Journal of Discrete Mathematics and Its Applications 10 (4) (2025) 393–401

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Research Paper

On the characterization of tricyclic graphs with Szeged complexity one

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Academic Editor: Abdullah Roslan Hasni

Abstract. This paper presents a classification of 12 out of 15 known families of tricyclic graphs based on their Szeged complexity. It is shown that only two of these families contain graphs with Szeged complexity equal to one. Building on previous structural analyses of unicyclic and bicyclic graphs, this study extends the classification framework to include a substantial portion of tricyclic configurations. The results contribute to a deeper understanding of graph complexity and lay the groundwork for further exploration of cyclic graph structures.

Keywords. Szeged complexity, Szeged contribution, tricyclic graphs.
Mathematics Subject Classification (2020): 05C92, 05C12, 05C30.

1 Introduction

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$.

The distance between any two vertices $x, y \in V(G)$ is denoted by $d(x,y)$. Based on this notation, the total distances of vertex $x \in V(G)$ is defined as

$$w_G(x) = \sum_{x \neq y \in V(G)} d(x,y).$$

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Received 13 September 2025 ; Revised 10 October 2025; Accepted 23 Nomember 2025
First Publish Date: 01 December 2025

The Wiener complexity of a graph G, as introduced in [3] (see also [1,5,11,14]), is given by

$$C_W(G) = |\{w_G(x) \ : \ x \in V(G)\}| .$$

Given an edge $e = xy \in E(G)$, define the following vertex subsets:

$$N_x(e) = \{z \in V(G) : d(x,z) < d(y,z)\} ,$$

$$N_y(e) = \{z \in V(G) : d(y,z) < d(x,z)\} ,$$

and

$$N_0(e) = \{z \in V(G) : d(y,z) = d(x,z)\} .$$

Let $n_x(e) = |N_x(e)|$ and $n_y(e) = |N_y(e)|$. The Szeged contribution of the edge e, denoted by $Sz(e)$, is defined as the product $n_x(e).n_y(e)$. Summing these contributions over all edges in G yields the Szeged index $Sz(G)$ of the graph, as introduced in [10]. The Szeged complexity of G, as discussed in [2,4,7–9], is defined as

$$C_{Sz}(G) = |\{Sz(e) \ : \ e \in E(G)\}| .$$

Studying complexity indices is beneficial as they provide insights into structural characteristics of graphs and facilitate graph classification. In [7–9] graphs with small Sz-complexity are classified and their W-complexity is calculated.

A graph G is considered μ-cyclic if it is connected and satisfied the relation $\mu = |E(G)| - |V(G)| + 1$. When $\mu = 1$, 2, or 3, the graph is referred to as unicyclic, bicyclic, or tricyclic, respectively. These classes of graphs play a fundamental role in structural graph theory and have applications in chemical graph and network design. In [6, 10, 12, 13] μ-cyclic graphs ($\mu = 1$, 2, or 3) with extremal Szeged index have been determined.

This research builds upon the foundational results reported in [7,8], where some μ-cyclic graphs with Sz-complexity equal to one were analyzed and classified. In continuation of that work, we extend the classification to tricyclic graphs and classify 12 out of 15 families with Sz-complexity one.

2 Related work and background results

In this section, we review relevant results from previous studies and present preliminary observations that will be used in the classification of tricyclic graphs.

Theorem 2.1. *[7] Let G be a graph with Sz-complexity equal to one. Then G does not contain a non-leaf cut edge.*

Theorem 2.2. *[7] Let G be a unicyclic graph. Then $C_{Sz}(G) = 1$ if and only if G is a cycle.*

A **μ-cyclic graph of type I** is formed by joining μ cycles at a common vertex, possibly with trees attached to some cycle vertices. The Dutch windmill graph $D(\mu,r)$ exemplifies this structure, comprising μ copies of the cycle C_r intersecting at a single shared vertex.

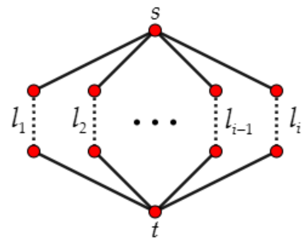

Figure 1. The graph $\Theta(l_1, l_2, ..., l_i)$.

