



Research Paper

The elliptic Sombor index of special graphs

Shahroud Azami^{1,*}, Hassan Izanloo²

¹ Department of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, I. R. Iran

² School of Mathematics, University of Leeds, Woodhouse, Leeds, LS2 9JT, UK.

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Abstract. The elliptic Sombor index serves as a topological index that is based on vertex degrees. Our research focuses on the elliptic Sombor index within various graph types, including m^k -graphs, Jahangir graphs, Barbell graphs, and Friendship graphs. Additionally, we derive this index for the subdivided m^k -graphs. Furthermore, we present several bounds for some of these indices.

Keywords. Sombor index, m^k -graphs, Jahangir graphs, Barbell graphs.

Mathematics Subject Classification (2020): 05C09, 05C07, 05C69.

1 Introduction

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ denote a finite, connected, and simple graph with an order of $|V(\mathcal{H})| = n$ and a size of $|E(\mathcal{H})| = m$. The vertex set of \mathcal{H} is represented as $V(\mathcal{H}) = \{v_1, v_2, \dots, v_n\}$, while the edge set is denoted by $E(\mathcal{H})$. The degree of a vertex v_i , which reflects the number of its adjacent vertices, is denoted as $d_{\mathcal{H}}(v_i)$ for $i = 1, 2, \dots, n$. Isolated and pendant vertices are vertices of degree zero and degree one, respectively. Let $\delta(\mathcal{H})$ represent the minimum vertex degree and $\Delta(\mathcal{H})$ signify the maximum vertex degree within the graph \mathcal{H} . A sequence $w = w_1, w_2, \dots, w_{n-1}, w_n = v$ in \mathcal{H} where $w_i w_{i+1} \in E(\mathcal{H})$, $i = 1, \dots, n - 1$ is said to be a $w - v$ walk in \mathcal{H} . If a walk with n vertices has not repeated vertex, then it is called a path and denoted by P_n . A closed path consisting of n edges is referred to as a cycle C_n . In the context

*Corresponding author (Email address: azami@sci.ikiu.ac.ir).

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of graph theory, a graph with n vertices is termed a complete graph if every pair of vertices is connected by an edge, and this is represented by the notation K_n .

Let A be a real bivariate function defined over $\mathbb{R} \times \mathbb{R}$ with condition $A(x, y) = A(y, x)$ for all non-negative real number x and y . The topological index based on vertex degrees, denoted as TI , for graph \mathcal{H} is defined as follows

$$TI(\mathcal{H}) = \sum_{w_i w_j \in E(\mathcal{H})} A(d_{\mathcal{H}}(w_i), d_{\mathcal{H}}(w_j)).$$

In [10], Gutman listed 26 types of topological indices based on vertex degrees, for instance, the different type of Zagreb indices [2], Randić indices [6], and Sombor index. A molecular graph [18] is defined as a connected graph in which the vertices represent atoms and the edges signify the covalent bonds that link these atoms together. Topological indices contain information on the atom-connectivity molecular refractivity, the nature of atoms, molecular volume, the bond multiplicity, etc.

The Sombor index, as defined by Gutman [10], is presented in the following manner

$$SO(\mathcal{H}) = \sum_{w_i w_j \in E(\mathcal{H})} \sqrt{d_{\mathcal{H}}^2(w_i) + d_{\mathcal{H}}^2(w_j)}, \tag{1}$$

which has the geometric notion of the degree radius associated with an edge $w_i w_j \in E(\mathcal{H})$ and the elliptic Sombor index was introduced by Gutman [10, 11]. This index is defined as follows

$$ESO(\mathcal{H}) = \sum_{w_i w_j \in E(\mathcal{H})} (d_{\mathcal{H}}(w_i) + d_{\mathcal{H}}(w_j)) \sqrt{d_{\mathcal{H}}^2(w_i) + d_{\mathcal{H}}^2(w_j)}. \tag{2}$$

In 1972, Gutman and Trinajstić [12] presented the first Zagreb index $M_1(\mathcal{H})$ and the second Zagreb index $M_2(\mathcal{H})$, defined as follows

$$M_1(\mathcal{H}) = \sum_{u \in w(\mathcal{H})} d_{\mathcal{H}}^2(u) = \sum_{uw \in E(\mathcal{H})} (d_{\mathcal{H}}(u) + d_{\mathcal{H}}(w)),$$

$$M_2(\mathcal{H}) = \sum_{uw \in E(\mathcal{H})} (d_{\mathcal{H}}(u)d_{\mathcal{H}}(w)).$$

In 2015, Furtula and Gutman introduced the concept of the F -index [9] by

$$F(\mathcal{H}) = \sum_{u \in w(\mathcal{H})} d_{\mathcal{H}}(u)^3 = \sum_{uw \in E(\mathcal{H})} (d_{\mathcal{H}}(u)^2 + d_{\mathcal{H}}(w)^2).$$

In 2004, Li and Zhao [15] presented the concept of the general first Zagreb index for a graph \mathcal{H} by

$$M_1^\alpha(\mathcal{H}) = \sum_{u \in w(\mathcal{H})} d_{\mathcal{H}}(u)^\alpha = \sum_{uw \in E(\mathcal{H})} (d_{\mathcal{H}}(u)^{\alpha-1} + d_{\mathcal{H}}(w)^{\alpha-1}), \quad \text{for } \alpha \in \mathbb{R} \setminus \{0, 1\}.$$

