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Research Paper

Generalized stepwise irregular graphs: graph operations and construction of 3-SI graphs

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Abstract. *Generalized stepwise irregular* (GSI) graphs are graphs in which the degree difference between every pair of adjacent vertices is positive constant. Specifically, a graph G is called a k-stepwise irregular (k-SI) graph if $|d_G(u) - d_G(v)| = k$, for each edge $uv \in E(G)$. In this paper, we examine the behavior of GSI graphs under some graph operations such as sum, corona product, complement, subdivision, line graph, and vertex deletion. An infinite family of 3-SI graphs with a given cyclomatic number and distinct cycles are constructed. Further, a lower bound on the size of the unicyclic 3-SI graphs is proposed.

Keywords. GSI graph, 3-SI graph, graph operations, cyclomatic number.
Mathematics Subject Classification (2020): 05C07, 05C09, 05C76, 05C90.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph. The notations $n(G)$, $d_G(v)$, $\delta(G)$ and $\Delta(G)$ are used for order of G, degree of vertex v, minimum and maximum of degrees in G, respectively. Let $d_1, d_2, \ldots, d_k$ denote the distinct degrees of the vertices in a graph. The degree sequence of the graph is defined as $d_1^{n_1}, d_2^{n_2}, \ldots, d_k^{n_k}$, where n_i represents the number of vertices of degree d_i in the graph. Distance between two vertices u and v, $d_G(u,v)$ is the length of shortest path connecting them. Maximum distance from a vertex u is called eccentricity of u and denoted by

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$\varepsilon(u)$. The center of G is the set of vertices of minimum eccentricity, which is denoted by $C(G)$. The complete bipartite and star graphs, are denoted by $K_{m,n}$ and S_n respectively. The imbalance degree of an edge uv, $Im(uv)$ is defined as $Im(uv) = |d(u) - d(v)|$. A graph is called *regular* if all vertices have the same degree; otherwise, it is *irregular*. The study of graph irregularities has attracted growing interest in recent years owing to its theoretical importance and applications in network theory, mathematical chemistry, and other fields [1–4,7,16]. Various measures have been proposed to quantify irregularities in graphs. In [12], Gutman introduced the concept of *stepwise irregular (SI)* graphs, in which the degree difference between every pair of adjacent vertices is exactly one; that is, $Im(uv) = 1$ holds for each edge uv of the graph. This concept can be extended to *k-stepwise irregular (k-SI)* graphs, where k is a positive integer, and $Im(uv) = k$ holds for each edge uv of the graph [5]. The classical SI graphs correspond to the case $k = 1$, whereas larger values of k define a more complex and less studied family of graphs. Recent studies have explored various properties of SI and k-SI graphs including degree bounds and extremal configurations [8–10]. In [8], it is established that if is an SI graph, then its subdivision does not necessarily retain the SI property. In contrast, the present study explores specific graph constructions in which the GSI-property is preserved. Furthermore, [8] demonstrates that the removal of a vertex from an SI graph, i.e., $G - v$, results in a graph that is no longer SI. However, we show that for certain graphs such as star graphs, the *GSI* property remains intact. Reference [10] investigates the corona product of two 2-SI graphs and concludes that the resulting graph does not preserve the 2-SI property. In this study, we establish the necessary and sufficient conditions for the corona product of two graphs to exhibit the *GSI* property. In [5], the necessary and sufficient conditions for the Cartesian product and composition of two graphs to be $k-$SI have been established. In this paper, we present the necessary and sufficient conditions for preserving the *GSI* property under other graph operations such as sum, corona product, subdivision, complement, line graph, and vertex deletions. The remainder of this paper is organized as follows. In Section 2, we analyze the behavior of generalized stepwise irregular (GSI) graphs under several standard graph operations. In Section 3, we concentrate on the class of *3-stepwise irregular (3-SI)* graphs, an infinite family of 3-SI graphs with a given cyclomatic number and distinct cycles are constructed. Finally, a lower bound on the size of unicyclic 3-SI graph in terms of the girth of graph is posed. recall that the girth of a graph G, is the length of shortest cycle in G.