Theorem 2.3. *[8] Suppose G is a μ-cyclic graph of type I. Then $C_{Sz}(G) = 1$ if and only if G is isomorphic to $D(\mu, r)$, for even integer r.*

A graph is referred to as a Θ-graph if it consists of $i \geq 3$ internally disjoint paths with length l_i connecting two vertices s and t, which is denoted by $\Theta(l_1, l_2, ..., l_i)$ as illustrated in Figure 1. The order of the lengths l_1, l_2, ..., l_i in this notation is not important. To make things easier, we often assume the lengths are sorted so that $1 \leq l_1 \leq l_2 \leq \cdots \leq l_i$.

Let a μ**-cyclic graph of type II** denote a graph derived from a Θ-graph by connecting trees to some vertices.

Theorem 2.4. *[8] Let $G \cong \Theta(l_1, l_2, ..., l_i)$ be a bipartite graph. Then $C_{Sz}(G) = 1$ if and only if $l_1 = l_2 = \cdots = l_{i-1} = 2$ and l_i is an arbitrary even integer.*

Theorem 2.5. *[8] Let G be a μ-cyclic graph of type II with $C_{Sz}(G) = 1$. Then G is leafless.*

3 Classification of tricyclic graphs

This part of the study outlines the classification of tricyclic graphs with Sz-complexity one, focusing on 12 distinct structural families identified through our analysis as depicted in Figure 2.

Corollary 3.1. *Consider a tricyclic graph G based on one of the structures (2-8) in Figure 2. Then $C_{Sz}(G) \neq 1$.*

Proof. According to Theorem 2.1, it is obvious. □

Theorem 2.3 leads us to the following result.

Corollary 3.2. *Let G be a tricyclic graph based on the structure of (12) in Figure 2. Then $C_{Sz}(G) = 1$ if and only if G is isomorphic to $D(3, r)$ that r is even.*

Consider the graph $\Theta(l_1, l_2, ..., l_i)$ on n vertices, which is non-bipartite, and suppose that k of lengths l_i are even, where $1 \leq k < i$. Let e and f denote the middle edges (see Figure 3 for an example) along an odd-length and an even-length path joining s and t, respectively, as illustrated in Figure 4. The Szeged contributions of these edges are given by:

$$Sz(e) = \left(\frac{n-k}{2}\right)^2 \text{ and } Sz(f) = \left(\frac{n-k}{2} + 1 - (i-k)\right)\left(\frac{n-k}{2} + (k-1)\right).$$

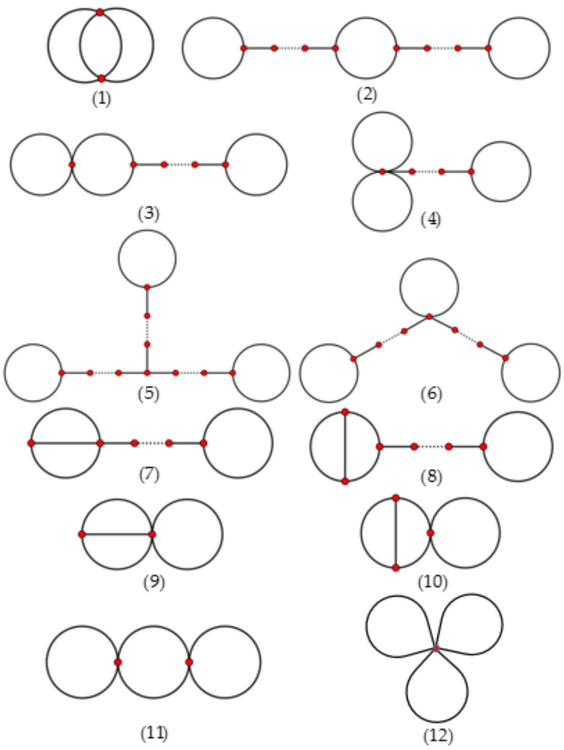

Figure 2. Twelve bases of tricyclic graphs.

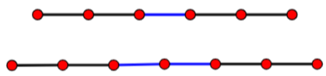

Figure 3. The middle edges in P_6 and P_7 are colored blue.