For especial case $\alpha = 4$, it is called Y-index and denoted by $Y(\mathcal{H})$, that is,

$$Y(\mathcal{H}) = \sum_{u \in w(\mathcal{H})} d_{\mathcal{H}}(u)^4.$$

The general Randić index [5] is represented by

$$M_2^\alpha(\mathcal{H}) = \sum_{uw \in E(\mathcal{H})} (d_{\mathcal{H}}(u)d_{\mathcal{H}}(w))^\alpha, \quad \text{for } \alpha \in \mathbb{R} \setminus \{0, 1\},$$

and the (α, β) -Zagreb indices are characterized by

$$Z_{(\alpha, \beta)}(\mathcal{H}) = \sum_{uw \in E(\mathcal{H})} (d_{\mathcal{H}}(u)^\alpha d_{\mathcal{H}}(w)^\beta + d_{\mathcal{H}}(u)^\beta d_{\mathcal{H}}(w)^\alpha), \quad \text{for } (\alpha, \beta) \neq (0, 0).$$

The Elliptic Sombor Index is a mathematically elegant and chemically relevant topological index that quantifies graph structures through a geometric lens. Its sensitivity to degree heterogeneity, strong correlations with molecular properties, and connections to classical indices make it a powerful tool in both theoretical and applied graph theory. Ongoing research continues to uncover its potential in chemical informatics, network science, and discrete mathematics.

The elliptic Sombor index has the mathematical properties and chemical applications. Kulli [14] provided elliptic Sombor index on some chemical graphs. Rada et al. [19] studied elliptic Sombor index of benzenoid systems. Espinal et al. [8] investigated elliptic Sombor index on some chemical graphs. Also, Alikhani et al. [1] studied elliptic Sombor energy.

Chanda and Iyer [4], studied the Sombor index of generalized Siperpiński graphs and generalized Mycielskian graphs and obtained some upper and lower bounds for them. Also, Liu [16] investigated multiplicative Sombor index on some graphs such as unicyclic graphs and trees. Jalali and Ghods [13] computed Y-index $Y(\mathcal{H}) = \sum_{v \in V(\mathcal{H})} d_{\mathcal{H}}^4(v)$ for m^k -graph and Jahangir graph.

While classical indices like Zagreb and Randić have been extensively studied on standard graph families, the ESO index remains underexplored for complex hierarchical structures like m^k -graphs and Jahangir graphs. This gap is significant because these graphs model real-world systems: m^k -graphs generalize interconnection networks in parallel computing [3], while Jahangir graphs appear in benzenoid molecular chains [7]. Our work addresses this void by providing the first systematic analysis of ESO on these graph classes, revealing how structural complexity manifests in this novel index.

Although the Sombor index has been computed for generalized Sierpiński graphs [4] and trees [16], and the elliptic Sombor index has been examined for chemical graphs [14], no prior work exists on ESO for m^k -graphs, Jahangir graphs, Barbell graphs, or Friendship graphs. This paper closes this gap while establishing new bounds and comparative insights.

Motivated as the above works, we study the elliptic Sombor index on some special graphs: m^k -graphs, Jahangir graphs, Barbell graphs, friendship graphs, and subdivided m^k -graphs.

2 Main results

In this section, we showcase and illustrate our key findings. We compute the elliptic Sombor index on m^k -graphs, Jahangir graphs, Barbell graphs, and friendship graphs.

2.1 The elliptic Sombor index on m^k -graphs

m^k -graph $m^k\mathcal{H}$ introduced by Ayache and Alameri [3] and constructed as follows. For a given simple graph \mathcal{H} and an integer $m \geq 2$, we examine m copies of \mathcal{H} , denoted as $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_m$. In this scenario, if each vertex x in the graph \mathcal{H}_j is adjacent to the corresponding vertex y in another graph \mathcal{H}_i , the resulting structure is referred to as mG . As an illustration, when \mathcal{H} consists of a single vertex, the graph mG corresponds to the complete graph K_m . For any non-negative integer k , the m^k -graph denoted by $m^k\mathcal{H}$ and is defined by

$$m^0(\mathcal{H}) = \mathcal{H}, m^1(\mathcal{H}) = mG, m^2(\mathcal{H}) = m(m(\mathcal{H})), \dots, m^{k+1} = m(m^k\mathcal{H}).$$

If we consider a graph \mathcal{H} with a single vertex, then for the case where $m = 2$, the m^k -graph corresponds to the cube graph Q_k of dimension k . Let us denote \mathcal{H} as a graph with an order of n and a size of p . For the m^k -graph of \mathcal{H} , we have

$$|V(m^k\mathcal{H})| = m^k n, |E(m^k\mathcal{H})| = m^k p + \binom{m}{2} n k m^{k-1}, d_{m^k\mathcal{H}}(v) = d_{\mathcal{H}}(v) + k(m - 1).$$

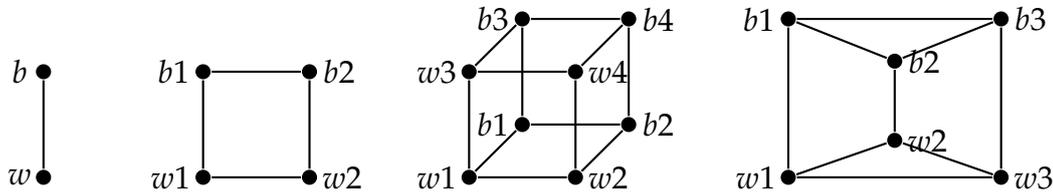


Figure 1. $\mathcal{H}, 2G, 2^2\mathcal{H}$ and $3G$.

