, we have $|G| = pq$. We consider the following cases.

Case 1: $p = q$. Since $|S| = p^2$, so S is isomorphic to either cyclic group \mathbb{Z}_{p^2} or elementary abelian group $\mathbb{Z}_p \times \mathbb{Z}_p$.

If $S \cong \mathbb{Z}_{p^2}$, then $\deg(e) = p - 1$, because the identity element is adjacent to elements of order p . Therefore, maximum degree of vertices in $\Gamma(G)$ is $p - 1$. Hence G has elements of composite order and since $|G| = p^2$, so $G \cong \mathbb{Z}_{p^2}$.

If $S \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $\deg(e) = p^2 - 1$, because all non-identity elements have prime order. Thus maximum degree of vertices in $\Gamma(G)$ is $p^2 - 1$. Hence all non-identity elements in G have prime order and since $|G| = p^2$, so $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Case 2: $p \neq q$. Since $|S| = |G| = pq$ and $p \neq q$, S and G should be the direct or semi-direct product of \mathbb{Z}_p and \mathbb{Z}_q . If one of them is the direct product and the other is the semi-direct product, then the maximum degree (that is, the degree of the identity element) of one of $\Gamma(S)$ and $\Gamma(G)$ is $p + q - 2$, and the maximum degree of the other is $pq - 1$, a contradiction. Thus, S is uniquely determined by $\Gamma(S)$. \square

3 Conclusion

In this paper, we prove that the alternating groups A_4 and A_5 , all groups of order pq , where p and q are prime numbers, and cyclic groups \mathbb{Z}_p of prime order are uniquely characterized by the prime order element graph. For future study, we can check the characterization of other simple groups, e.g., A_n , for $n \geq 6$ or $\text{PSL}(2, q)$ by $\Gamma(G)$.

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Data Availability Statement

Data is contained within the article.

Conflicts of Interests

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