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Research Paper

Counting vertices among all higher-dimensional plane trees

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Abstract. In this paper, we study the enumeration of vertices in d-dimensional plane trees with respect to their levels and degrees. This class of trees generalizes both ordinary plane trees and noncrossing trees. Our approach builds upon a decomposition framework that extends the butterfly decomposition of plane trees introduced by Chen, Li and Shapiro, as well as that of noncrossing trees studied by Oduol and Okoth. We derive both explicit and asymptotic formulas for the enumeration of vertices, eldest children, first children, non-first children and non-leaves at specified levels and degrees. The results are obtained through a combination of generating function techniques, refined butterfly decompositions and bijective methods. This work extends previous enumeration results on ordinary plane trees and noncrossing trees and provides new insights into the combinatorial structure of their higher-dimensional analogues.

Keywords. level, degree, eldest child, first child, leaf.
Mathematics Subject Classification (2020): $05A15$, $05C05$, $52B05$.

1 Introduction

The enumeration of tree vertices by level and degree has attracted considerable attention across several tree families. Notable examples include Cayley trees [11], plane trees [3,4,8,17], t-ary trees [1, 13, 16] and more recently noncrossing trees [10]. In the case of plane and t-ary trees, many enumeration results are derived using the butterfly decomposition method

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introduced by Chen, Li and Shapiro [2]. This decomposition was modified by Oduol and Okoth [10] to enumerate vertices of noncrossing trees. A defining characteristic of any tree is that a unique path of edges connects each pair of vertices. In a rooted tree, the *level* of a vertex corresponds to the length of the path from the root to that vertex, with the root itself at level 0.

In plane trees, two adjacent vertices u and v are considered to be in a *parent-child relationship* if one is situated at a higher level than the other. Specifically, v is a child of u (and u the parent of v) if v is adjacent to u and lies one level below. Children of a common parent are termed *siblings* and the leftmost among them is designated the *first child*. The *leftmost path* is the path formed by tracing eldest children down the tree. A vertex's *degree* denotes the number of its children, with a vertex of degree zero referred to as a *leaf* and any other as a *non-leaf*.

This paper examines a generalization of plane trees that includes both ordinary plane trees and noncrossing trees. In noncrossing trees, vertices are placed around a circle and labeled in counterclockwise order, with straight line edges that do not intersect. Introduced by Noy [7] in 1998, noncrossing trees have since been extensively studied under various structural statistics, including path length [10,11]. Panholzer and Prodinger [15] later introduced an (l,r)-labeling system for these trees: for any edge (i,j) where i is closer to the root than j, the vertex j is labeled by l if $j < i$, and r if $j > i$. The resulting tree is a plane tree where the root remains unlabeled, its children are labeled by r and all other vertices are labeled by l or r. An example of a noncrossing tree alongside its corresponding (l,r)-labeling is provided in Figure 1.

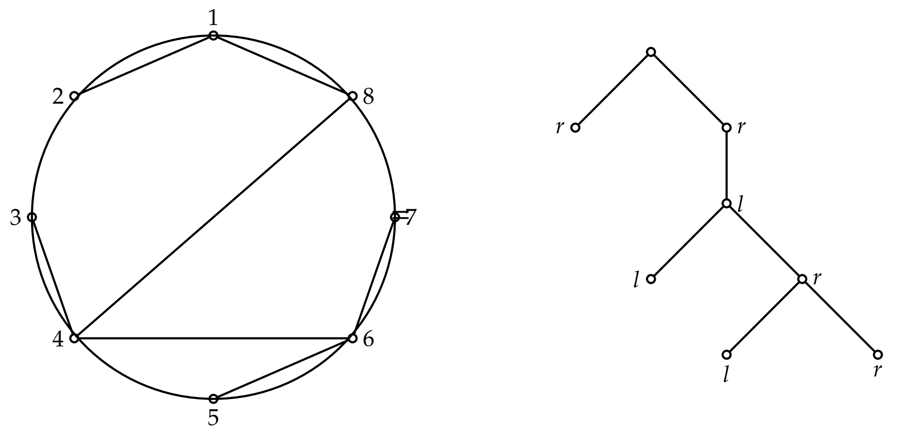

Figure 1. A noncrossing tree on 8 vertices alongside its (l,r)-labeling.

For over 25 years, the butterfly decomposition by Flajolet and Noy [6] has been a central tool for analysing noncrossing trees. A *butterfly* is defined as a pair of noncrossing trees sharing a vertex.For a non-root vertex i, a butterfly rooted at i is naturally divided into two parts, called *wings*: (i) the left wing (vertices have labels less than i) and (ii) the right wing (vertices have labels greater than i). When the butterfly is rooted at the root of the tree, only the right wing exists; the left wing is empty. When the root serves as the shared vertex, only the right wing is present. This concept was later generalized by Okoth and Kasyoki [12][a], who extended the idea of two wings to d wings, giving rise to what they called *d-dimensional plane trees*. In this generalization, the wings are indexed from left to right as $1,2,\ldots,d$. We remark that the root has only one wing, indexed d. So, all the children of the root are labelled d. Notably, in this framework: ordinary plane trees correspond to the 1-dimensional plane trees, noncrossing trees correspond to the 2-dimensional plane trees and the labels l and r in the (l,r)-labeling of noncrossing trees correspond to wings numbered 1 and 2, respectively. An illustration of a 3-dimensional plane tree is provided in Figure 2.

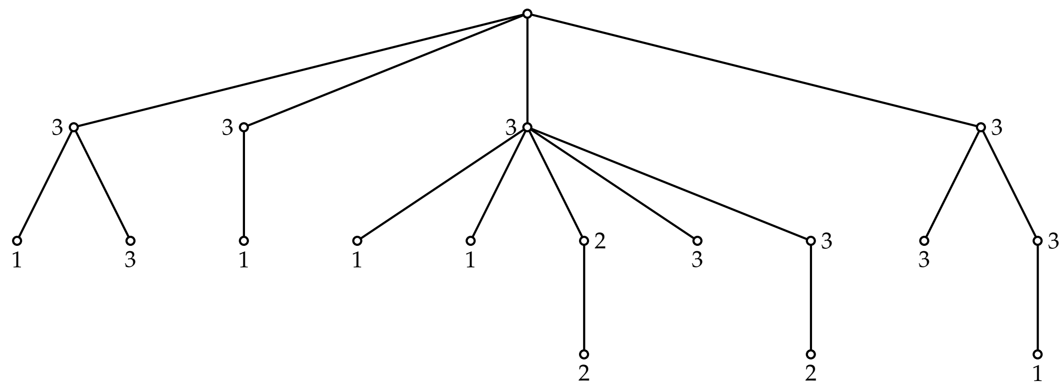

Figure 2. A 3-dimensional plane tree on 19 vertices.