Theorem 2.1. Let \mathcal{H} represent a graph characterized by an order of n and a size of p . For any integer $m \geq 2$ and any non-negative integer k , we get

$$\begin{aligned} ESO(m^k\mathcal{H}) &\leq 2 \binom{m}{2} k m^{k-1} \sqrt{2} \left(M_1(\mathcal{H}) + 4pk(m - 1) + k^2 n(m - 1)^2 \right) \\ &\quad + m^k \left(ESO(\mathcal{H}) + \sqrt{2} k(m - 1) (M_1(\mathcal{H}) + \sqrt{2} SO(\mathcal{H}) + 2kp(m - 1)) \right), \\ ESO(m^k\mathcal{H}) &\geq 2 \binom{m}{2} k m^{k-1} \sqrt{2} \left(M_1(\mathcal{H}) + 4pk(m - 1) + k^2 n(m - 1)^2 \right) \\ &\quad + m^k (ESO(\mathcal{H}) + k(m - 1) (M_1(\mathcal{H}) + 2SO(\mathcal{H}) + 2kp(m - 1))). \end{aligned}$$

Proof. Suppose $V(\mathcal{H}) = \{v_1, v_2, \dots, v_n\}$, $V(\mathcal{H}_{i_1 i_2 \dots i_k}) = \{v_{1i_1 i_2 \dots i_k}, \dots, v_{ni_1 i_2 \dots i_k}\}$ for i_1, i_2, \dots, i_k such that $0 \leq i_1, i_2, \dots, i_k \leq m$, and

$$V(m^k\mathcal{H}) = \bigcup_{i_1, i_2, \dots, i_k} \{V(\mathcal{H}_{i_1 i_2 \dots i_k}) \mid 0 \leq i_1, i_2, \dots, i_k \leq m\}.$$

By definition of the elliptic Sombor index and $d_{m^k\mathcal{H}}(v) = d_{\mathcal{H}}(v) + k(m - 1)$, we obtain

$$\begin{aligned}
 & ESO(m^k\mathcal{H}) \tag{3} \\
 &= \sum_{uv \in E(m^k\mathcal{H})} (d_{m^k\mathcal{H}}(u) + d_{m^k\mathcal{H}}(v)) \sqrt{d_{m^k\mathcal{H}}^2(u) + d_{m^k\mathcal{H}}^2(v)} \\
 &= \sum_{v_{i_1 i_2 \dots i_k} v_{j_1 j_2 \dots j_k} \in E(m^k\mathcal{H})} \left\{ (d_{\mathcal{H}}(v_i) + d_{\mathcal{H}}(v_j) + 2k(m - 1)) \right. \\
 &\quad \left. \times \sqrt{(d_{\mathcal{H}}(v_i) + (m - 1)k)^2 + (d_{\mathcal{H}}(v_j) + (m - 1)k)^2} \right\} \\
 &= \sum_{v_{i_1 i_2 \dots i_k} v_{j_1 j_2 \dots j_k} \in E(m^k\mathcal{H})} 2\sqrt{2} (d_{\mathcal{H}}(v_i) + k(m - 1))^2 \\
 &\quad + \sum_{v_{i_1 i_2 \dots i_k} v_{j_1 j_2 \dots j_k} \in E(m^k\mathcal{H}), i \neq j} \left\{ (d_{\mathcal{H}}(v_i) + d_{\mathcal{H}}(v_j) + 2k(m - 1)) \right. \\
 &\quad \left. \times \sqrt{(d_{\mathcal{H}}(v_i) + (m - 1)k)^2 + (d_{\mathcal{H}}(v_j) + (m - 1)k)^2} \right\}.
 \end{aligned}$$

Applying inequality $\sqrt{(x + z)^2 + (y + z)^2} \leq \sqrt{x^2 + y^2} + \sqrt{2}z$ for $x, y, z \geq 0$ in (3), we get

$$\begin{aligned}
 ESO(m^k\mathcal{H}) &\leq 2 \binom{m}{2} km^{k-1} \sqrt{2} \sum_{v_i \in V(\mathcal{H})} (d_{\mathcal{H}}(v_i) + k(m - 1))^2 \\
 &\quad + m^k \sum_{v_i v_j \in E(\mathcal{H})} \left\{ (d_{\mathcal{H}}(v_i) + d_{\mathcal{H}}(v_j) + 2k(m - 1)) \right. \\
 &\quad \left. \times \left(\sqrt{d_{\mathcal{H}}^2(v_i) + d_{\mathcal{H}}^2(v_j)} + \sqrt{2}k(m - 1) \right) \right\}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 ESO(m^k\mathcal{H}) &\leq 2 \binom{m}{2} km^{k-1} \sqrt{2} \left(M_1(\mathcal{H}) + 4pk(m - 1) + k^2n(m - 1)^2 \right) \\
 &\quad + m^k \left(ESO(\mathcal{H}) + \sqrt{2}k(m - 1)(M_1(\mathcal{H}) + \sqrt{2}SO(\mathcal{H}) + 2kp(m - 1)) \right).
 \end{aligned}$$

