Journal of Discrete Mathematics and Its Applications 10 (4) (2025) 305–319

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Shahid Rajaee Teacher Training University

Research Paper

On a class of skew Dyck paths

Yvonne Wakuthii Kariuki[1], **Isaac Owino Okoth**[2,*]

[1]Department of Mathematics, Kibabii University, Bungoma, Kenya

[2]Department of Pure and Applied Mathematics, Maseno University, Maseno, Kenya

Academic Editor: Amir Hossein Ghodrati

Abstract. This paper introduces the set of skew 2-Dyck paths - Dyck-like lattice paths that allow unit up-steps, down-steps of length 2, and left-steps of length 2, provided the paths remain non-intersecting. An explicit enumeration formula for these paths is derived using the symbolic method and the Lagrange Inversion Formula. In addition, the paper defines three related combinatorial structures: 2-labeled box paths, 3-leaf-labeled plane trees, and 2-edge-labeled plane trees. Bijections are constructed between the set of skew 2-Dyck paths and the set of each of these three structures, thereby demonstrating their enumerative equivalence.

Keywords. box, bijection, binary, log-convex, plane tree.
Mathematics Subject Classification (2020): $05A15$, $05A19$.

1 Introduction

Skew Dyck paths were introduced by Deutsch, Munarini and Rinaldi [2] in 2010. These are Dyck paths with extra left steps $(-1,-1)$ that do not cross up steps ($(1,1)$ steps) and down steps ($(1,-1)$ steps). In the same paper, the authors showed that the number of skew-Dyck paths of semi-length n is given by

$$\sum_{i=0}^{n-1} c_i \tag{1}$$

*Corresponding author (Email address:ookoth@maseno.ac.ke).
Received 21 June 2025; Revised 10 September 2025; Accepted 27 November 2025
First Publish Date: 01 December 2025

where c_i is the i^{th} Catalan number,

$$c_i = \binom{2i}{i}\frac{1}{i+1}.$$

They showed that Equation (1) also enumerates marked plane trees, hex trees, and Motzkin paths in which there are no horizontal steps on the x-axis, while the remaining horizontal steps come in two colours. Bijections between the set of skew-Dyck paths and other combinatorial structures were later constructed by Prodinger in [7]. The same authors of [2] further derived expressions for the area enclosed between these paths and the x-axis in [3]. In 2019, Selkirk studied a generalization of k-Dyck paths in her thesis [12], defining them as Dyck paths with steps $(1,1)$, and $(1,-k)$ that start at the origin, remain in the first quadrant and end on the x-axis. She further obtained enumerative formula for k-Dyck paths based on path length. Interest in the enumeration of skew-Dyck paths and their generalizations has remained strong, as evidenced by subsequent publications [1,4,6,8–10].

We define a skew k-Dyck path as a lattice path that starts at the origin $(0,0)$ and ends on the x-axis, consisting of up-steps $(1,1)$, down-steps $(k,-k)$, and left-steps $(-k,-k)$. We define the semi-length of a skew k-Dyck path as the total number of down-steps and left-steps in the path. Two examples of skew 2-Dyck paths of semi-length 3 are shown in Figure 1.

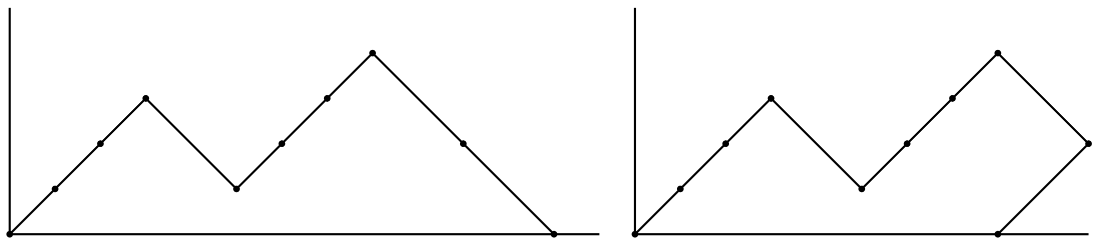


Figure 1. Skew 2-Dyck paths of semi-length 3.

In this paper, we enumerate the set of skew 2-Dyck paths in Section 2. In Section 3, we construct bijections between the set of skew 2-Dyck paths and three distinct combinatorial structures introduced therein. Section 4 presents two additional combinatorial structures that are enumerated by the same sequence as the skew 2-Dyck paths. Finally, Section 5 concludes the paper by summarizing the main results and proposing further questions related to enumeration.

2 Enumeration of skew 2-Dyck paths

Let $S(z)$ be the generating function for skew 2-Dyck paths, where z marks a down-step or a left-step. The paths are decomposed according to the first down-step or left-step that touches the x-axis. This is depicted in Figure 2.