Let $D(x)$ and $B(x)$ be respectively the generating function for d-dimensional plane trees and butterflies where x marks non-root vertex. Then,

$$D(x) = \frac{1}{1-B(x)} \quad \text{and} \quad B(x) = \frac{(xD(x))^d}{x^{d-1}}.$$

The generating function $D(x)$ can thus be expressed as

$$D(x) = \frac{1}{1 - xD(x)^d}. \tag{1}$$

Setting $D(x) = \frac{F(x)}{\sqrt[d]{x}}$, then (1) becomes

$$F(x) = \frac{\sqrt[d]{x}}{1 - F(x)^d},$$

which allows for use of Lagrange inversion formula [18] to extract the coefficient of the monomial x^n. The theorem is stated as follows without proof.

Theorem 1.1. *Let $\phi(x)$ be a generating function that satisfies the equation $\phi(x) = x\lambda(\phi(x))$, where $\phi(0)$ is non-negative. The coefficient of x^n in $\phi(x)$ is given as, $[x^n]\phi(x)^k = \frac{k}{n}[p^{n-k}]\lambda(p)^n$.*

[a] The paper was accepted for publication in The Bulletin of the Institute of Combinatorics and its Applications

In [12], the authors enumerated d-dimensional plane trees based on various parameters, including vertices, root degree, vertices in a given wing, descents, forests, and leaves. However, they did not address enumeration by levels. In this paper, we extend the butterfly decomposition of plane trees to the setting of d-dimensional plane trees. Consider a vertex v labeled by i whose parent is vertex u in a d-dimensional plane tree. The children of u (excluding v) can then be partitioned into $d+1$ subsets: vertices labeled by j comprising each set for $j = 1, 2, \ldots, i-i, i+1, \ldots, d$, vertices labeled by i that appear to the left of v, and vertices labeled by i that appear to the right of v. The decomposition is depicted in Figure 3.

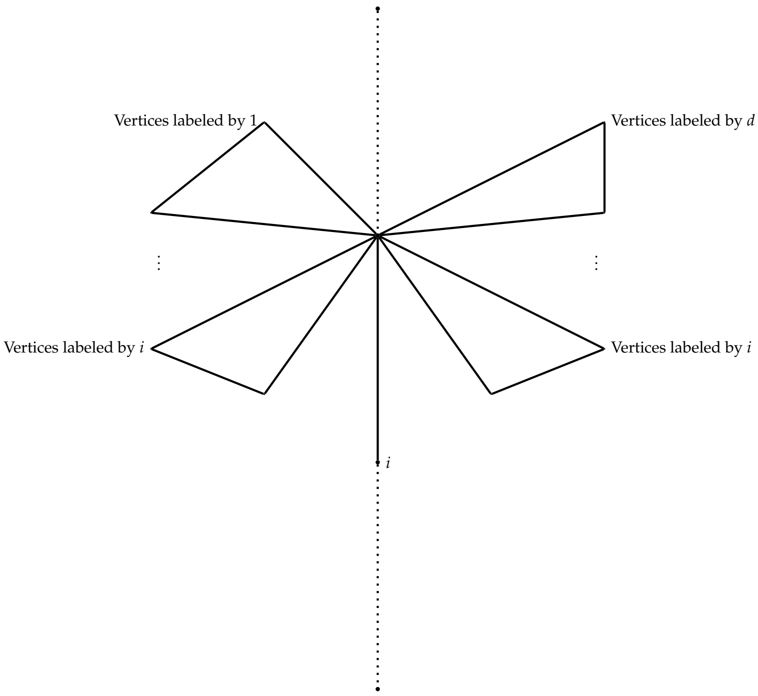

Figure 3. Decomposition of d-dimesional plane trees based on the label of a child vertex.

This study is important because reachability in d-ary trees has significant applications in phylogenetics, role hierarchies (or delegation trees), game state trees, and algorithm analysis. For example, in phylogenetic trees, reachability enables the computation of genetic relationships, identification of common ancestors, and tracing of mutation propagation. Obtaining closed-form expressions for the number of vertices in d-dimensional plane trees enhances the analysis of parent-child relationships within these structures. The present work enumerates vertices of given kinds across all d-dimensional plane trees. The specific cases for $d = 1$ and $d = 2$ were previously obtained by Chen, Li, and Shapiro in [2], and by Oduol and Okoth in [10], respectively. This study thus provides a unified framework that generalizes and connects the results in articles [2] and [10].

The remainder of the paper is structured as follows. In Section 2, we derive both explicit and asymptotic formulas for the enumeration of various structural features of vertices at fixed levels and degrees. These include: vertices by level and degree (Subsection 2.1), eldest children (Subsection 2.2), first children (Subsection 2.3), non-first children (Subsection 2.4), and non-leaves (Subsection 2.5). Subsection 2.6 presents bijections between relevant tree structures. The concluding section (Section 3) summarizes the results and highlights open problems.

2 Counting vertices in all trees

In this section, we derive both explicit and asymptotic formulas for counting structural characteristics of vertices located at specific levels in d-dimensional plane trees. The focus lies on the enumeration of vertices based on level and degree, as well as the identification and enumeration of particular vertex roles - eldest children, first children, non-first children and non-leaves. We also present bijective relations that connect various configurations within these structures. Each subsection addresses one of these aspects, beginning with vertices classified by their level and degree.

2.1 Levels and degrees

We begin by formulating an expression that counts vertices having a given degree and appearing at a specified level across all d-dimensional plane trees with a fixed number of vertices.

Theorem 2.1. *The number of vertices at level $\ell \geq 1$ with degree r in d-dimensional plane trees on n vertices is given by*

$$d^{\ell-1}\binom{d+r-1}{r}\cdot\frac{d(r+\ell-1)+\ell+1}{dn-2d+\ell+1}\binom{(d+1)(n-1)-d-r}{n-\ell-r-1}. \tag{2}$$

Proof. Consider d-dimensional plane trees in which a vertex of degree r lies at level $\ell \geq 1$. These trees can be decomposed as shown in Figure 4.