2 *GSI* graphs under graph operations

In this section, we analyze the preservation of the *GSI* property when graphs undergo classical operations. These operations include the sum, corona product, subdivision, complement, line graph, and vertex deletions. First, we recall the basic property of a k-SI graph, which is also stated in [5].

Lemma 2.1. *Every k-SI graph is bipartite.*

In [5], it was proven that the Cartesian product of the two graphs is *GSI* if and only if both are *GSI*. In this section, we examine the *GSI* property of other composite graphs. The *Sum* of two graphs G and H, denoted $G + H$, is a graph with

$$V(G + H) = V(G) \cup V(H),$$
$$E(G + H) = E(G) \cup E(H) \cup \{ uv \mid u \in V(G), v \in V(H) \}.$$

Theorem 2.2. *The sum of the two graphs G and H, $G + H$, is k-SI if and only if G and H are both edgeless, and $|n(G) - n(H)| = k$.*

Proof. If G or H contains any edge, since all vertices in G are connected to all vertices in H, any internal edge within G or H would induce a triangle, contradicting Lemma 2.1. Therefore, both G and H must be edgeless. It follows that $G + H$ forms a complete bipartite graph with two partitions of orders $n(G)$ and $n(H)$, whereas $|n(G) - n(H)| = k$ must hold for $G + H$ to be a k-SI graph. Conversely, if G and H are edgeless graphs such that $|n(G) - n(H)| = k$, then their sum $G + H$ is isomorphic to the complete bipartite graph $K_{n(G),n(H)}$, which is evidently a $k-$SI graph. □

The *Corona product* of two graphs G and H, $G \circ H$, is a graph constructed by taking a copy of G and $n(G)$ copies of H. Each vertex $v \in V(G)$ is connected to every vertex of the corresponding copy of H, which is denoted by H_v.

Theorem 2.3. *The corona product $G \circ H$ is a k-SI graph if and only if G and H are both edgeless graphs with $n(H) = k + 1$.*

Proof. If H has an edge uv, then in some copy H_x of H (attached to a vertex $x \in V(G)$), the vertices u, v and x form a triangle, contradicting Lemma 2.1. Therefore, H must be edgeless. In addition, if G contains an edge xy and $u \in V(H)$, we consider the edges xu in H_x and yu in H_y. Since the imbalance of the edges xu and yu must be equal, we have

$$Im(xu) = Im(yu),$$
$$d_{G \circ H}(x) - d_{G \circ H}(u) = d_{G \circ H}(y) - d_{G \circ H}(u).$$

This implies $d_{G \circ H}(x) = d_{G \circ H}(y)$ violates the k-SI condition. Therefore, G must be edgeless. Moreover, to satisfy the k-SI condition, each vertex in H must differ in degree k from its adjacent vertex in G, which requires $n(H) = k + 1$. Conversely, if G and H are edgeless graphs such that $n(H) = k + 1$, then the corona product $G \circ H$ is isomorphic to disjoint union $\cup_{v \in V(G)} S_{k+1}$. Since each component S_{k+1} is a $k-$SI graph it follows that $G \circ H$ is also a $k-$SI graph. □

The *subdivision graph* of G, denoted $S(G)$, is obtained by inserting a new vertex into each edge of G.

Theorem 2.4. *If G is a connected* GSI *graph, its subdivision graph $S(G)$ is* GSI *if and only if $G \cong K_{1,3}$.*

Proof. Let G be a k-SI graph and let v be a vertex of the minimum degree $\delta(G)$. Suppose that w is a vertex adjacent to v. Clearly, $d(w) = \delta(G) + k$. If $S(G)$ is a *GSI* graph and z is the inserted vertex on the edge vw, then

$$|d(v) - 2| = |d(w) - 2|,$$

and thus

$$|\delta(G) - 2| = |\delta(G) + k - 2|.$$

Consequently, $\delta(G) = 2 - \frac{k}{2}$. Hence, $\delta(G) = 1$, $k = 2$ and $\deg(w) = 3$. This means G is a 2-SI graph and $S(G)$ is a SI graph. Note that if w has any adjacent vertex of degree 5, then $S(G)$ cannot be a *GSI* graph. This observation implies that $G \cong K_{1,3}$. Conversely, if $G \cong K_{1,3}$ which is a $2-$SI graph. It can be verified that subdivision graph $S(G)$ is a SI graph. So both G and $S(G)$ retain the *GSI* property. □