Based on the decompositions, the generating function $S(z) = S$ satisfies the functional

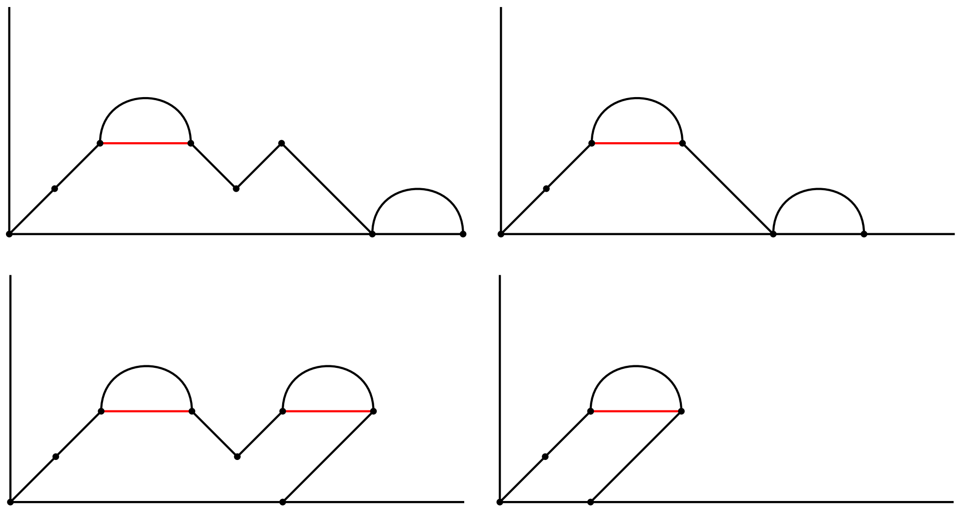


Figure 2. Decomposition of skew 2-Dyck paths

equation

$$S(z) = 3zS(z)(S(z)+1) + z(S(z)+1) = z\left(3S^2 + 4S + 1\right). \tag{2}$$

We apply the Lagrange Inversion Formula [14,15] to extract the coefficient of z^n in the generating function S.

$$\begin{aligned}
[z^n]\, S(z) &= \frac{1}{n}\left[s^{n-1}\right]\left(3s^2 + 4s + 1\right)^n \\
&= \frac{1}{n}\left[s^{n-1}\right](3s+1)^n(s+1)^n \\
&= \frac{1}{n}\left[s^{n-1}\right]\sum_{i=0}^{n}\sum_{j=0}^{n}\binom{n}{i}\binom{n}{j}3^i s^{i+j} \\
&= \frac{1}{n}\sum_{i=0}^{n-1}\binom{n}{i}\binom{n}{n-i-1}3^i. \qquad (3)
\end{aligned}$$

Since the Narayana or Runyon number is given by

$$N(n,i) = \frac{1}{n}\binom{n}{i}\binom{n}{i+1} \tag{4}$$

then Equation (3) can be written in terms of the Narayana numbers as

$$[z^n]\, S(z) = \sum_{i=0}^{n-1} 3^i N(n,i).$$

We now advertise this result as a theorem.

Theorem 2.1. *The number, s_n, of skew 2-Dyck paths of semi-length n is given by*

$$s_n = \sum_{i=0}^{n-1} 3^i N(n,i) \tag{5}$$

where $N(n,i)$ is the Narayana number given by Equation (4).

By writing Equation (2) as $S = z\left((1+S)^2 + 2S(1+S)\right), S = z\left(3S^2 + (4S+1)\right)$ and $S = z\left((1+2S)^2 - S^2\right)$ and extracting the coefficient of z^n in S, we realize that Equation (5) can also be written as

$$s_n = \frac{1}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{2n-i}{n+1}2^i,$$
$$s_n = \frac{1}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{n-i}{i+1}3^i4^{n-2i-1}$$

and

$$s_n = \frac{1}{n}\sum_{i=0}^{n}\binom{n}{i}\binom{2(n-i)}{n+1}(-1)^i2^{n-2i-1}$$

respectively.

Theorem 2.2. *The sequence of numbers s_n, for $n \geq 3$, satisfies the recurrence relation*

$$(n+1)s_n = (8n-4)s_{n-1} - (4n-8)s_{n-2}, \tag{6}$$

with initial conditions $s_1 = 1$ and $s_2 = 4$.