According to this decomposition, the generating function for such trees is given as follows:

$$(xD(x)^2)(xD(x)^{d+1})^{\ell-1}x(xD(x)^d)^r = x^{\ell+r+1}D(x)^{(d+1)\ell+d(r-1)+1}.$$

We then extract the coefficient of x^n from the resulting generating function using the Lagrange inversion formula (Theorem 1.1).

$$\begin{aligned}[x^n]x^{\ell+r+1}D(x)^{(d+1)\ell+d(r-1)+1} &= [x^{n-\ell-r-1}]D(x)^{(d+1)\ell+d(r-1)+1}\\ &= [x^{\frac{d(n-2)+\ell+1}{d}}]F(x)^{(d+1)\ell+d(r-1)+1}\\ &= \frac{(d+1)\ell+d(r-1)+1}{d(n-2)+\ell+1}[f^{d(n-\ell-r-1)}](1-f^d)^{-(d(n-2)+\ell+1)}.\end{aligned}$$

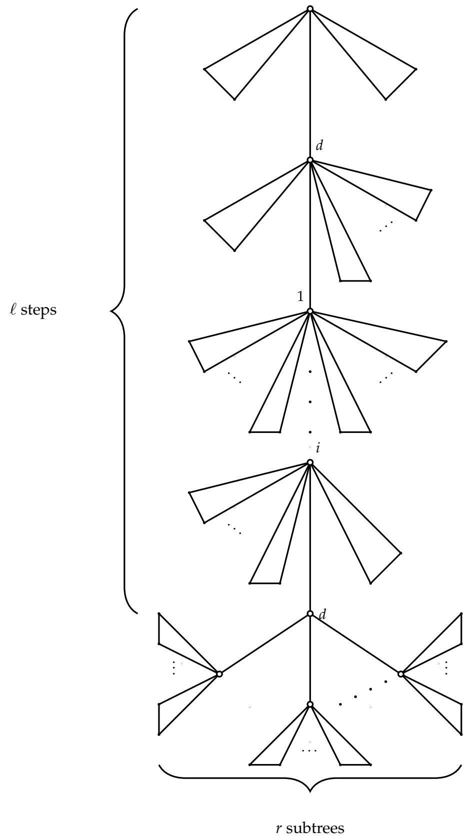

Figure 4. Decomposition of a d-dimensional plane tree containing a vertex of degree r at level $\ell \geq 1$. At each level from 1 to $\ell - 1$, exactly $d + 1$ subtrees are attached to the corresponding vertex along the unique path from the root to the vertex at level ℓ.

Binomial theorem gives

$$\begin{aligned}
&[x^n]x^{\ell+r+1}D(x)^{(d+1)\ell+d(r-1)+1} \\
&\quad = \frac{(d+1)\ell + d(r-1)+1}{d(n-2)+\ell+1}[f^{d(n-\ell-r-1)}]\sum_{i\geq 0}\binom{-(d(n-2)+\ell+1)}{i}(-f^d)^i \\
&\quad = \frac{(d+1)\ell + d(r-1)+1}{d(n-2)+\ell+1}\binom{(d+1)n-2d-r-1}{n-\ell-r-1}.
\end{aligned}$$

The vertex immediately adjacent to the root is labeled by d, while all other non-root vertices receive labels from the set $\{1,2,\ldots,d\}$. This leads to $d^{\ell-1}$ possible labelings for the path vertices. Additionally, the final vertex (of degree r) on the path can distribute its children across d wings in $\binom{d+r-1}{r}$ ways. Applying the product rule yields the stated formula. □

By summing over all valid values of r in Equation (2), we obtain the following result:

Corollary 2.2. *The number of vertices at level $\ell \geq 1$ in all d-dimensional plane trees on n vertices is* *given by*

$$d^{\ell-1}\cdot\frac{(d+1)\ell+1}{dn-d+\ell+1}\binom{(d+1)(n-1)}{n-\ell-1}. \tag{3}$$

Setting $d=1$ and $d=2$ in Equation (3) recovers earlier results on ordinary plane and noncrossing trees:

Corollary 2.3 ([8]). *There are*

$$\frac{2\ell+1}{n+\ell}\binom{2n-2}{n-\ell-1}$$

vertices at level $\ell\geq 1$ in all plane trees on n vertices.

Corollary 2.4 ([10]). *There are*

$$2^{\ell-1}\cdot\frac{3\ell+1}{2n+\ell-1}\binom{3n-3}{n-\ell-1}$$

vertices at level $\ell\geq 1$ in all noncrossing trees on n vertices.

Taking $r=0$ in Equation (2) gives the number of leaves at level $\ell\geq 1$ in all d-dimensional plane trees with n vertices:

$$d^{\ell-1}\cdot\frac{(d+1)\ell-d+1}{dn-2d+\ell+1}\binom{(d+1)(n-1)-d}{n-\ell-1} \tag{4}$$

Dividing Equation (4) by Equation (3) yields

$$\frac{((d+1)\ell-d+1)\cdot(d(n-1)+\ell+1)(d(n-1)+\ell)\cdots(d(n-1)+\ell-d+2)}{((d+1)\ell+1)\cdot(d+1)(n-1)((d+1)(n-1)-1)\cdots((d+1)(n-1)-d+1)}$$

as the average number of leaves per a d-dimensional plane tree. Furthermore, as $n\to\infty$, the proportion of leaves among all vertices tends toward the expression given in Equation (5):

$$\frac{(d+1)\ell-d+1}{(d+1)\ell+1}\cdot\left(\frac{d}{d+1}\right)^{d}. \tag{5}$$

By substituting $\ell=1$ into Equation (4), we obtain the total number of leaves that are immediate children of the root,

$$\frac{2}{dn-2d+2}\binom{(d+1)n-2d-1}{n-2},$$

where n is the number of vertices in the d-dimensional plane trees. Setting $\ell=1$ in Equation (5), we find that as $n\to\infty$, the proportion of leaves which are children of the root approaches:

$$\frac{2}{d+2}\cdot\left(\frac{d}{d+1}\right)^{d}.$$

Finally, if we avoid fixing labels along the path, that is, if we omit specifying the labeling pattern, we arrive at the following result:

Corollary 2.5. *The number of vertices at level $\ell \geq 1$ in d-dimensional plane trees on n vertices such that all vertices on the path from the root to the vertex under consideration are labeled by i is given by*

$$\frac{(d+1)\ell+1}{dn-d+\ell-1}\binom{(d+1)(n-1)}{n-\ell-1}. \tag{6}$$

Letting $\ell = 1$ in Equation (6) gives:

Corollary 2.6. *There are*

$$\frac{d+2}{d(n-1)}\binom{(d+1)(n-1)}{n-2}$$

children of the root in d-dimensional plane trees on n vertices.

2.2 Eldest children

In the sequel, we focus on counting eldest children that have a specified degree and appear at a particular level in d-dimensional plane trees.

Theorem 2.7. *The number of eldest children at level $\ell \geq 1$ with root degree r in d-dimensional plane trees on n vertices, such that for each eldest child there are k_i vertices labeled by i on the leftmost path from the root to the eldest child, is given by*

$$\binom{d+r-1}{r} \cdot \frac{d\ell-t+dr}{dn-d-t}\binom{(d+1)(n-1)-t-\ell-r-1}{n-\ell-r-1}\binom{\ell-1}{k_1,\ldots,k_{d-1},k_d-1}, \tag{7}$$

where $t := k_2 + 2k_3 + \cdots + (d-1)k_d$.