The *complement graph* of G, denoted by $\overline{G}$, is a graph with the same vertex set as G and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *line graph* $L(G)$ of a graph G is defined such that each vertex of $L(G)$ represents an edge of G, and two vertices in $L(G)$ are adjacent if and only if their corresponding edges in G share a common endpoint. Recall that, for $e = uv \in E(G)$, the degree of e in $L(G)$ is given by

$$d_{L(G)}(e) = d(u) + d(v) - 2.$$

Theorem 2.5. *If G is a* GSI *graph, neither $\overline{G}$ nor $L(G)$ is a* GSI *graph.*

Proof. Let G be a k-SI graph. It is well known that each connected component of a graph has at least two vertices of the same degree. Since G is a k-SI graph, such vertices cannot be adjacent; however, they are adjacent in $\overline{G}$ with the same degree. This follows that $\overline{G}$ is not a *GSI* graph. Furthermore, if $L(G)$ is *GSI*, Lemma 2.1 implies that $\Delta(G) \leq 2$; that is, G is a path or a cycle. This result contradicts the fact that G is a *GSI* graph. □

Let G be a graph and $v \in V(G)$. The graph $G - v$ is obtained by removing v and all incident edges.

Theorem 2.6. *Let T be a k-SI tree and $v \in V(T)$. Then $T' = T - v$ is* GSI *if and only if either $T \cong S_{k+2}$ with $d(v) = 1$, or T is a tree with center $C(T) = \{v\}$ and v is connected to the centers of $2k + 1$ copies of S_{k+1}.*

Proof. Suppose X is the set of vertices of T with an even distance from v and $Y = V(T) \setminus X$. Since T' is a *GSI*, removing v alters the degree of all adjacent vertices of v. We claim that $\varepsilon(v) = 2$. Suppose, for contradiction, that there exists a vertex w such that the distance between v and w is 3. Let $P : v - x - y - w$ be the shortest path connecting v to w. Then we have $d_{T'}(x) = d_T(x) - 1$ but $d_{T'}(y) = d_T(y)$ and $d_{T'}(w) = d_T(w)$. This implies that $Im_{T'}(xy) \neq Im_T(xy) = Im_T(yw) = Im_{T'}(yw)$. This contradicts the assumption that T' is a *GSI* graph.

Hence, v is adjacent to all vertices in the partition Y. If v is a pendant vertex, then $|Y| = 1$ and this means $T \cong S_{k+2}$. If $d(v) \geq 2$, Since T is a tree and $\varepsilon(v) = 2$ then all vertices at a distance 2 from v are pendant vertices. Hence, all the vertices in Y are of degree $k+1$ since T is a $k-$SI tree. Consequently, v is a central vertex of T with $d(v) = 2k+1$ which is connected to the centers of $2k+1$ copies of S_{k+1}. Let T_k denotes such tree. Conversely, suppose that v is a pendant vertex of the graph S_{k+2}. Then, by removing v, we obtain $S_{k+2} - v \cong S_{k+1}$. Similarly, if w denotes the central vertex of the tree T_k, then removing w yields $T_k - w \cong \bigcup S_{k+1}$, where the resulting graph is a disjoint union of S_{k+1} components. In both scenarios, the resulting graphs preserve the *GSI* property. □

Figures 1 and 2 illustrate the two possible structures of the tree T described in Theorem 1. In Figure 1, the tree T is isomorphic to S_6 with the pendant vertex v. In Figure 2, the vertex v is the unique center connected to the centers of 5 copies of S_3. These figures visually demonstrate the two configurations characterized in the Theorem 2.6.

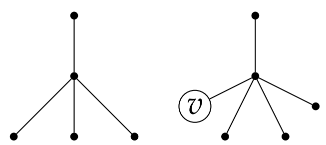

Figure 1. The tree $T \cong S_6$ where the vertex v is pendant.