Also, using inequality $\sqrt{(x + w)^2 + (y + w)^2} \geq \sqrt{x^2 + y^2} + w$ for $x, y, w \geq 0$, we obtain

$$\begin{aligned}
 ESO(m^k\mathcal{H}) &\geq 2 \binom{m}{2} km^{k-1} \sqrt{2} \sum_{w \in V(\mathcal{H})} (d_{\mathcal{H}}(w) + k(m - 1))^2 \\
 &\quad + m^k \sum_{wv \in E(\mathcal{H})} \left\{ (d_{\mathcal{H}}(w) + d_{\mathcal{H}}(v) + 2k(m - 1)) \right. \\
 &\quad \left. \times \left(\sqrt{d_{\mathcal{H}}^2(w) + d_{\mathcal{H}}^2(v)} + (m - 1)k \right) \right\} \\
 &= 2\sqrt{2} \binom{m}{2} km^{k-1} \left(M_1(\mathcal{H}) + 4pk(m - 1) + k^2n(m - 1)^2 \right) \\
 &\quad + m^k (ESO(\mathcal{H}) + k(m - 1)(M_1(\mathcal{H}) + 2SO(\mathcal{H}) + 2kp(m - 1))).
 \end{aligned}$$

□

Using equation (3), we derive the subsequent corollary.

Corollary 2.2. *Let \mathcal{H} denote a graph of order n that is r -regular. Then for any integer $m \geq 2$ and any non-negative integer k , we get*

$$ESO(m^k \mathcal{H}) = 2\sqrt{2}k(r + k(m - 1))^2 \left(\binom{m}{2} knm^{k-1} + m^k \frac{rn}{2} \right).$$

Epecially,

$$\begin{aligned} ESO(m^k C_n) &= 2\sqrt{2}(2 + k(m - 1))^2 \left(\binom{m}{2} knm^{k-1} + nm^k \right), \\ ESO(m^k K_n) &= 2\sqrt{2}((m - 1)k + n - 1)^2 \left(\binom{m}{2} knm^{k-1} + m^k \frac{n(n - 1)}{2} \right), \\ ESO(Q_k) &= \sqrt{2}k^3 2^k. \end{aligned}$$

Also, applying equation (3), we state the subsequent corollary.

Corollary 2.3. *For any integer $m \geq 2$ and any non-negative integer k , we have*

$$\begin{aligned} ESO(m^k P_n) &= 2 \binom{m}{2} km^{k-1} \sqrt{2} \left\{ 2(1 + (m - 1)k)^2 + (n - 2)(2 + (m - 1)k)^2 \right\} \\ &+ 2m^k \left\{ (3 + k(m - 1)) \sqrt{(1 + (m - 1)k)^2 + (2 + (m - 1)k)^2} + (n - 3)\sqrt{2}(2 + (m - 1)k)^2 \right\}. \end{aligned}$$

A wheel graph, denoted as W_n , is constructed by linking a central vertex to each vertex of a cycle graph C_n . Equation (3) leads to

$$\begin{aligned} ESO(m^k W_n) &= 2 \binom{m}{2} km^{k-1} \sqrt{2} \sum_{u \in V(W_n)} (d_{W_n}(u) + (m - 1)k)^2 \\ &+ m^k \sum_{uv \in E(W_n)} \{ (d_{W_n}(u) + d_{W_n}(v) + 2k(m - 1)) \\ &\times \sqrt{(d_{W_n}(u) + (m - 1)k)^2 + (d_{W_n}(v) + (m - 1)k)^2} \}. \end{aligned}$$

By considering

$$V(C_n) = \{v_1, v_2, \dots, v_n\}, \quad V(W_n) = V(C_n) \cup \{w\},$$

and

$$E(W_n) = \left\{ \bigcup_{i=1}^{n-1} \{v_i v_{i+1}, wv_i\} \right\} \cup \{v_n v_1, wv_n\},$$

equation (3) gives

$$\begin{aligned}
 ESO(m^k W_n) &= 2\sqrt{2} \binom{m}{2} km^{k-1} \sum_{u \in V(W_n)} (d_{W_n}(u) + (m-1)k)^2 \\
 &\quad + m^k \sum_{i=1}^n \left\{ (d_{W_n}(w) + d_{W_n}(v_i) + 2(m-1)k) \right. \\
 &\quad \quad \left. \times \sqrt{(d_{W_n}(w) + (m-1)k)^2 + (d_{W_n}(v_i) + (m-1)k)^2} \right\} \\
 &\quad + m^k \sum_{i=1}^{n-1} \left\{ (d_{W_n}(v_i) + d_{W_n}(v_{i+1}) + 2(m-1)k) \right. \\
 &\quad \quad \left. \times \sqrt{(d_{W_n}(v_i) + (m-1)k)^2 + (d_{W_n}(v_{i+1}) + (m-1)k)^2} \right\} \\
 &\quad + m^k \left\{ (d_{W_n}(v_1) + d_{W_n}(v_n) + 2k(m-1)) \right. \\
 &\quad \quad \left. \times \sqrt{(d_{W_n}(v_1) + k(m-1))^2 + (d_{W_n}(v_n) + k(m-1))^2} \right\}.
 \end{aligned}$$

Since

$$d_{W_n}(u) = \begin{cases} n & \text{if } u = w, \\ 3 & \text{if } u \neq w. \end{cases}$$

We present the subsequent corollary.