Proof. Since there is only one skew 2-Dyck path of semi-length 1 with one down-step then $s_1 = 1$. There are 3 skew 2-Dyck paths of semi-length 2 with two downs and 1 skew 2-Dyck path of semi-length 2 with one down-step and one left-step. So $s_2 = 4$. Now, from (2), we have the quadratic equation

$$3zS(z)^2 + (4z-1)S(z) + z = 0$$

which simplifies to

$$S(z) = \frac{(1-4z) - \sqrt{4z^2 - 8z + 1}}{6z} \tag{7}$$

by means of quadratic formula. The first few terms of the sequence are 1,4,19,100,562,..., which is sequence A007564 in [13]. From (7), we have

$$1 - 6zS(z) - 4z = \sqrt{4z^2 - 8z + 1}. \tag{8}$$

Differentiating Equation (8) with respect to z, and simplifying further gives

$$-6zS'(z) - 6S(z) - 4 = \frac{8z-8}{2\sqrt{4z^2-8z+1}}.$$

So,

$$\left(6zS'(z) + 6S(z) + 4\right)\sqrt{4z^2 - 8z + 1} = 4 - 4z.$$

Multiplying through by $\sqrt{4z^2-8z+1}$ results in

$$\left(6zS'(z)+6S(z)+4\right)\left(4z^2-8z+1\right)=(4-4z)\sqrt{4z^2-8z+1}. \tag{9}$$

Substituting (8) into (9), results in

$$z\left(4z^2-8z+1\right)S'(z)+(1-4z)S(z)-2z=0. \tag{10}$$

To obtain the recurrence equation, let $S(z)=\sum_{n=0}^{\infty}s_nz^n$ so that $S'(z)=\sum_{n=1}^{\infty}ns_nz^{n-1}$. We substitute $S(z)$ and $S'(z)$ into (10) to obtain

$$\left(4z^2-8z+1\right)\sum_{n=1}^{\infty}ns_nz^n+(1-4z)\sum_{n=1}^{\infty}s_nz^n-2z=0$$

which is the same as

$$\sum_{n=1}^{\infty}4ns_nz^{n+2}-\sum_{n=1}^{\infty}(8n+4)s_nz^{n+1}+\sum_{n=1}^{\infty}(n+1)s_nz^n-2z=0.$$

This results in the recurrence relation

$$(n+1)s_n=(8n-4)s_{n-1}-4(n-2)s_{n-2}$$

for the number of skew 2-Dyck paths of semi-length $n\geq 3$. □

Definition 2.3 ([5]). *A sequence a_k is log-convex if and only if the sequence $\frac{a_{k+1}}{a_k}$ is increasing.*

For example, $1,4,19,\ldots$ is log-convex since $\frac{4}{1},\frac{19}{4}\ldots$ is increasing. We shall use the following lemma to show that the sequence of s_n is log-convex.

Lemma 2.4 ([5, Theorem 3.10]). *Let $z_n, n\geq 0$, be a sequence of positive numbers and satisfies the three-term recurrence*

$$(\alpha_1n+\alpha_0)z_{n+1}=(\beta_1n+\beta_0)z_n-(\gamma_1n+\gamma_0)z_{n-1} \tag{11}$$

for $n\geq 1$, where $\alpha_1+\alpha_0,\beta_1+\beta_0,\gamma_1+\gamma_0$ are positive for $n\geq 1$. Suppose that z_0,z_1,z_2 is log-convex, then the sequence $z_n(n\geq 0)$ is log-convex if one of the following conditions holds:

(i) $B,C\geq 0$,

(ii) $B<0, C>0, AC\geq B^2$ *and* $z_0B+z_1C\geq 0$ *or*

(iii) $B>0, C<0, AC\leq B^2$ *and* $z_0B+z_1C\geq 0$,

where

$$A=\begin{vmatrix}\beta_0 & \beta_1\\ \gamma_0 & \gamma_1\end{vmatrix}, B=\begin{vmatrix}\gamma_0 & \gamma_1\\ \alpha_0 & \alpha_1\end{vmatrix} \text{ and } C=\begin{vmatrix}\alpha_0 & \alpha_1\\ \beta_0 & \beta_1\end{vmatrix}.$$

Theorem 2.5. *The sequence of numbers s_n is log-convex, $\frac{s_{n+1}}{s_n}$ is bounded above by 8 and* $\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = 4 + 2\sqrt{3}$.

Proof. Rewrite the recurrence relation (6) as

$$(n+2)s_{n+1} = (8n+4)s_n - (4n-4)s_{n-1}$$

for $n \geq 2$. Now, comparing this equation with Equation (11), results in $z_0 = 1, z_1 = 4, \alpha_0 = 2$, $\alpha_1 = 1\beta_0 = 4, \beta_1 = 8, \gamma_0 = -4$ and $\gamma_1 = 4$ with $A = 48, B = -12$ and $C = 12$. Since $B < 0, C > 12, AC = 576 > 144 = B^2$ and $1(-12) + 4(12) = 36 > 0$, then by option (ii) in Lemma 2.4, s_n is log-convex.