Theorem 3.3. *Assume G is a tricyclic graph based on the structure (1) in Figure 2. Then $C_{Sz}(G) = 1$ if and only if $G \cong \Theta(l_1, ..., l_4)$ that $l_1 = l_2 = l_3 = 2$ and l_4 is an arbitrary even integer.*

Proof. Given that $C_{Sz}(G) = 1$, Theorem 2.5 implies that $G \cong \Theta(l_1, ..., l_4)$. We now consider the case where G is non-bipartite. Based on the previously computed Szeged contributions of the middle edges along the st-paths, we demonstrate that $C_{Sz}(G) \neq 1$. The analysis proceeds through the following three cases:

Case 1. One of the lengths l_i is odd. Then

$$Sz(e) = \left(\tfrac{n-3}{2}\right)^2 \text{ and } Sz(f) = \left(\frac{n+1}{2}\right)\left(\tfrac{n-3}{2}\right).$$

Case 2. Two of the lengths l_i are odd. Then

$$Sz(e) = \left(\tfrac{n-2}{2}\right)^2 \text{ and } Sz(f) = \left(\tfrac{n}{2}\right)\left(\frac{n-4}{2}\right).$$

Case 3. Three of the lengths l_i are odd. Hence

$$Sz(e) = \left(\frac{n-1}{2}\right)^2 \text{ and } Sz(f) = \left(\frac{n-1}{2}\right)\left(\tfrac{n-5}{2}\right).$$

In each case, it is evident that $Sz(e) \neq Sz(f)$, and therefore $C_{Sz}(G) \neq 1$.

If G is bipartite, then the result follows directly from Theorem 2.4. $\square$

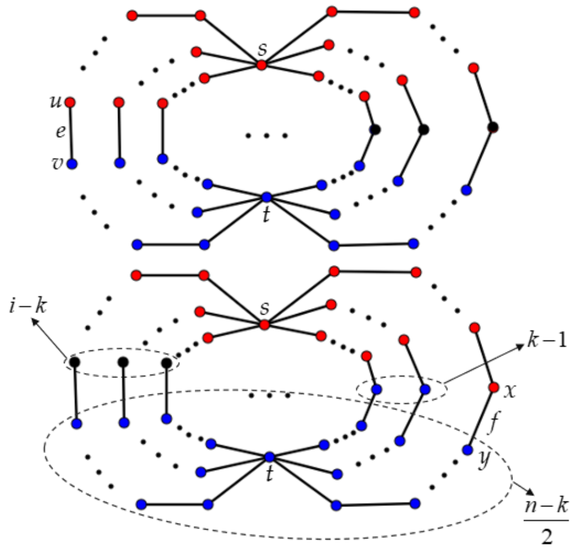

Figure 4. The vertices in N_u and N_x are colored red, those in N_v and N_y are colored blue, and the vertices belonging to $N_0(e)$ and $N_0(f)$ are colored black.

We proceed to show that for tricyclic graphs based on structures (9), (10), or (11) in Figure 2, there exists no graph with Sz-complexity equal to one.

Now, let $G[C_r]$ denote the graph obtained by attaching a cycle C_r to a vertex of G.

Theorem 3.4. *[8] Let G be a graph and consider the construction $G[C_r]$, where r is odd. Then $C_{Sz}(G[C_r]) \neq 1$.*

Theorem 3.5. *Let H be the graph formed by extending $G[C_r]$ through the attachment of trees to some of its vertices. If $C_{Sz}(H) = 1$, then H is leafless.*

Proof. Assume $|V(H)| = n$ and e is a leaf of H. Then $Sz(e) = n - 1$. Consider $f \in E(C_r)$. If r is even, then clearly $Sz(f) \neq n - 1$. Also, if r is odd, then similar to the proof of Theorem 2.2, it is not possible for all edges belonging to the cycle C_r to simultaneously have a Szeged contribution equal to $n - 1$ that is a contradiction. Hence H is leafless. □

From Theorem 3.5, we obtain the following result.

Corollary 3.6. *Let G be a tricyclic graph based on structures (9), (10), or (11) in Figure 2. If $C_{Sz}(G) = 1$, then G is leafless.*

Theorem 3.7. *Let G be a tricyclic graph based on structure (11) in Figure 2. Then $C_{Sz}(G) \neq 1$.*

Proof. If G contains a leaf, then by Corollary 3.6 $C_{Sz}(G) \neq 1$. Hence consider G is leafless. Let G consists of C_p, C_q, and C_r as illustrated in Figure 5 and $x, y \in V(G)$ are cut vertices. If p or r is odd, then by Theorem 3.4 $C_{Sz}(G) \neq 1$. Thus p and r are even. Assume $e \in E(C_p)$ and $f \in E(C_r)$. Then $Sz(e) = \left(\frac{p}{2}\right)\left(|V(G)| - \frac{p}{2}\right)$ and $Sz(f) = \left(\frac{r}{2}\right)\left(|V(G)| - \frac{r}{2}\right)$. It follows that $p = r$. Now we consider two cases:

Case 1. Suppose that q is odd. Then C_q consists of two xy-paths, one of odd length and the other even. Suppose g lies at the center of the odd xy-path (see Figure 6) and $h \in E(C_q)$

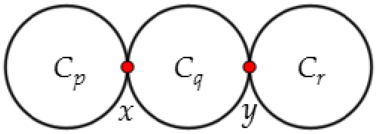

Figure 5. The graph G composed of three cycles C_p, C_q, and C_r, with cut vertices x and y.