Corollary 2.4. *Let m be an integer such that $m \geq 2$, and let k represent a non-negative integer value. For any $n \geq 3$, we get*

$$\begin{aligned}
 ESO(m^k W_n) &= 2\sqrt{2} \binom{m}{2} km^{k-1} \left((n + (m-1)k)^2 + n(3 + (m-1)k)^2 \right) \\
 &\quad + m^k n(3 + n + (m-1)k) \sqrt{(n + (m-1)k)^2 + (3 + (m-1)k)^2} \\
 &\quad + 2m^k n \sqrt{2} (3 + (m-1)k)^2.
 \end{aligned}$$

The bounds reveal that $ESO(m^k \mathcal{H})$ grows polynomially with k and m . This dual sensitivity reflects the graph’s hierarchical complexity: exponential growth captures the recursive edge additions between copies, while polynomial scaling relates to intra-copy connections. Notably, for regular graphs (Corollary 2.2), ESO exhibits superlinear growth in k , contrasting with the linear growth of the standard Sombor index [11]. This suggests ESO better captures structural amplification in recursive networks.

For any graph \mathcal{H} , if we remove the edge $uv \in E(\mathcal{H})$ and introduce a new vertex w along with two new edges uw and wv in the graph \mathcal{H} , the resulting graph is referred to as the subdivision graph of \mathcal{H} .

Theorem 2.5. *Suppose $m \geq 2$ is an integer and k is a non-negative integer. Let us denote the graph \mathcal{H} as a subdivided version of the graph $m^k P_n$. In this context, the elliptic Sombor index of this graph is*

represented by

$$\begin{aligned} ESO(\mathcal{H}) &= 4 \binom{m}{2} nkm^{k-1} (3 + k(m - 1)) \sqrt{4 + (1 + (m - 1)k)^2} \\ &\quad + 2 \binom{m}{2} nkm^{k-1} (n - 2) (4 + (m - 1)k) \sqrt{4 + (2 + (m - 1)k)^2} \\ &\quad + 2m^k (n - 1) (3 + k(m - 1)) \sqrt{4 + (1 + (m - 1)k)^2} \\ &\quad + 2(n - 2)m^k (n - 1) (4 + k(m - 1)) \sqrt{4 + (2 + (m - 1)k)^2}. \end{aligned}$$

Proof. Suppose $V(P_n) = \{v_1, v_2, \dots, v_n\}$, $V(P_{ni_1i_2\dots i_k}) = \{v_{1i_1i_2\dots i_k}, \dots, v_{ni_1i_2\dots i_k}\}$ for i_1, i_2, \dots, i_k such that $0 \leq i_1, i_2, \dots, i_k \leq m$, and

$$V(m^k P_n) = \bigcup_{i_1, i_2, \dots, i_k} \{V(P_{ni_1i_2\dots i_k}) \mid 0 \leq i_1, i_2, \dots, i_k \leq m\}.$$

Let we omit the edge $v_{ii_1i_2\dots i_k}v_{jj_1j_2\dots j_k} \in E(m^k P_n)$ and added a new vertex w and two new edges $v_{ii_1i_2\dots i_k}w$ and $wv_{jj_1j_2\dots j_k}$ in graph $m^k \mathcal{H}$. Let

$$E_1 = \bigcup_{i=1}^m \{ \{wv_{ii_1i_2\dots i_k} \mid 0 \leq i_1, i_2, \dots, i_k \leq m\} \cup \{v_{ij_1j_2\dots j_k}w \mid 0 \leq j_1, j_2, \dots, j_k \leq m\} \}$$

and

$$E_2 = \bigcup_{i,j=1}^m \{ \{wv_{ii_1i_2\dots i_k} \mid 0 \leq i_1, i_2, \dots, i_k \leq m\} \cup \{v_{jj_1j_2\dots j_k}w \mid 0 \leq j_1, j_2, \dots, j_k \leq m\} \mid v_i v_j \in E(\mathcal{H}) \}.$$

Then $|E_1| = 2 \binom{m}{2} nkm^{k-1}$ and $|E_2 \setminus E_1| = 2m^k(n - 1)$. By definition of subdivision graph and the elliptic Sombor index, we deduce

$$\begin{aligned} ESO(\mathcal{H}) &= \sum_{uv \in E_1} (d_{\mathcal{H}}(u) + d_{\mathcal{H}}(v)) \sqrt{d_{\mathcal{H}}^2(u) + d_{\mathcal{H}}^2(v)} \\ &\quad + \sum_{uv \in E_2 \setminus E_1} (d_{\mathcal{H}}(u) + d_{\mathcal{H}}(v)) \sqrt{d_{\mathcal{H}}^2(u) + d_{\mathcal{H}}^2(v)}. \end{aligned}$$

Since $d_{\mathcal{H}}^2(w) + d_{\mathcal{H}}(v_{ii_1i_2\dots i_k}) = 4 + (d_{P_n}(v_i) + k(m - 1))^2$, we conclude that

$$\begin{aligned} ESO(\mathcal{H}) &= 2 \binom{m}{2} nkm^{k-1} \sum_{i=1}^n (2 + d_{P_n}(v_i) + (m - 1)k) \sqrt{4 + (d_{P_n}(v_i) + (m - 1)k)^2} \\ &\quad + m^k (n - 1) \sum_{v_i v_j \in E(P_n)} \left((2 + d_{P_n}(v_i) + (m - 1)k) \sqrt{4 + (d_{P_n}(v_i) + k(m - 1))^2} \right. \\ &\quad \left. + (2 + d_{P_n}(v_j) + (m - 1)k) \sqrt{4 + (d_{P_n}(v_j) + (m - 1)k)^2} \right). \end{aligned}$$