Proof. The decomposition of the trees is illustrated in Figure 5. According to this decomposition, the generating function for trees where each eldest child lies on the leftmost path and satisfies the labeling pattern defined by the k_i's is:

$$\begin{aligned}
&(xD(x))^{k_d}(xD(x)^2)^{k_{d-1}}\cdots(xD(x)^d)^{k_1}x(xD(x)^d)^r\\
&\quad= x^{(k_1+k_2+\cdots+k_d)+r+1}D(x)^{k_d+2k_{d-1}+\cdots+dk_1+dr}\\
&\quad= x^{\ell+r+1}D(x)^{d(k_1+k_2+\cdots k_d)-(k_2+2k_3+\cdots+(d-1)k_d)+dr}\\
&\quad= x^{\ell+r+1}D(x)^{d\ell-(k_2+2k_3+\cdots+(d-1)k_d)+dr}.
\end{aligned}$$

Setting $t := k_2 + 2k_3 + \cdots + (d-1)k_d$, we simplify the exponent of $D(x)$ to obtain

$$x^{\ell+r+1}D(x)^{d\ell-t+dr}.$$

We then apply the Lagrange inversion formula to extract the coefficient of x^n:

$$\begin{aligned}
[x^n]x^{\ell+r+1}D(x)^{d\ell-t+dr} &= [x^{n-\ell-r-1}]D(x)^{d\ell-t+dr} = [x^{\frac{dn-d-t}{d}}]F(x)^{d\ell-t+dr}\\
&= \frac{d\ell-t+dr}{dn-d-t}\binom{(d+1)(n-1)-t-\ell-r-1}{n-\ell-r-1}.
\end{aligned}$$

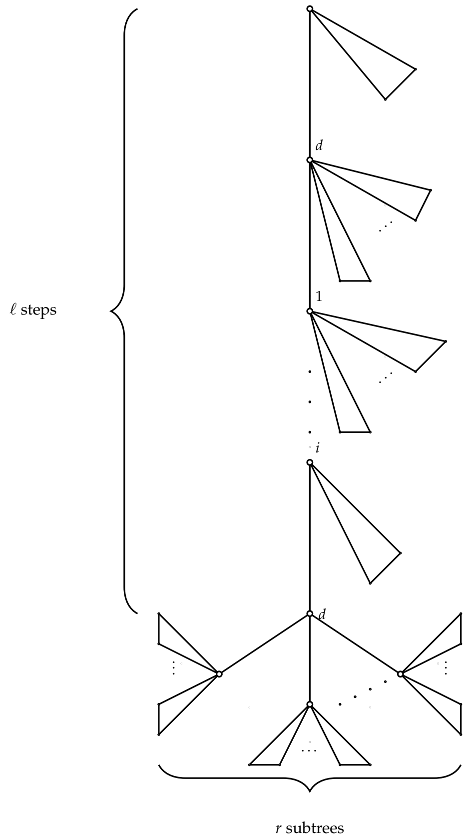

Figure 5. Decomposition of a d-dimensional plane tree with an eldest child of degree r at level $\ell \geq 1$.

The multinomial coefficient $\binom{\ell-1}{k_1,\ldots,k_{d-1},k_d-1}$ counts the number of ways to label the vertices along the leftmost path (excluding the root), where each label i appears k_i times for $i = 1,2,\ldots,d$, with the root's child necessarily labeled by d. Additionally, the labels of the children of the eldest child can be distributed in $\binom{d+r-1}{r}$ ways. These factors, when combined using the product rule, yield the desired count. □

By summing over all possible values of r in (7), we obtain the total number of eldest children at a given level satisfying the specified labeling conditions.

Corollary 2.8. *The total number of eldest children that reside on level $\ell \geq 1$ in d-dimensional plane trees on n vertices such that for each eldest child, there are k_i vertices labeled by i on the leftmost path from the root to the eldest child is*

$$\frac{d\ell - t + d}{dn - t}\binom{(d+1)n - \ell - t - 2}{n - \ell - 1}\binom{\ell - 1}{k_1,\ldots,k_{d-1},k_d - 1}, \tag{8}$$

where $t := k_2 + 2k_3 + \cdots + (d-1)k_d$.

Taking $r = 0$ in Equation (7) gives the number of eldest children who are also leaves and reside at level $\ell \geq 1$:

Corollary 2.9. *There are*

$$\frac{d\ell - t}{dn - d - t}\binom{(d+1)(n-1) - t - \ell - 1}{n - \ell - 1}\binom{\ell - 1}{k_1, \ldots, k_{d-1}, k_d - 1}$$

eldest children, which are also leaves, that reside on level $k \geq 1$ in d-dimensional plane trees on n vertices such that for each eldest child, there are k_i vertices labeled by i on the leftmost path from the root to the eldest child. Again, $t := k_2 + 2k_3 + \cdots + (d-1)k_d$.

Further, by setting all $k_i = 0$ for $i = 1,2,\ldots,d-1$ and letting $k_d = \ell$, we ensure that every vertex along the leftmost path is labeled by d. In this case, $t = (d-1)k_d$ and Equation (7) simplifies to Equation (9):

Corollary 2.10. *The number of eldest children at level $\ell \geq 1$ with degree r in d-dimensional plane trees on n vertices, such that all the vertices on the leftmost path from the root to the eldest child are labeled by d, is given by*

$$\binom{d+r-1}{r} \cdot \frac{\ell + dr}{dn - (d-1)\ell - d}\binom{(d+1)(n-1) - d\ell - r - 1}{n - \ell - r - 1}. \tag{9}$$

Summing over all valid values of r in Equation (9) or letting $k_i = 0$ for all $i = 1,2,\ldots,d-1$, $k_d = \ell$ and $t = (d-1)\ell$ in Equation (8), one gets:

Corollary 2.11. *The total number of eldest children residing at level $\ell \geq 1$ in d-dimensional plane trees on n vertices, such that all the vertices on the leftmost path from the root to each eldest child are labeled by d, is given by*

$$\frac{\ell + d}{dn - (d-1)\ell}\binom{(d+1)n - d\ell - 2}{n - \ell - 1}.$$

Setting $r = 0$ in Equation (9) gives the number of eldest children which are also leaves, where the entire leftmost path consists of vertices labeled by d:

Corollary 2.12. *There are*

$$\frac{\ell}{dn - (d-1)\ell - d}\binom{(d+1)(n-1) - d\ell - 1}{n - \ell - 1}$$

eldest children, which are also leaves, residing at level $\ell \geq 1$ in d-dimensional plane trees on n vertices, such that all the vertices on the leftmost path from the root to the eldest child are labeled by d.

Proof. This follows by substituting $r = 0$ into Equation (9). □

Finally, evaluating Corollary 2.12 at $\ell = 1$ provides the number of eldest children of the root that are also leaves. This value,

$$\frac{1}{d(n-2)+1}\binom{(d+1)(n-2)}{n-2},$$

corresponds to the count of d-dimensional plane trees on $n-1$ vertices and of complete d-ary trees with $n-2$ internal vertices.