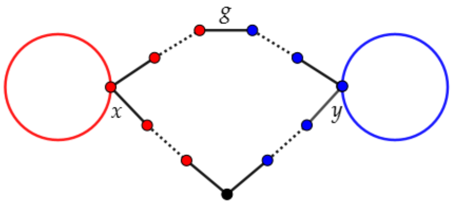

Figure 6. The middle edge g of the xy-path.

such that $x \in N_0(h)$. Therefore $Sz(g) = \left(\frac{q-1}{2} + p - 1\right)^2$ and $Sz(h) = \left(\frac{q-1}{2} + p - 1\right)\left(\frac{q-1}{2}\right)$. Hence, we infer that $Sz(g) \neq Sz(h)$.

Case 2. Assume q is even. Thus C_q consists of two even (or odd) xy-paths. Let $g \in E(C_q)$ be a middle edge of an xy-path. Then $Sz(g) = (\frac{q}{2} + p - 1)^2$. Hence $Sz(g) \neq Sz(e)$ that $e \in E(C_p)$.

Therefore $C_{Sz}(G) \neq 1$.

□

Theorem 3.8. *Suppose G is a tricyclic graph based on structures (9) or (10) in Figure 2. Then $C_{Sz}(G) \neq 1$.*

Proof. If G has a leaf, then by Corollary 3.6 $C_{Sz}(G) \neq 1$. Now, assume G is leafless and constructed from $\Theta(l_1, l_2, l_3)$ that C_r with a cut vertex connected to it. We analyze the following four cases:

Case 1. One of the lengths l_i is odd. Consider Figure 7. In analogy with the proof of Theorem 3.3, we analyze the Szeged contributions of the middle edges e and f, contingent upon the vertex where the cycle C_r is connected:

(i) C_r is connected to w. Thus $Sz(e) = (\frac{n-2}{2})^2$ and $Sz(f) = (\frac{n-2}{2})(\frac{n}{2} + r - 1)$.

(ii) C_r is connected to y. Then $Sz(e) = (\frac{n-2}{2})^2$ and $Sz(f) = (\frac{n-2}{2} + r - 1)(\frac{n}{2})$.

(iii) C_r is connected to v. It follows that

$$Sz(e) = \left(\tfrac{n-2}{2}\right)\left(\tfrac{n-2}{2} + r - 1\right) \text{ and } Sz(f) = \left(\tfrac{n-2}{2}\right)\left(\tfrac{n}{2}\right).$$

(iv) C_r is connected to the blue vertices except for v in Figure 7 (1). Hence

$$Sz(e) = \left(\tfrac{n-2}{2}\right)\left(\tfrac{n-2}{2} + r - 1\right) \text{ and } Sz(f) = \left(\tfrac{n}{2}\right)\left(\tfrac{n-2}{2} + r - 1\right).$$

(v) C_r is connected to the red vertices in Figure 7 (1). Consequently,

$$Sz(e) = \left(\tfrac{n-2}{2}\right)\left(\tfrac{n-2}{2} + r - 1\right) \text{ and } Sz(f) = \left(\tfrac{n}{2} + r - 1\right)\left(\tfrac{n-2}{2}\right).$$

Case 2. Two of the lengths l_i are odd. Refer to Figure 8. Similar to case 1, we examine the following items:

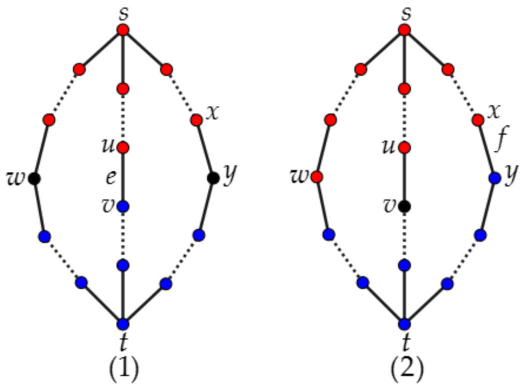

Figure 7. Vertex colors correspond to the same partitioning as in Figure 4.