Therefore

$$\begin{aligned}
 ESO(\mathcal{H}) &= 4 \binom{m}{2} n k m^{k-1} (3 + (m - 1)k) \sqrt{4 + (1 + (m - 1)k)^2} \\
 &\quad + 2 \binom{m}{2} n k m^{k-1} (n - 2) (4 + (m - 1)k) \sqrt{4 + (2 + (m - 1)k)^2} \\
 &\quad + 2 m^k (n - 1) (3 + (m - 1)k) \sqrt{4 + (1 + (m - 1)k)^2} \\
 &\quad + 2 (n - 2) m^k (n - 1) (4 + (m - 1)k) \sqrt{4 + (2 + (m - 1)k)^2}.
 \end{aligned}$$

□

Subdivided graphs model molecular structures where atoms are replaced by functional groups [19]. The graph \mathcal{H} obtained by subdividing each edge in $m^k P_n$ represents a hierarchical polymer chain with inserted heteroatoms. Computing ESO here reveals how subdivision affects topological descriptors in recursive nanostructures.

2.2 Jahangir graphs

For every integer $n \geq 2$ and $m \geq 3$, a graph that comprises a cycle denoted as C_{nm} , along with a central vertex that is adjacent to m vertices of C_{nm} , where these vertices are situated at a distance of n from one another on the cycle, is referred to as a Jahangir graph, represented by the notation $J_{n,m}$. For more details of Jahangir graphs see [7,20]. Then $|V(J_{n,m})| = nm + 1$ and $|E(J_{n,m})| = m(n + 1)$.

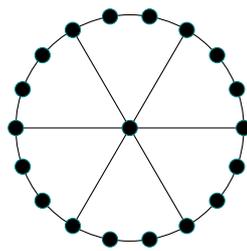


Figure 2. $J_{3,6}$

Theorem 2.6. For all $m \geq 3$ and $n \geq 2$, we have

$$ESO(J_{n,m}) = m(m + 3) \sqrt{m^2 + 9} + 8\sqrt{2}m(n - 2) + 10\sqrt{13}m.$$

Proof. Let $V_i = \{v_{i1}, v_{i2}, \dots, v_{in}\}$ for $1 \leq i \leq m$, $V(C_{nm}) = V_1 \cup V_2 \cup \dots \cup V_m$, $V(J_{n,m}) = V(C_{nm}) \cup \{w\}$, and $E(J_{n,m}) = \{\cup_{i=1}^m \{wv_{i1}\}\} \cup E(C_{nm})$. Set $N(w) = \{v_{11}, v_{21}, \dots, v_{m1}\}$. By definitions of

the elliptic Sombor index and Jahangir graph, we get

$$\begin{aligned}
 ESO(J_{n,m}) &= \sum_{i=1}^m (d_{J_{n,m}}(w) + d_{J_{n,m}}(a_{i1})) \sqrt{d_{J_{n,m}}^2(w) + d_{J_{n,m}}^2(a_{i1})} \\
 &+ \sum_{\substack{ua \in E(C_{nm}) \\ u, a \notin N(w)}} (d_{J_{n,m}}(u) + d_{J_{n,m}}(a)) \sqrt{d_{J_{n,m}}^2(u) + d_{J_{n,m}}^2(a)} \\
 &+ \sum_{\substack{ua \in E(C_{nm}) \\ u \in N(w)}} (d_{J_{n,m}}(u) + d_{J_{n,m}}(a)) \sqrt{d_{J_{n,m}}^2(u) + d_{J_{n,m}}^2(a)} \\
 &= m(m + 3) \sqrt{m^2 + 9} + 8\sqrt{2}m(n - 2) + 10\sqrt{13}m.
 \end{aligned}$$

□

The above formula shows ESO scales linearly with cycle segments m but sublinearly with cycle length n . This indicates ESO is more sensitive to radial connections (central vertex attachments) than to cycle elongation a non-intuitive result since cycle length increases total edges. Compared to the first Zagreb index $M_1(J_{n,m}) = 6m + 2mn$ [17], ESO’s slower growth in n suggests it prioritizes vertex-degree heterogeneity over pure edge count.

Theorem 2.7. *Suppose $m \geq 2$ is an integer and k is a non-negative integer. For all $r \geq 2$ and $s \geq 3$, we have*

$$\begin{aligned}
 ESO(m^k J_{r,s}) &= 2 \binom{m}{2} km^{k-1} \sqrt{2} s (r - 1) (2 + (m - 1)k)^2 \\
 &+ 2 \binom{m}{2} km^{k-1} \sqrt{2} (s(3 + (m - 1)k)^2 + (s + (m - 1)k)^2) \\
 &+ m^k s (s + 3 + 2k(m - 1)) \sqrt{(s + (m - 1)k)^2 + (3 + (m - 1)k)^2} \\
 &+ 2m^k (r - 2) s \sqrt{2} (2 + (m - 1)k)^2 \\
 &+ 2m^k s (5 + 2k(m - 1)) \sqrt{(3 + (m - 1)k)^2 + (2 + (m - 1)k)^2}.
 \end{aligned}$$