We now find the bounds of s_n. Rewrite the recurrence relation,

$$(n+1)s_n = (8n-4)s_{n-1} - (4n-8)s_{n-2},$$

as

$$(n+2)s_{n+1} - (8n+4)s_n + 4(n-1)s_{n-1} = 0.$$

Since $4(n-1)s_{n-1}$ is non-negative, it follows that

$$(n+2)s_{n+1} - (8n+4)s_n \leq 0.$$

So,

$$(n+2)s_{n+1} \leq (8n+4)s_n.$$

This means that

$$\frac{s_{n+1}}{s_n} \leq \frac{8n+4}{n+2} \leq \frac{8n}{n} = 8.$$

Finally, we strive to find the limit of $\frac{s_{n+1}}{s_n}$ as n tends to infinity.

According to [16, Theorems 9 and 10], a three-term recursive relation,

$$(n+2)s_{n+1} = (2dn+d)s_n + \left(4c - d^2\right)(n-1)s_{n-1} \tag{12}$$

has the limit,

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = d + 2\sqrt{c} \tag{13}$$

The recurrence under investigation is

$$(n+2)s_{n+1} = (8n+4)s_n - 4(n-1)s_{n-1}.$$

So, comparing this recurrence relation with Equation (12), we have $2dn + d = 8n + 4$ and $4c - d^2 = -4$. This implies that $d = 4$ and $c = 3$. Thus, by (13),

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = d + 2\sqrt{c} = 4 + 2\sqrt{3}$$

as desired. □

3 Bijections

In this section, we construct bijections between the set of skew 2-Dyck paths and the sets of three other combinatorial structures.

3.1 2-Labeled box paths

We recall the definition of a subset of skew Dyck paths that was recently introduced by Zhang and Zhuang in [17].

Definition 3.1. *[17] A* k-box path *of size* n *is a skew Dyck path of semi-length* $(k+2)n-1$ *comprised of* n UD^kL *factors where* U *represents a* $(1,1)$ *step (up-step),* D *is a* $(1,-1)$ *step (down-step) and* L *is a* $(-1,-1)$ *step (left-step).*

Definition 3.2. *[17] A* box *is a factor in a skew Dyck path that consists of three consecutive steps* UDL*. These* UDL *factors protrude from the path in a box-like shape hence the motivation behind their name.*

We introduce a new class of box paths in which the factors are of the form $UDLD$. We shall call a path of the form $DUDLD$ a *box*.

Definition 3.3. *A* 2-labeled box path *is a 1-box path whose boxes are labeled 1 or 2 except those that touch the x-axis with another box path on its right which receives only label 1.*

Figure 3 depicts a 2-labeled box path of size 2. We establish a bijection between the set of

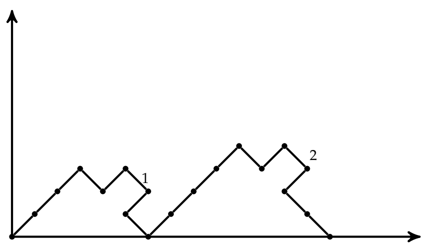


Figure 3. A 2-box labeled path with two factors.

2-labeled box paths and skew 2-Dyck paths in the following theorem.

Theorem 3.4. *There exists a bijection between the set of skew 2-Dyck paths of semi-length* n *and the set of 2-labeled box paths with* n *factors.*

Proof. Given a skew 2-Dyck path, we obtain the corresponding 2-labeled box path by performing the following transformation. For every $(-2,-2)$ step (left-step) or $(2,-2)$ step (down-step) encountered, delete the step and draw a unit up-step, followed by a unit down-step, then a unit left-step and finally a unit down-step, i.e., draw a factor $UDLD$. Label any box corresponding to a down-step (respectively, left-step) as 1 (respectively, 2). See Figure 4 for an example.

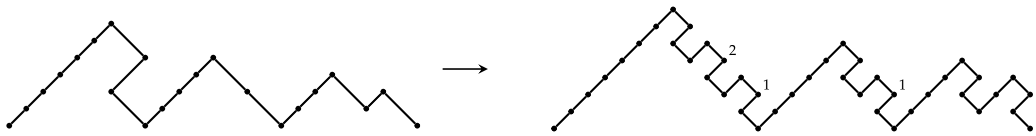


Figure 4. A procedure of obtaining a 2-labeled box path on 21 up-steps from a skew 2-Dyck path of semi-length 7.

We obtain the reverse procedure: Consider a 2-labeled box path with n factors, that is, with $3n$ unit up-steps. For each factor $UDLD$, replace it with a down-step of length 2. If the box was labeled 2, then label the down-step as 2. Now, replace each down-step labeled 2 with a left-step of length 2. This procedure is depicted in Figure 5