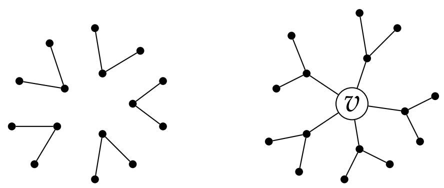

Figure 2. The tree T_2 with central vertex v connected to the centers of 5 copies of S_3.

Table 1 presents graph operations along with the necessary and sufficient conditions for preserving the *GSI* property on graphs

3 Construction of 3-SI graphs with given cyclomatic number

In this section, we focus on constructing 3-SI graphs with a specified cyclomatic number, such that the cycles are distinct. By using single-cycle graphs U_m and applying appropriate transformations, families of 3-SI graphs with desired properties are constructed.

Theorem 3.1. *For every even integer $m \geq 4$, there is a 3-SI graph of the girth m.*

Proof. Let U_m be a unicyclic graph with the cycle $C : v_1, v_2, \cdots v_m$, constructed as follows: Each vertex of odd index is connected to two pendant vertices and each vertex of even index is connected to the center of five star graphs S_4. Hence, U_m is a unicyclic 3-SI graph with the

Table 1. Graph operations that preserve the *GSI* property

Graph Operation	**The necessary and sufficient conditions on** G **and** H
Corona Product: $G \circ H$	G and H are both edgeless graphs with $n(H) = k+1$
Sum operation: $G + H$	G and H are both edgeless graphs with $\lvert n(G) - n(H) \rvert = k$
Subdivision: $S(G)$	$G \cong K_{1,3}$
Removing vertex: $T - v$	either $T \cong S_{k+2}$ with $d(v) = 1$ or $T \cong T_k$ while v is the unique central vertex
Cartesian product: $G \square H$	G and H are both k-SI graphs (see [5])
Composition: $G[H]$	either $G \cong \overline{K}_n$ and H is a k-SI graph, or $H \cong \overline{K}_t$ and G is a k'-SI graph where $k' = \frac{k}{t}$ (see [5])

girth m. Note that the order of U_m $12m$ and its degree sequence is given by $1^{\frac{17}{2}m}, 4^{3m}, 7^{\frac{m}{2}}$. The smallest unicyclic graph 3-SI graph is illustrated in Figure3. □

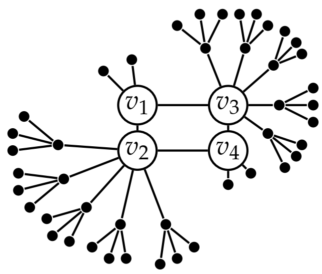

Figure 3. The graph U_4 with order 48, size 96 and degree sequence $1^{34}, 4^{12}, 7^{2}$.

Next, we construct a 3-SI graph of a given cyclomatic number t in which the cycles are distinct. The cyclomatic number of a graph G of order $n(G)$ and size $e(G)$ are formulated as $e(G) - n(G) + 1$.

Theorem 3.2. *For any sequence of even integers $m_1, m_2, \ldots, m_t$, where $m_i \geq 4$, there is a 3-SI graph with cyclomatic number t containing t distinct cycles of length $m_1, m_2, \ldots, m_t$.*

Proof. Let v_1 and v_2 be two vertices of degree 4 in the graphs U_{m_1} and U_{m_2}, respectively. Construct the graph $G(m_1, m_2)$ by first removing the pendant vertices adjacent to v_1 and v_2, and then identifying these two vertices. Denote the resulting vertex by $v_{1,2}$. Analogously, the graph $G(m_1, m_2, m_3)$ is constructed by applying the same transformation to the graphs $G(m_1, m_2)$ and U_{m_3}. Continuing this process inductively, we obtain a graph $G(m_1, m_2, \ldots, m_t)$, which is a 3-SI graph with cyclomatic number t that contains distinct cycles of length $m_1, m_2, \ldots, m_t$. See Figure 4 for the structure of the graphs $G(m_1, \cdots, m_t)$. Applying a similar transformation on the graphs $U_{m_1}, \cdots U_{m_t}$, illustrated in Figure 5, another family of 3-SI graph, $H(m_1, \cdots m_t)$ is constructed such that its cycles are vertex disjoint. □