Proof. Using equation (3), we get

$$\begin{aligned}
 ESO(m^k J_{r,s}) &= 2 \binom{m}{2} km^{k-1} \sqrt{2} \sum_{u \in V(J_{r,s})} (d_{J_{r,s}}(u) + k(m - 1))^2 \\
 &+ m^k \sum_{uv \in E(J_{r,s})} \{ (d_{J_{r,s}}(u) + d_{J_{r,s}}(v) + k(m - 1)) \\
 &\quad \times \sqrt{(d_{J_{r,s}}(u) + k(m - 1))^2 + (d_{J_{r,s}}(v) + k(m - 1))^2} \}. \tag{4}
 \end{aligned}$$

Now, if $V_i = \{v_{i1}, v_{i2}, \dots, v_{ir}\}$ for $1 \leq i \leq s$, $V(C_{rs}) = V_1 \cup V_2 \cup \dots \cup V_s$, $V(J_{r,s}) = V(C_{rs}) \cup \{w\}$,

$N(w) = \{v_{11}, v_{21}, \dots, v_{s1}\}$, and $E(J_{r,s}) = \{\cup_{i=1}^s \{wv_{i1}\}\} \cup E(C_{rs})$, then

$$\begin{aligned}
 ESO(m^k J_{r,s}) &= 2 \binom{m}{2} km^{k-1} \sqrt{2} \sum_{u \in V(J_{r,s})} (d_{J_{r,s}}(u) + k(m-1))^2 \\
 &\quad + m^k \sum_{i=1}^s \{ (d_{J_{r,s}}(u) + d_{J_{r,s}}(v_{i1}) + k(m-1)) \\
 &\quad \quad \times \sqrt{(d_{J_{r,s}}(w) + k(m-1))^2 + (d_{J_{r,s}}(v_{i1}) + k(m-1))^2} \} \\
 &\quad + m^k \sum_{\substack{uv \in E(C_{rs}) \\ u, v \notin N(w)}} \{ (d_{J_{r,s}}(u) + d_{J_{r,s}}(v) + k(m-1)) \\
 &\quad \quad \times \sqrt{(d_{J_{r,s}}(u) + k(m-1))^2 + (d_{J_{r,s}}(v) + k(m-1))^2} \} \\
 &\quad + m^k \sum_{\substack{uv \in E(C_{rs}) \\ u \in N(w)}} \{ (d_{J_{r,s}}(u) + d_{J_{r,s}}(v) + k(m-1)) \\
 &\quad \quad \times \sqrt{(d_{J_{r,s}}(u) + k(m-1))^2 + (d_{J_{r,s}}(v) + k(m-1))^2} \}.
 \end{aligned}$$

Hence

$$\begin{aligned}
 ESO(m^k J_{r,s}) &= 2\sqrt{2} \binom{m}{2} km^{k-1} (s(r-1)(2 + (m-1)k)^2 + s(3 + (m-1)k)^2 + (s + (m-1)k)^2) \\
 &\quad + m^k s (s + 3 + 2(m-1)k) \sqrt{(s + (m-1)k)^2 + (3 + (m-1)k)^2} \\
 &\quad + 2m^k (r-2) s \sqrt{2} (2 + (m-1)k)^2 \\
 &\quad + 2m^k s (5 + 2(m-1)k) \sqrt{(3 + (m-1)k)^2 + (2 + k(m-1))^2}. \quad \square
 \end{aligned}$$

2.3 Barbell graphs

A m -Barbell graph, denoted as B_m , is defined as a simple graph created by linking two instances of a complete graph K_m through a connecting edge, commonly referred to as a bridge [17].

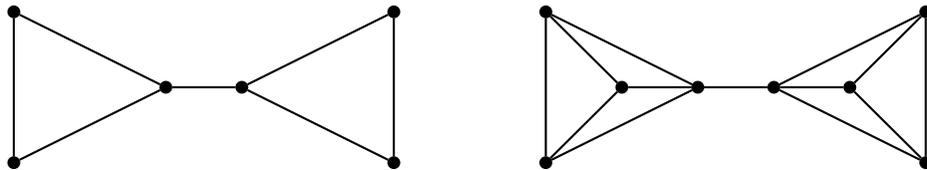


Figure 3. B_3, B_4

It is evident from the structure of graph B_m that it contains $2m - 2$ vertices, each having a degree of $m - 1$, along with 2 vertices that possess a degree of m .

Theorem 2.8. For any integer $m \geq 2$, we have

$$ESO(B_m) = 2\sqrt{2}(m - 2)(m - 1)^3 + 2(2m - 1)(m - 1)\sqrt{m^2 + (m - 1)^2} + 2\sqrt{2}m^2.$$