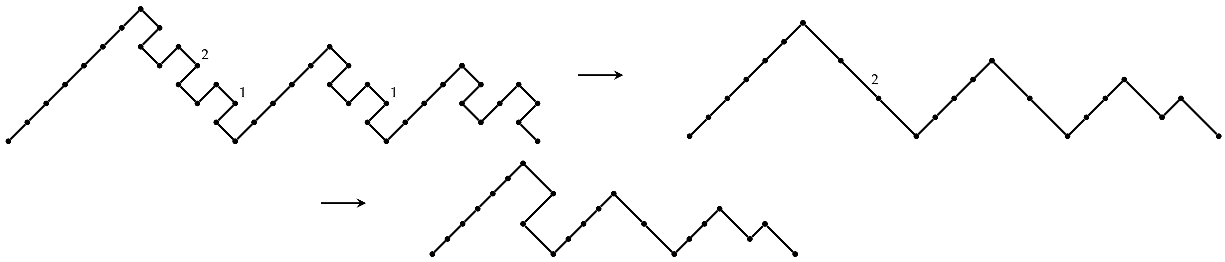


Figure 5. A procedure of obtaining a skew 2-Dyck path of semi-length 7 from a 2-labeled box path with 21 unit up-steps.

□

3.2 3-Leaf-labeled plane trees

We introduce a new class of plane trees with the *length* of a path defined as the number of edges on the path.

Definition 3.5. *A* 3-leaf-labeled plane tree *is a plane tree with all leaves at lengths multiples of* 3 *from the root and a leaf at length 1 from its immediate branching points is labeled as follows: If the rightmost sub-tree rooted at its root is a leaf or is empty, then that leaf is labeled either 1 or 2 and if that sub-tree has at least three vertices, then the leaf is strictly labeled 1.*

An example of a 3-leaf-labeled plane tree is shown in Figure 6.

Theorem 3.6. *There exists a bijection between the set of 2-labeled box paths with n factors and the set of 3-leaf-labeled plane trees with* $3n + 1$ *vertices.*

Proof. Consider a 2-labeled box path with n up-steps. Draw a vertex which is the root of the 3-leaf-labeled plane tree under construction. We traverse the path starting at the origin. For each up-step encountered, draw a new vertex. If a down-step is encountered, move up the tree along the right hand side towards the root. If the down-step was labeled 1 (respectively, 2), then label the vertex encountered as one moves up the tree as 1 (respectively, 2). Since

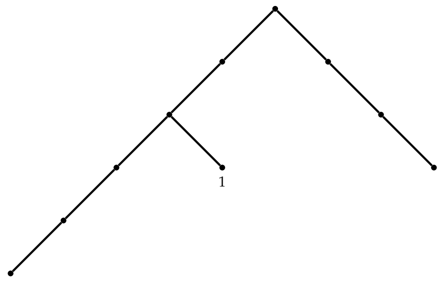


Figure 6. A 3-leaf-labeled plane tree on 10 vertices.

a factor $(UDLD)$ corresponds to three down-steps, we have that each branching point is a multiple of 3 from the last vertex drawn. For each up-step traversed, there is a corresponding vertex in the constructed tree. In addition, an initial vertex (root) was introduced. So, the constructed tree has $3n+1$ vertices and is a 3-leaf-labeled plane tree.

Conversely, given a 3-leaf-labeled plane tree, traverse the tree in preorder and build a 2-labeled box path as follows. For each vertex encountered as one moves away from the root, draw an up-step. Now, for three consecutive vertices encountered as one moves towards the root, draw a down-step, a left-step and a down-step in this order. If a leaf labeled 1 (respectively, 2) is encountered, label the corresponding down-step as 1 (respectively, 2). These are the labels of the boxes. The Dyck path obtained is a 2-labeled box path. The labeling of the boxes agrees with the description in the definition of a 2-labeled box path. An illustration of this bijection is shown in figure 7.

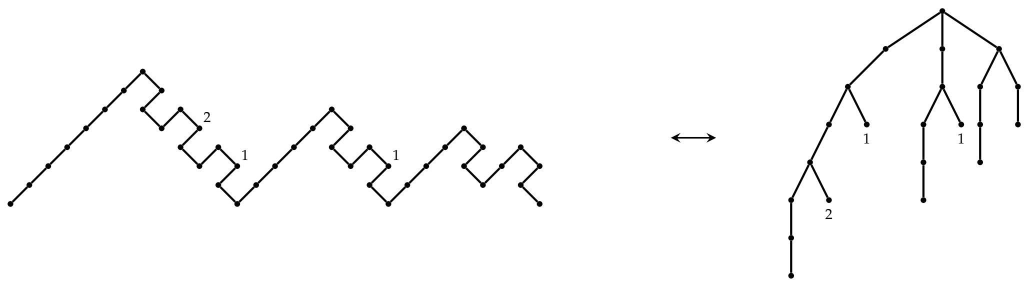


Figure 7. A 2-labeled box path with 7 factors and its corresponding 3-leaf-labeled plane tree on 22 vertices.

□

By Theorems 3.4 and 3.6, we get the following corollary.