2.3 First children

In this part, we focus on counting first children within all d-dimensional plane trees having a fixed number of vertices. Specifically, we identify first children with a particular label and degree located at a given level.

Theorem 2.13. *The number of first children of degree r labeled by i, residing at level $\ell \geq 2$ in d-dimensional plane trees on n vertices, is given by*

$$d^{\ell-2}\binom{d+r-1}{r} \cdot \frac{(d+1)\ell + d(r-1) - i + 1}{dn - 2d + \ell - i + 1}\binom{(d+1)n - r - 2d - i - 1}{n - \ell - r - 1}. \tag{10}$$

Proof. We consider d-dimensional plane trees that contain a path of length $\ell \geq 2$ terminating at a vertex which is a first child, has degree r, and is labeled by i. The structural decomposition for this scenario is depicted in Figure 6. Based on this decomposition, the generating

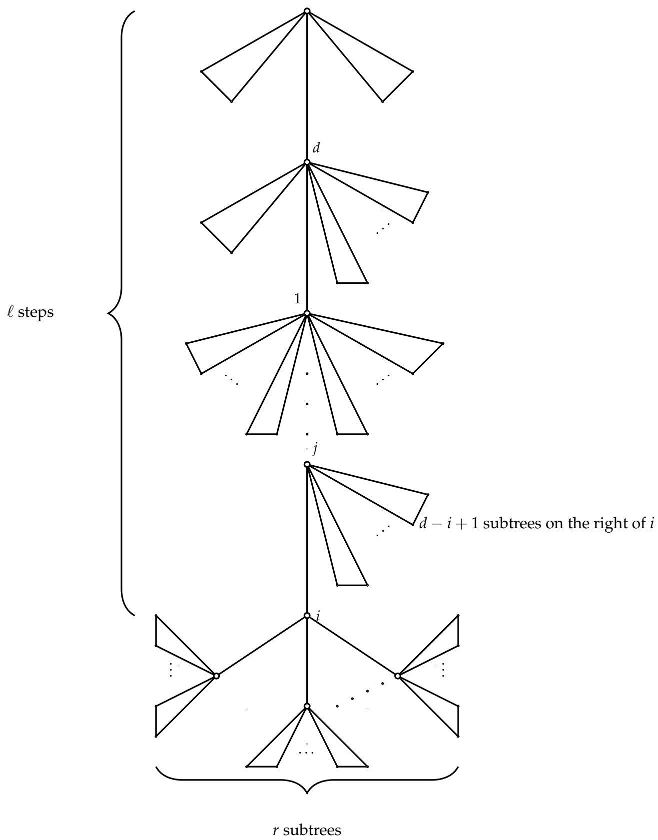

Figure 6. Decomposition of a d-dimensional plane tree containing a first child labeled by i with degree r at level $\ell \geq 1$.

function for such trees is constructed as:

$$(xD(x)^2)(xD(x)^{d+1})^{\ell-2}(xD(x)^{d-i+1})x(xD(x)^d)^r = x^{\ell+r+1}D(x)^{(d+1)\ell+d(r-1)-i+1}.$$

Using the Lagrange inversion formula, we extract the coefficient of x^n from this generating function:

$$\begin{aligned}
[x^n]x^{\ell+r+1}D(x)^{(d+1)\ell+d(r-1)-i+1} &= [x^{n-\ell-r-1}]D(x)^{(d+1)\ell+d(r-1)-i+1} \\
&= [x^{\frac{dn-2d+\ell-i+1}{d}}]F(x)^{(d+1)\ell+d(r-1)-i+1} \\
&= \frac{(d+1)\ell+d(r-1)-i+1}{dn-2d+\ell-i+1}\binom{(d+1)n-r-2d-i-1}{n-\ell-r-1}.
\end{aligned}$$

The first vertex after the root on the path is labeled 1. All remaining vertices on the path, except for the terminal vertex (which is labeled by i), are assigned labels from $\{1,2,\ldots,d\}$, resulting in $d^{\ell-2}$ possible labelings. Additionally, the r children of the terminal vertex can be labeled in $\binom{d+r-1}{r}$ ways. Applying the product rule, we arrive at the stated expression. □

Summing Equation (10) across all valid values of r gives:

Corollary 2.14. *The total number of first children labeled by i at level $\ell \geq 2$ in d-dimensional plane trees on n vertices is*

$$d^{\ell-2}\cdot\frac{(d+1)\ell-i+1}{d(n-1)+\ell-i+1}\binom{(d+1)(n-1)-i}{n-\ell-1}. \tag{11}$$

Taking $r=0$ in Equation (10) yields the number of first children that are also leaves, with label i, located at level $\ell \geq 2$:

Corollary 2.15. *There are*

$$d^{\ell-2}\cdot\frac{(d+1)\ell-d-i+1}{dn-2d+\ell-i+1}\binom{(d+1)n-2d-i-1}{n-\ell-1}$$

first children, which are also leaves, labeled by i and residing at level $\ell \geq 2$ in d-dimensional plane trees on n vertices.

To determine the total number of first children of degree r at level $\ell \geq 2$, we sum Equation (10) over all values of i. Likewise, summing Equation (11) over all i yields the total number of first children (regardless of label) at that level.

Proposition 2.16. *There are*

$$\binom{d+r-1}{r}\frac{dr+1}{dn-2d+1}\binom{(d+1)(n-2)-r}{n-r-2} \tag{12}$$

first children of degree r that are children of the root in d-dimensional plane trees on n vertices.

Proof. Figure 7 shows the decomposition of trees in which a first child of the root has degree r.

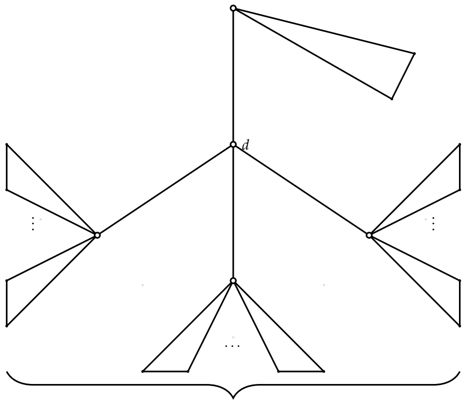

Figure 7. Decomposition of a d-dimensional plane tree with a first child of degree r that is also a child of the root.