Proof. Let $V(B_m) = \{w_1, w_2, \dots, w_m, u_1, u_2, \dots, u_m\}$ such that $\{w_1, w_2, \dots, w_m\}$ and $\{u_1, u_2, \dots, u_m\}$ are vertices of complete graphs and edge w_1u_1 be bridge of B_m . Now, we calculate the elliptic Sombor index of B_m as follows

$$\begin{aligned} ESO(B_m) &= \sum_{\substack{w_i w_j \in E(B_m) \\ w_i, w_j \notin \{w_1\}}} (d_{B_m}(w_i) + d_{B_m}(w_j)) \sqrt{d_{B_m}^2(w_i) + d_{B_m}^2(w_j)} \\ &+ \sum_{w_i w_1 \in E(B_m)} (d_{B_m}(w_i) + d_{B_m}(w_1)) \sqrt{d_{B_m}^2(w_i) + d_{B_m}^2(w_1)} \\ &+ \sum_{\substack{u_i u_j \in E(B_m) \\ u_i, u_j \notin \{u_1\}}} (d_{B_m}(u_i) + d_{B_m}(u_j)) \sqrt{d_{B_m}^2(u_i) + d_{B_m}^2(u_j)} \\ &+ \sum_{u_i u_1 \in E(B_m)} (d_{B_m}(u_i) + d_{B_m}(u_1)) \sqrt{d_{B_m}^2(u_i) + d_{B_m}^2(u_1)} \\ &+ (d_{B_m}(u_1) + d_{B_m}(w_1)) \sqrt{d_{B_m}^2(u_1) + d_{B_m}^2(w_1)}. \end{aligned}$$

Since

$$d_{B_m}(w) = \begin{cases} m & \text{if } w \in \{u_1, w_1\}, \\ m - 1 & \text{if } w \notin \{u_1, w_1\}, \end{cases}$$

we obtain

$$ESO(B_m) = 2\sqrt{2}(m - 2)(m - 1)^3 + 2(2m - 1)(m - 1)\sqrt{2m^2 - 2m + 1} + 2\sqrt{2}m^2. \quad \square$$

The term $2\sqrt{2}m^2$ dominates $ESO(B_m)$, highlighting the critical role of the bridge edge. Unlike the Sombor index [10], where bridge contributions scale linearly, ESO’s quadratic term emphasizes how high-degree vertices (degree m) disproportionately influence the index. This aligns with ESO’s geometric interpretation as an elliptic radius, where large degrees amplify contributions nonlinearly.

2.4 Friendship graphs

An m -Friendship graph is defined as a simple graph formed by connecting m copies of the cycle graph C_3 at a shared vertex. This graph is represented as F_m [17].

Theorem 2.9. For any integer $m \geq 2$, we get

$$ESO(F_m) = 8m(m + 1)\sqrt{m^2 + 1} + 8m\sqrt{2}.$$

Proof. Let $V(C_3) = \{v_1, v_2, v_3\}$, $V(F_m) = \{v_1, v_2^1, v_3^1, v_2^2, v_3^2, \dots, v_2^m, v_3^m\}$, and

$$E(F_m) = \cup_{k=1}^m \{v_1 v_2^k, v_1 v_3^k, v_2^k v_3^k\}.$$

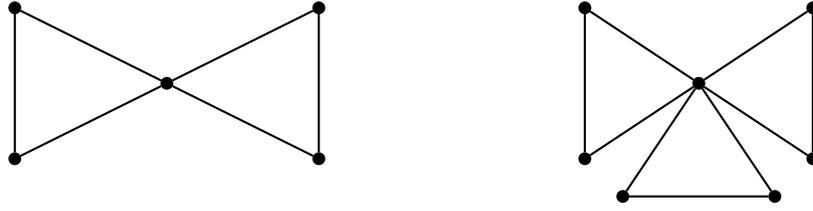


Figure 4. F_2, F_3

From the construction of graph F_m , we have

$$\begin{aligned}
 ESO(F_m) &= \sum_{v_1 v_2^k \in E(F_m)} \left(d_{F_m}(v_1) + d_{F_m}(v_2^k) \right) \sqrt{d_{F_m}^2(v_1) + d_{F_m}^2(v_2^k)} \\
 &+ \sum_{v_1 v_3^k \in E(F_m)} \left(d_{F_m}(v_1) + d_{F_m}(v_3^k) \right) \sqrt{d_{F_m}^2(v_1) + d_{F_m}^2(v_3^k)} \\
 &+ \sum_{v_2^k v_3^k \in E(F_m)} \left(d_{F_m}(v_2^k) + d_{F_m}(v_3^k) \right) \sqrt{d_{F_m}^2(v_2^k) + d_{F_m}^2(v_3^k)}.
 \end{aligned}$$

Since $d_{F_m}(v_1) = 2m, d_{F_m}(v_2^k) = d_{F_m}(v_3^k) = 2$, we find

$$ESO(F_m) = 8m(m + 1)\sqrt{m^2 + 1} + 8m\sqrt{2}. \quad \square$$

Conclusion

This paper provides the first comprehensive analysis of the elliptic Sombor index on mk-graphs, Jahangir graphs, Barbell graphs, and Friendship graphs. Our key contributions include:

1. Exact formulas and bounds for ESO on these graph families, revealing exponential sensitivity to recursion in mk-graphs and quadratic scaling in star-dominated structures.
2. Proof that ESO outperforms classical indices in detecting hierarchical complexity and critical edges, as demonstrated by comparative analysis.
3. Novel results on subdivided mk-graphs, with implications for modeling functionalized polymers.

Future work should explore ESO’s correlation with physicochemical properties in molecular graphs and its behavior under dynamic graph operations.

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All data produced or examined throughout this study are incorporated within this published article.

Conflicts of Interests

We affirm that we possess no commercial or associative interests that could potentially create a conflict of interest regarding the submitted work.

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