Corollary 3.7. *There is a bijection between the set of skew 2-Dyck paths of semi-length n and the set of 3-leaf-labeled plane trees with $3n+1$ vertices.*

3.3 2-Edge-labeled plane trees

We introduce 2-edge-labeled plane trees defined as follows:

Definition 3.8. *A* 2-edge-labeled plane tree *is a plane tree whose rightmost sub-tree is of even length from the root to the leaf while all other sub-trees are of even length from their branching point such that any two consecutive edges of the rightmost sub-tree are labeled either 1 or 2 and any of other sub-trees are labeled 1, excluding the pair farthest from the root (respectively, branching point).*

Consider an example of a 2-edge-labeled plane tree in Figure 8.

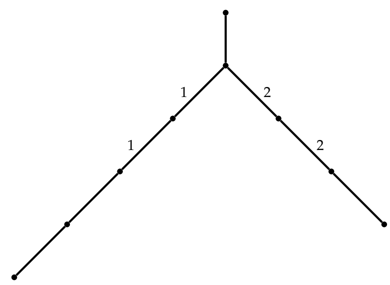


Figure 8. A 2-edge labeled plane tree on 9 vertices.

Theorem 3.9. *There exists a bijection between the set of skew 2-Dyck paths of semi-length n and the set of 2-edge-labeled plane trees with* $2n + 1$ *vertices.*

Proof. Given a skew 2-Dyck path, we obtain its corresponding 2-edge-labeled plane tree by performing the following transformation. First, convert the down-steps (have length 2) and left-steps (have length 2) of the path into pairs of unit down-steps and unit left-steps respectively. Starting with an initial vertex which will be the root of the tree, we build the tree as follows: For every up-step encountered, draw a vertex. For every two consecutive down-steps encountered, move up two steps towards the root in the constructed tree and label the two edges encountered as 1 (do not label the first two). For every two consecutive left-steps encountered, move up two steps towards the root in the constructed tree and label the two consecutive edges as 2. We obtain a 2-edge-labeled plane tree with $2n + 1$ vertices since each up-step ($2n$ in total) corresponds to a non-root vertex in the tree.

For the reverse procedure, consider a 2-edge-labeled plane tree. We traverse the tree in preorder to obtain the corresponding skew 2-Dyck path as follows: Moving away from the root, for any vertex encountered, draw an up-step. Moreover, moving toward the root for any unlabeled edges or edges labeled 1 encountered, draw a down-step and for any edge labeled 2 encountered, draw a left-step. From the skew path obtained, transform it into a skew 2-Dyck path by combining any pair of consecutive down-steps or left-steps into a new step of length 2. The resulting structure is a skew 2-Dyck path of semi-length n.

In Table 1, we get all the four skew 2-Dyck paths of semi-length 2 listed against their corresponding 2-edge-labeled plane trees with 5 vertices.

□

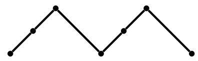
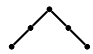
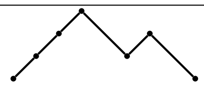
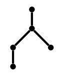
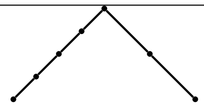
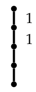
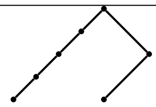
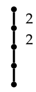
Skew 2-Dyck paths	2-edge-labeled plane trees
	
	
	
	

Table 1. The four skew 2-Dyck paths of semi-length 2 and their corresponding 2-edge labelled plane trees on 5 vertices.

4 Two combinatorial structures counted by the sequence

The sequence $1, 4, 19, 100, \ldots$ appears as entry A007564 in the Online Encyclopedia of Integer Sequences (OEIS) [13]. The following combinatorial structures are listed therein as being counted by this sequence:

(i) The number of Schröder paths of semi-length n in which there are no $(2,0)$-steps at level 0, and where such steps at higher levels come in two colours;

(ii) The number of Schröder paths of semi-length $n-1$ in which the $(2,0)$-steps at level 0 come in three colours, and those at higher levels come in two colours;

(iii) The number of (left) planted binary trees with n edges in which each vertex has a designate favourite neighbour.

In addition to skew 2-Dyck paths and the three combinatorial structures discussed in Section 3, we now present two more structures that are enumerated by this sequence.

4.1 Partially labeled plane trees with outdegree at most 2

Consider plane trees in which internal vertices have outdegree (i.e., number of children) at most 2. Moreover, let each vertex of outdegree 1 (respectively, 2) be labeled with an integer from the set $\{1,2,3,4\}$ (respectively, $\{1,2,3\}$). We refer to these as *partially labeled plane trees with outdegree at most 2*. Figure 9 displays all 19 such plane trees with 3 vertices.

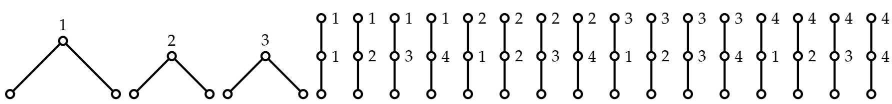


Figure 9. Partially labeled plane trees with 3 vertices.