The corresponding generating function is: $(xD(x))x(xD(x)^d)^r = x^{r+2}D(x)^{dr+1}$. We again apply the Lagrange inversion formula to extract the coefficient of x^n:

$$[x^n]x^{r+2}D(x)^{dr+1} = [x^{n-r-2}]Dx)^{dr+1} = [x^{\frac{dn-2d+1}{d}}]F(x)^{dr+1}$$
$$= \frac{dr+1}{dn-2d+1}\binom{(d+1)(n-2)-r}{n-r-2}. \tag{13}$$

There are $\binom{d+r-1}{r}$ ways to label the children of the first child of the root. Combining this with (13) yields the result. □

When Equation (12) is summed across all values of r, we obtain the total number of first children of the root in all d-dimensional plane trees with n vertices. This quantity simplifies to

$$\frac{1}{n-1}\binom{(d+1)(n-1)}{n-2},$$

which shows that each such tree has exactly one first child of the root. Setting $r = 0$ in Equation (12) provides the number of first children of the root which are also leaves:

$$\frac{1}{d(n-2)+1}\binom{(d+1)(n-2)}{n-2}.$$

This formula also gives the total number of d-dimensional plane trees with $n-1$ vertices.

2.4 Non-first children

In this section, we turn our attention to counting non-first children within d-dimensional plane trees. Specifically, we consider those that have a given degree and satisfy certain labeling constraints.

Theorem 2.17. *The number of non-first children of degree r at level $\ell \geq 2$ in d-dimensional plane trees on n vertices, such that the sibling immediately to the left of the non-first child has the same label as the non-first child, is given by*

$$d^{\ell-1}\binom{d+r-1}{r} \cdot \frac{(d+1)\ell+dr+1}{dn-2d+\ell+1}\binom{(d+1)(n-2)-r}{n-\ell-r-2}. \tag{14}$$

Proof. The decomposition of trees under consideration is shown in Figure 8.

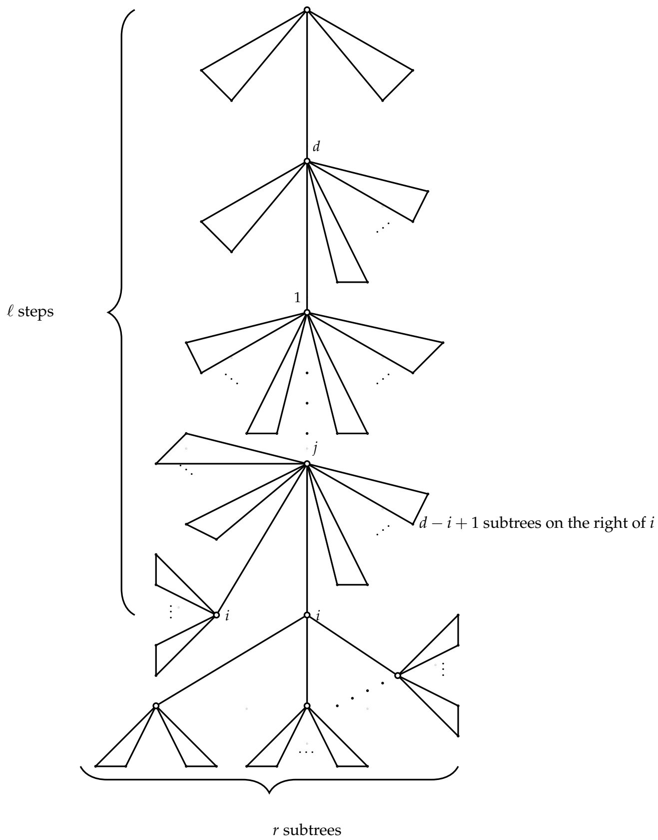

Figure 8. Decomposition of a d-dimensional plane tree with a non-first child of degree r at level $\ell \geq 1$, where the label of the non-first child matches that of its immediate left sibling.

Here, the non-first child appears at level $\ell \geq 2$, has degree r and shares its label with the sibling immediately to its left. From this structure, we derive the corresponding generating function:

$$(xD(x)^2)(xD(x)^{d+1})^{\ell-1}(xD(x)^d)x(xD(x)^d)^r = x^{\ell+r+2}D(x)^{(d+1)\ell+dr+1}.$$

We then use the Lagrange inversion formula to extract the coefficient of x^n:

$$[x^n]x^{\ell+r+2}D(x)^{(d+1)\ell+dr+1} = [x^{n-\ell-r-2}]D(x)^{(d+1)\ell+dr+1} = [x^{\frac{dn-2d+\ell+1}{d}}]F(x)^{(d+1)\ell+dr+1}$$
$$= \frac{(d+1)\ell+dr+1}{dn-2d+\ell+1}\binom{(d+1)(n-2)-r}{n-\ell-r-2}.$$

The first vertex adjacent to the root is labeled 1 and the remaining vertices on the path (except for the final non-first child) are labeled using integers from $\{1,2,\ldots,d\}$. This allows for $d^{\ell-1}$ labeling combinations. Moreover, the children of the final vertex can be labeled in $\binom{d+r-1}{r}$ distinct ways. The final expression is obtained by multiplying these counts together. □

Setting $r=0$ in Equation (14) gives the number of non-first children that are also leaves and share a label with their left sibling:

Corollary 2.18. *There are*

$$\frac{d\ell+\ell+1}{dn-2d+\ell+1}\binom{(d+1)(n-2)}{n-\ell-2}$$

non-first children, which are also leaves, at level $\ell \geq 2$ in d-dimensional plane trees on n vertices, such that the sibling immediately to the left of each non-first child has the same label as the non-first child.

Summing Equation (14) over all values of r, we derive the total number of non-first children (not necessarily leaves) at level $\ell \geq 2$ that satisfy the label-equality condition with the sibling to the left:

Corollary 2.19. *The total number of non-first children at level $\ell \geq 2$ in d-dimensional plane trees on n vertices, such that the sibling immediately to the left of the non-first child has the same label as the non-first child, is given by*

$$d^{\ell-1}\cdot\frac{\ell+1}{n-1}\binom{(d+1)(n-1)}{n-\ell-2}. \tag{15}$$

The following theorem addresses the more general case where the non-first child and its left sibling have different labels:

Theorem 2.20. *The number of non-first children of degree r at level $\ell \geq 2$ in d-dimensional plane trees on n vertices, such that the non-first child is labeled by i and its immediate left sibling is labeled by j, is given by*

$$d^{\ell-2}\binom{d+r-1}{r}\cdot\frac{(d+1)\ell+dr+j-i+1}{dn-2d+\ell+j-i+1}\binom{(d+1)(n-2)+j-i-r}{n-\ell-r-2}. \tag{16}$$

Proof. The relevant decomposition is shown in Figure 9. Here, the non-first child at level $\ell \geq 1$ is labeled by i and its immediate left sibling is labeled by j where $j \neq i$. The subtree rooted at this vertex has degree r.
The generating function corresponding to this decomposition is given as:

$$(xD(x)^2)(xD(x)^{d+1})^{\ell-2}(xN(x)^{j+d-i+1})(xD(x)^d)x(xD(x)^d)^r = x^{\ell+r+2}D(x)^{(d+1)\ell+dr+j-i+1}.$$