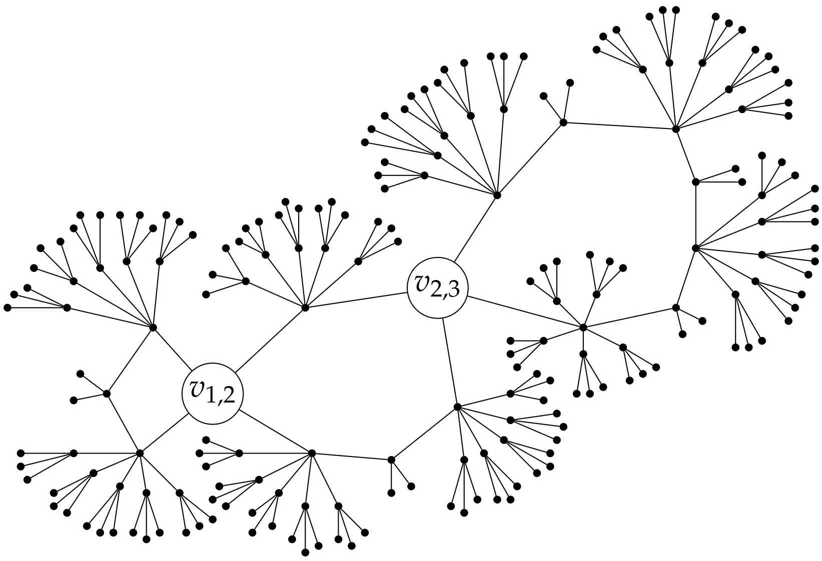

Figure 4. Structure of the graph $G(4,6,8)$ with order 416, size 208 and degree sequence $1^{145}, 4^{52}, 7^{9}$.

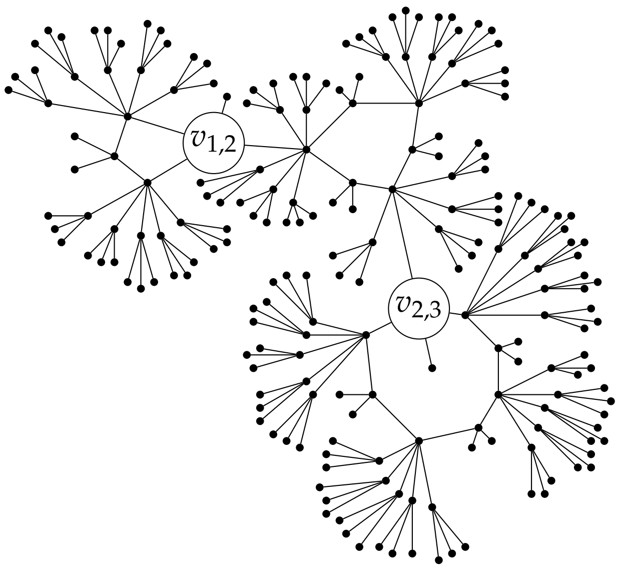

Figure 5. Structure of the graph $H(4,6,8)$ with order 416, size 208 and degree sequence $1^{145}, 4^{52}, 7^{9}$.

Corollary 3.3. *For every sequence of even integers $m_1, m_2, \cdots m_t$, where $m_i \geq 4$, there is an infinite family of $3-SI$ graphs of the given girth t and distinct cycles.*

Proof. Assume that T be a tree with a central vertex v of degree 6 connecting to the centers of 6 copies of star graphs S_4. Let $\Gamma \in \{G(m_1, m_2, \cdots m_t), H(m_1, m_2, \cdots m_t)\}$. Consider the following transformation on Γ as: coinciding the center of T on a pendant vertex connecting to a vertex of degree 4 in Γ. The new constructed graph is a 3-SI graph of the cyclomatic number t. Thus, we can obtain a family of graphs by applying the transformation to the resulting graph. We show this graph by $T.\Gamma$. The structure of $T.H(4,6,8)$ is illustrated in Figure6. □