Let $\mathcal{P}$ denote the family of these trees. Let $P(z)$ be the generating function for trees in $\mathcal{P}$, where z marks each vertex. Symbolically, we obtain the functional equation:

$$P(z) = z + 4zP(z) + 3zP(z)^2. \tag{14}$$

Equation (14) is also the generating function for skew 2-Dyck paths, where z marks down-steps and left-steps. Hence, the number of such plane trees with n vertices coincides with the number of skew 2-Dyck paths of semi-length n.

4.2 Partially labeled binary trees

Next, consider binary trees in which non-root vertices are labeled according to the following rules:

(i) A right child is labeled in three ways (i.e., 1, 2, or 3) if it has a left sibling, and in two ways (i.e., 1 or 2) if it does not.

(ii) A left child is labeled in two ways (i.e., 1 or 2) if it does not have a right sibling. If it has a right sibling, it is not labeled.

We refer to these as *partially labeled binary trees*. All 19 such binary trees on 3 vertices are shown in Figure 9.

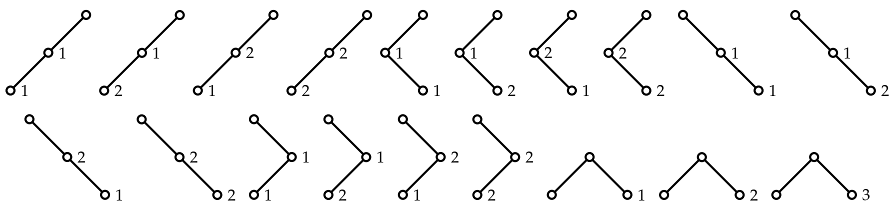


Figure 10. Partially labeled binary trees with 3 vertices.

Let $B(z)$ be the generating function for these trees, where z marks a vertex. Each internal vertex may have left and/or right children, and labels are assigned according to the rules above. This yields the functional equation:

$$B(z) = z + 2zB(z) + 2B(z) + 3B(z)^2 = z + 4zB(z) + 3B(z)^2,$$

which is identical to Equation (14). Therefore, the number of such binary trees with n vertices also coincides with the number of skew 2-Dyck paths of semi-length n.

We now describe a bijection between the two structures introduced above.

Proposition 4.1. *There exists a bijection between the set of partially labeled plane trees with n vertices and outdegree at most 2, and the set of partially labeled binary trees with n vertices.*

Proof. Let us begin with a partially labeled plane tree with n vertices and outdegree at most 2. We transform it into a partially labeled binary tree using the following rules:

(i) For each internal vertex of outdegree 2 labeled $i \in \{1,2,3\}$, assign the label i to its right child.

(ii) For each internal vertex of outdegree 1 labeled $i \in \{1,2\}$, assign the label i to its child and make that child a left child.

(iii) For each internal vertex of outdegree 1 labeled $i \in \{3,4\}$, assign the label $i-2$ to its child and make that child a right child.

This process preserves the number of vertices and produces a valid partially labeled binary tree.

To obtain the reverse transformation:

(i) For each right child labeled $i \in \{1,2,3\}$ that has a left sibling, assign the label i to its parent.

(ii) For each left child labeled $i \in \{1,2\}$ without a right sibling, assign the label i to its parent.

(iii) For each right child labeled $i \in \{1,2\}$ without a left sibling, assign the label $i+2$ to its parent.

This also retains the number of vertices and yields a valid partially labeled plane tree with outdegree at most 2. The bijection is shown in Figure 11.

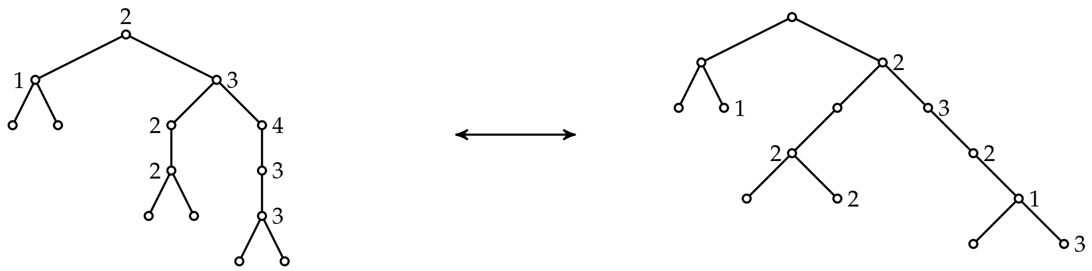


Figure 11. Bijection between partially labelled plane tree with outdegree at most 2 and partially labelled binary tree.

□

It would be of interest to construct explicit bijections between the set of skew 2-Dyck paths and each of the two combinatorial structures introduced in this section.