Applying the Lagrange inversion formula again yields:

$$\begin{aligned}[x^n]x^{\ell+r+2}D(x)^{(d+1)\ell+dr+j-i+1} &= [x^{n-\ell-r-2}]D(x)^{(d+1)\ell+dr+j-i+1}\\ &= [x^{\frac{dn-2d+\ell+j-i+1}{d}}]F(x)^{(d+1)\ell+dr+j-i+1}\\ &= \frac{(d+1)\ell+dr+j-i+1}{dn-2d+\ell+j-i+1}\binom{(d+1)(n-2)+j-i-r}{n-\ell-r-2}.\end{aligned}$$

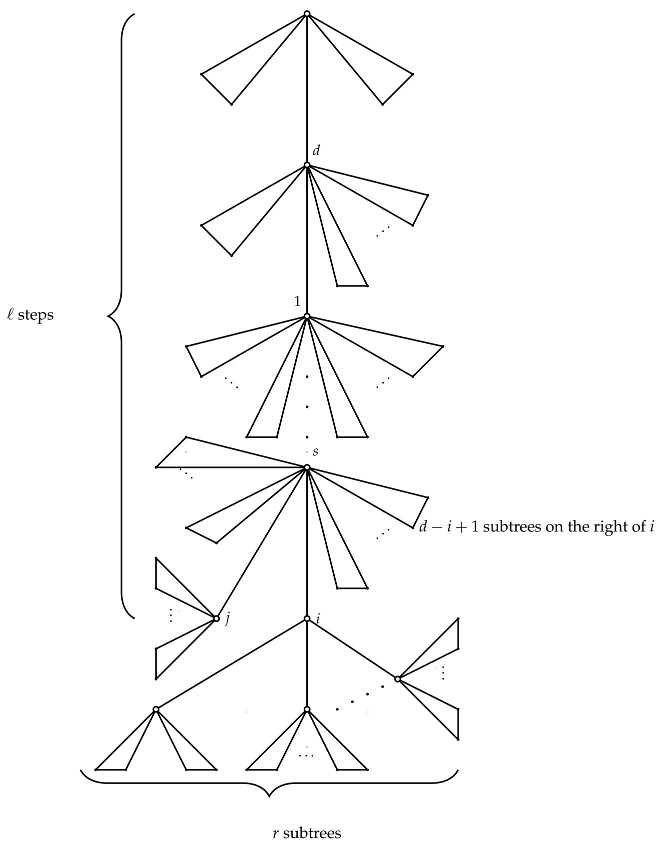

Figure 9. Decomposition of a d-dimensional plane tree with a non-first child of degree r at level $\ell \geq 1$, where the label of the non-first child differs from that of its immediate left sibling.

The labeling of the path allows for $d^{\ell-2}$ combinations. The distribution of labels to the children of the final vertex is again counted using $\binom{d+r-1}{r}$. Combining these factors yields the result. □

Taking $r = 0$ in Equation (16) yields the number of non-first children who are leaves and whose left sibling is labeled differently:

Corollary 2.21. *There are*

$$d^{\ell-2} \cdot \frac{(d+1)\ell + j - i + 1}{dn - 2d + \ell + j - i + 1} \binom{(d+1)(n-2) + j - i}{n - \ell - 2}$$

non-first children labeled by i, which are also leaves, at level $\ell \geq 2$ in d-dimensional plane trees on n vertices, such that the sibling immediately to the left of each non-first child is labeled by j.

Summing over all valid values of r in Equation (16), we obtain:

Corollary 2.22. *The total number of non-first children at level $\ell \geq 2$ in d-dimensional plane trees on n vertices, such that each non-first child is labeled by i and its immediate left sibling is labeled by j, is* *given by*

$$d^{\ell-2}\cdot\frac{(d+1)(\ell+1)+j-i}{dn-d+\ell+j-i+1}\binom{(d+1)(n-1)+j-i-1}{n-\ell-2}. \tag{17}$$

We now consider the case where the non-first child is a direct child of the root:

Proposition 2.23. *There are*

$$\binom{d+r-1}{r}\cdot\frac{dr+d+2}{dn-2d+2}\binom{(d+1)(n-2)-r}{n-r-3} \tag{18}$$

non-first children of the root of degree r in d-dimensional plane trees on n vertices.

Proof. Figure 10 depicts the decomposition for trees in which a non-first child of the root has degree r.

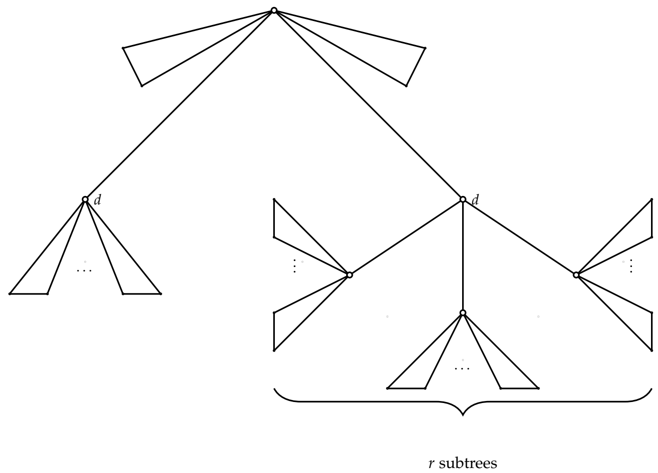

Figure 10. Decomposition of a d-dimensional plane tree with a non-first child of degree r that is also a child of the root.

The corresponding generating function is: $(xD(x)^2)(xD(x)^d)x(xD(x)^d)^r = x^{r+3}D(x)^{dr+d+2}$. Once again, we apply the Lagrange inversion technique to extract coefficients:

$$\begin{aligned}[x^n]x^{r+3}D(x)^{dr+d+2} &= [x^{n-r-3}]D(x)^{dr+d+2} = [x^{\frac{dn-2d+2}{d}}]F(x)^{dr+d+2} \\ &= \frac{dr+d+2}{dn-2d+2}\binom{(d+1)(n-2)-r}{n-r-3}.\end{aligned}$$

As usual, the labeling possibilities for the children are counted using the binomial coefficient $\binom{d+r-1}{r}$. The final result thus follows by product rule of counting □

Summing over all values of r in Equation (18), we get the total number of non-first children of the root in d-dimensional plane trees with n vertices:

$$\frac{2}{n-1}\binom{(d+1)(n-1)}{n-3} \tag{19}$$

Dividing (19) by

$$\frac{1}{n-1}\binom{(d+1)(n-1)}{n-2},$$

gives the average number of non-first children of the root in a randomly selected d-dimensional plane tree with n vertices: $\frac{2n-4}{d(n-1)+2}$. Asymptotically, this implies that a random d-dimensional plane tree has approximately $2/d$ non-first children of the root.