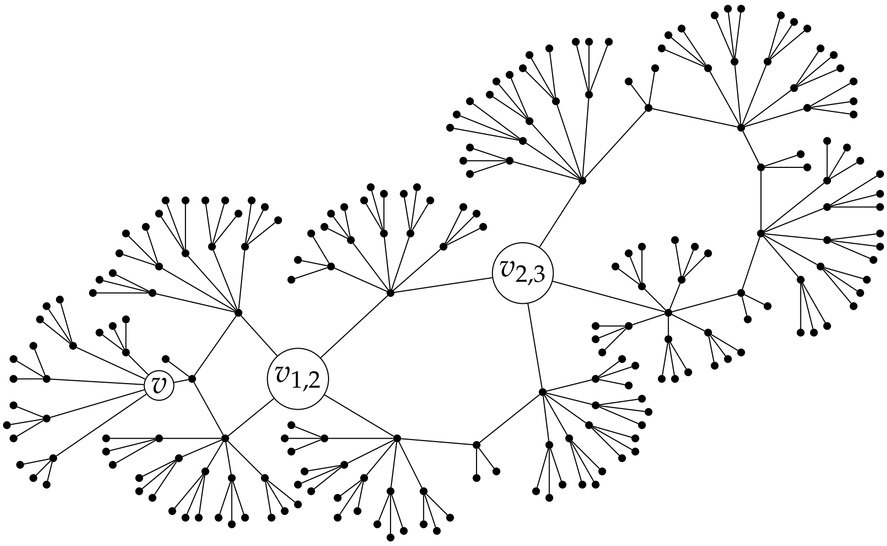

Figure 6. Structure of the graph $T.H(4,6,8)$ with order 464, size 232 and degree sequence: $1^{162}, 4^{58}, 7^{10}$.

Theorem 3.4. *Let G be a unicyclic 3-SI graph with a girth m. Then*

$$|E(G)| \geq 12m.$$

with equality holds if and only if $G \cong U_m$

Proof. Without loss of generality, we assume that G is a unicyclic 3-SI graph of minimum size. Let $C : v_1v_2 \ldots v_mv_1$ be the unique cycle of G. Since G is a 3-SI graph, for each vertex v, $d(v) \equiv \delta(G) \pmod 3$. On the other hand G is unicyclic graph and $G \ncong C_m$ which implies $\delta(G) = 1$. Thus $d(v) \equiv 1 \pmod 3$ for all $v \in V(G)$. Given that G has minimum size, the vertices on the cycle C must have degrees either 4 or 7. Let V_4 and V_7 denote the subsets of vertices on C with degrees 4 and 7, respectively. Each vertex of degree 4 is adjacent to exactly two vertices of degree 1 located outside the cycle C, while each vertex of degree 7 is adjacent to precisely five vertices of degree 4 outside C. Consequently, the vertices external to C must have degrees either 1 or 4. Given the structural characteristics of U_m, it follows that G is isomorphic to U_m. A direct computation yields that $|E(U_m)| = 12m$. Therefore, $|E(G)| \geq 12m$. □

4 Irregularity of *GSI* graphs

Graph irregularity measures aim to quantify the deviation of a graph from regularity, typically by evaluating differences in vertex degrees. A notable class of graphs in this study are *GSI* graphs in which imbalance of every edge is constant. Such graphs exhibit a uniform pattern of irregularity, which contrasts with the randomness or heterogeneity found in highly irregular graphs. For instance, the Albertson irregularity index-defined as the sum of imbalance degree across all edges-would yield a predictable value in these graphs, proportional to the number of edges and the fixed degree difference. Similarly, entropy-based measures, which assess the distribution of vertex degrees, may reflect moderate irregularity depending on the degree sequence. These graphs serve as structured examples that bridge the gap between regular and irregular graphs, offering insight into how controlled degree variation influences global irregularity metrics. Investigating various graph irregularity measures-such as irregularity indices, total irregularity, and sigma irregularity-in the context of graphs with a constant degree difference across all edges (*GSI* graphs) presents an intriguing direction for structural analysis and may reveal novel insights into their combinatorial properties.

5 Conclusion

The main goal of the paper is to examine the behavior of *GSI* graphs under graph operations and constructing several families of 3-SI graphs with a given cyclomatic number. Naturally, it is interesting to investigate two problems: First, which numbers can be the order of k-SI graphs and the second, to construct k-SI graphs with a given parameter such as cyclomatic number, diameter, etc.

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Data is contained within the article.

Conflicts of Interests

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