5 Conclusion and future work

In this article, we introduced the class of skew 2-Dyck paths and enumerated them with respect to the number of down-steps and left-steps. A recurrence relation for counting these paths was derived, and we demonstrated that the resulting sequence is log-convex. Future directions include refining the enumeration by considering additional parameters such as the number of hills or valleys, the type of the final step, and the distribution between down-steps and left-steps. The study can also be extended to skew k-Dyck paths for $k \geq 3$, potentially revealing deeper structural insights and further enumerative patterns. We constructed bijections between the set of skew 2-Dyck paths and the set of three other combinatorial structures: 2-labeled box paths, 3-leaf-labeled plane trees, and 2-edge-labeled plane trees. Identifying additional combinatorial families that are equinumerous with skew 2-Dyck paths remains an open and promising area of research. Additionally, we introduced two new families of combinatorial objects - partially labeled plane trees with outdegree at most 2 and partially labeled binary trees - and proved that they are counted by the same sequence as skew 2-Dyck paths. A bijection was established between these two tree families. Developing explicit bijections between these trees and skew 2-Dyck paths would be a natural and enriching continuation of this work.

Funding

This research received no external funding.

Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

References

[1] J. L. Baril, J. L. Ramirez, L. Simbaqueba, Counting prefixes of skew Dyck paths, J. Integer Seq. 24 (2021) Article 21.8.2. https://cs.uwaterloo.ca/journals/JIS/VOL24/Ramirez/ramirez10.pdf

[2] E. Deutsch, E. Munarini, S. Rinaldi, Skew Dyck paths, J. Stat. Plan. Infer. 140(8) (2010) 2191–2203. https://doi.org/10.1016/j.jspi.2010.01.015

[3] E. Deutsch, E. Munarini, S. Rinaldi, Skew Dyck paths, area, and superdiagonal bargraphs, J. Stat. Plan. Infer. 140(6) (2010) 1550–1562. https://doi.org/10.1016/j.jspi.2009.12.013

[4] C. Heuberger, S. J. Selkirk, S. Wagner, Enumeration of generalized Dyck paths based on the height of down-steps modulo k, Electronic J. Combin. 30 (2023) P1.26. https://doi.org/10.37236/11218

[5] L. L. Liu, Y. Wang, On the log-convexity of combinatorial sequences, Adv. in Appl. Math. 39(4) (2007) 453–476. https://doi.org/10.1016/j.aam.2006.11.002

[6] Q. L. Lu, Skew Motzkin paths, Acta Math. Sin. Engl. Ser. 33 (2017) 657–667. https://doi.org/10.1007/s10114-016-5292-y

[7] H. Prodinger, A walk in my lattice path garden, arXiv preprint arXiv:2111.14797 (2021).

https://doi.org/10.48550/arXiv.2111.14797
[8] H. Prodinger, Partial skew Dyck paths: a kernel method approach, Graphs Comb. 38(5) (2022) 135. https://doi.org/10.48550/arXiv.2108.09785
[9] H. Prodinger, Skew Dyck paths having no peaks at level 1, J. Integer Seq. 25 (2022) Article 22.2.3. https://doi.org/10.48550/arXiv.2201.00640
[10] H. Prodinger, Skew Dyck paths with catastrophes, arXiv preprint arXiv:2201.02518 (2022). https://doi.org/10.48550/arXiv.2201.02518
[11] F. Qi, B. N. Guo, Some explicit and recursive formulas of the large and little Schroder numbers, Arab J. Math. Sci. 23(2) (2017) 141–147. https://doi.org/10.1016/j.ajmsc.2016.06.002
[12] S. J. Selkirk, On a generalisation of k-Dyck paths, MSc Thesis, Stellenbosch University, 2019. https://math.sun.ac.za/2019/11/15/selkirk.html
[13] N. J. A. Sloane, The on-line fncyclopedia of integer sequences, 2019. https://oeis.org
[14] R. P. Stanley, Enumerative combinatorics, Vol. 2, volume 62 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999. https:// doi.org/ 10.1017/ CBO9780511609589
[15] H. S. Wilf, Generatingfunctionology, A. K. Peters, Ltd., Natick, MA, USA, 2006. https:// www2. math .upenn.edu/ wilf/gfology2.pdf
[16] W.-J. Woan, A recursive relation for weighted Motzkin sequences, J. Integer Seq. 8 (2005) Article 05.1.6. https://cs.uwaterloo.ca/journals/JIS/VOL8/Woan/woan11.html
[17] Y. Zhang, Y. Zhuang, A subfamily of skew Dyck paths related to k-ary trees, J. Integer Seq. 27 (2024) Article 24.2.3. https://doi.org/10.48550/arXiv.2306.15778

Citation: Y. W. Kariuki, I. O. Okoth, On a class of skew Dyck paths, J. Disc. Math. Appl. 10(4) (2025) 305–319.
https://doi.org/10.22061/jdma.2025.12170.1141