Finally, setting $r = 0$ in Equation (18) gives the number of non-first children of the root that are leaves in d-dimensional plane trees with n vertices:

$$\frac{d+2}{dn-2d+2}\binom{(d+1)(n-2)}{n-3}.$$

2.5 Non-leaves

In this part of the paper, we calculate the total number of non-leaf vertices at a given level across all d-dimensional plane trees.

Theorem 2.24. *The number of non-leaf vertices at level $\ell \geq 1$ in d-dimensional plane trees on n vertices is given by*

$$d^{\ell} \cdot \frac{(d+1)(\ell+1)+1}{dn-d+\ell+2}\binom{(d+1)(n-1)}{n-\ell-2}. \tag{20}$$

Proof. The tree decomposition corresponding to this enumeration is illustrated in Figure 11. We focus on vertices at level $\ell \geq 1$ that are not leaves, meaning they must have at least one child.

From the structure in the figure, we derive the generating function:

$$(xD(x)^2)(xD(x)^{d+1})^{\ell}(xD(x)^d) = x^{\ell+2}D(x)^{(d+1)(\ell+1)+1}.$$

Applying the Lagrange inversion formula, we extract the coefficient of x^n from the generating function as follows:

$$\begin{aligned}[x^n]x^{\ell+2}D(x)^{(d+1)(\ell+1)+1} &= [x^{n-\ell-2}]D(x)^{(d+1)(\ell+1)+1} \\ &= [x^{\frac{dn-d+\ell+2}{d}}]F(x)^{(d+1)(\ell+1)+1} \\ &= \frac{(d+1)(\ell+1)+1}{dn-d+\ell+2}\binom{(d+1)(n-1)}{n-\ell-2}.\end{aligned}$$

The labeling of the vertices on the path (excluding the root) admits d^{ℓ} possibilities. Combining this with the coefficient extraction gives the result. □

Setting $\ell = 1$ in Equation (20), we isolate the count of non-leaf children of the root:

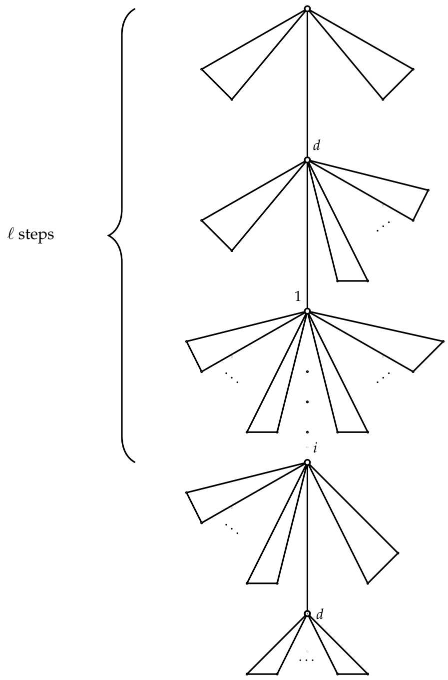

Figure 11. Decomposition of a d-dimensional plane tree with a non-leaf vertex at level $\ell \geq 1$.

Corollary 2.25. *The number of non-leaf vertices that are children of the root in d-dimensional plane trees on n vertices is given by*

$$\frac{2d^2 + 3d}{dn - d + 3}\binom{(d+1)(n-1)}{n-3}.$$

This result quantifies how many of the root's immediate children are not leaves in all d-dimensional plane trees on n vertices.

2.6 Bijections

By modifying the bijections established in [10], we derive the following structural configurations within d-dimensional plane trees:

(i) There exists a bijection between:

  (a) the set of d-dimensional plane trees with n vertices and root degree $\ell + 1$, and

  (b) the set of eldest children located at level $d\ell \geq 2$ in d-dimensional plane trees with $n + d\ell - \ell - 1$ vertices, where all the vertices on the leftmost path from the root to the eldest child are labeled by d.

  The bijection is constructed by modifying proof of Proposition of 3.1 in [10].

(ii) The set of first children labeled 1, which are also leaves, at level $\ell \geq 2$ in d-dimensional plane trees on n vertices is in bijection with the set of non-first children, which are also leaves, at level $\ell - 1$ in d-dimensional plane trees with n vertices such that for each non-first child, the sibling on the immediate left of the non-first child has the same label as the non-first child. The proof of this result follows a similar argument as in the proof of Corollary 3.2 in [10].

(iii) The set of first children labeled by i of degree $r + 1$ at level $\ell \geq 2$ in d-dimensional plane trees on n vertices such that the first child of the first child is labeled by $j \leq i$ is in a one-to-one correspondence with the set of non-first children of degree r, labeled by $i + 1$, at level ℓ in d-dimensional plane trees with n vertices such that for each non-first child, the sibling on the immediate left of the non-first child is labeled by j. The result is arrived by adapting proof of Proposition 3.5 in [10].

(iv) The set of vertices that reside at level $\ell + 1$ in d-dimensional plane trees on n vertices is in bijection with the set of non-leaves that reside at level ℓ in d-dimensional plane trees on n vertices. The proof is established by adapting proof of Proposition 3.6 in [10].

Moreover, the following three collections are equinumerous.

(i) First children labeled 1 and of degree r appearing at level $\ell \geq 2$.

(ii) Vertices of degree $r + 1$ at level $\ell - 1$, where the first child is labeled 1

(iii) Non-first children of degree r at level $\ell - 1$, where each such child shares its label with the sibling on its immediate left.

The correspondence between (i) and (ii) is obtained by modifying proof of Proposition 3.2 in [10]. Also, the bijection relating (i) and (iii) is arrived at by adapting proof of Proposition 3.3 in [10].

3 Conclusion

In this work, we extended the study of vertex enumeration within plane trees to a broader class known as d-dimensional plane trees, which generalizes both ordinary plane trees and noncrossing trees. Using a combination of generating functions, enhanced butterfly decompositions, and bijective constructions, we established both exact and asymptotic formulas for counting vertices according to level and degree. In particular, we enumerated the number of vertices, eldest children, first children, non-first children, and non-leaves occurring at a given level across all d-dimensional plane trees. These results extend prior work on 1- and 2-dimensional plane trees and demonstrate the versatility of the butterfly decomposition in higher-dimensional plane tree enumeration. In addition, we established bijections between certain vertex configurations and their related structures, thereby offering deeper insight into the structural regularities present in d-dimensional plane trees. Several directions for further research remain open. These include:

- Extending the enumeration to trees with additional constraints, such as bounded degree or fixed height;
- Investigating analogous decompositions for rooted forests in higher dimensions;
- Exploring the enumeration of vertices by levels and degrees in other structured tree families, such as k-plane trees (introduced by Gu, Prodinger, and Wagner [5]), k-noncrossing trees (studied by Pang and Lv [14]), and their d-dimensional analogues [9].

We hope that the results and methods developed in this paper will contribute to a broader understanding of multidimensional tree structures in enumerative combinatorics.

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