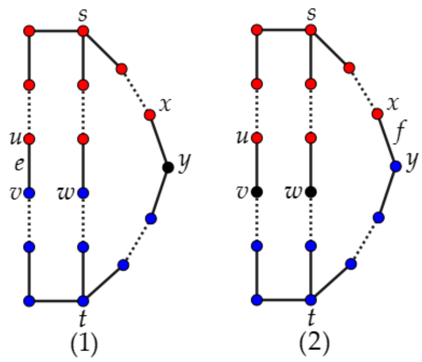

Figure 8. Colors are assigned as previously described.

(i) C_r is connected to v or w. Then $Sz(e) = \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+r-1\right)$ and $Sz(f) = (\frac{n-1}{2})(\frac{n-3}{2})$.

(ii) C_r is connected to y. Therefore $Sz(e) = \left(\frac{n-1}{2}\right)^2$ and $Sz(f) = (\frac{n-3}{2}+r-1)\left(\frac{n-1}{2}\right)$.

(iii) C_r is connected to the blue vertices except for v and w in Figure 8 (1). It follows that

$$Sz(e) = \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+r-1\right) \text{ and } Sz(f) = \left(\frac{n-1}{2}\right)(\frac{n-3}{2}+r-1).$$

(iv) C_r is connected to the red vertices in Figure 8 (1). Then we infer that

$$Sz(e) = \left(\frac{n-1}{2}\right)\left(\frac{n-1}{2}+r-1\right) \text{ and } Sz(f) = \left(\frac{n-1}{2}+r-1\right)(\frac{n-3}{2}).$$

Case 3. None of the lengths l_i is odd. Consider Figure 9. Similar to the previous cases, we investigate the following items:

(i) C_r is connected to the red vertices except for m and y in Figure 9 (1). Consequently,

$$Sz(e) = \left(\frac{n+1}{2}+r-1\right)\left(\frac{n-1}{2}\right) \text{ and } Sz(f) = \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}+r-1\right).$$

(ii) C_r is connected to the blue vertices except for v in Figure 9 (1). Thus

$$Sz(e) = \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2}+r-1\right) \text{ and } Sz(f) = \left(\frac{n+1}{2}+r-1\right)\left(\frac{n-1}{2}\right).$$

(iii) C_r is connected to y. Then

$$Sz(e) = \left(\frac{n+1}{2}+r-1\right)\left(\frac{n-1}{2}\right) \text{ and } Sz(g) = \left(\frac{n-1}{2}+r-1\right)\left(\frac{n+1}{2}\right).$$

(iv) C_r is connected to v. Hence

$$Sz(e) = \left(\frac{n+1}{2}\right)\left(\frac{n-1}{2} + r - 1\right) \text{ and } Sz(g) = \left(\frac{n+1}{2} + r - 1\right)\left(\frac{n-1}{2}\right).$$

(v) C_r is connected to m. It follows that

$$Sz(e) = \left(\frac{n+1}{2} + r - 1\right)\left(\frac{n-1}{2}\right) \text{ and } Sz(h) = \left(\frac{n-1}{2} + r - 1\right)\left(\frac{n+1}{2}\right).$$

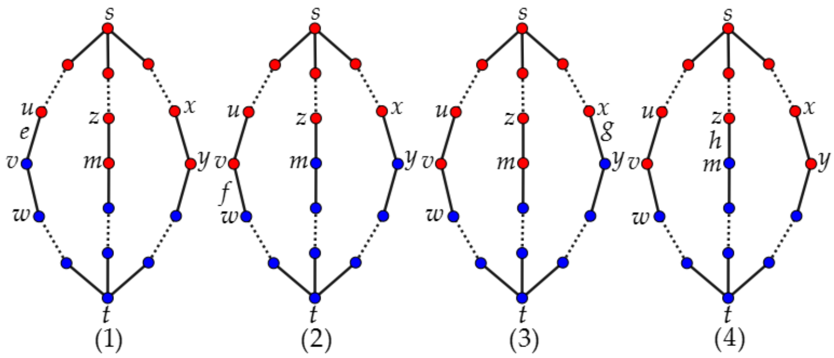

Figure 9. Four middle edges.

Case 4. Three of the lengths l_i are odd. Based on the argument presented in the proof of Theorem 2.4, if $e = uv$ is a middle edge of an st-path in G, then $|n_u - n_v| = r - 1$. In contrast, for an edge $f = xy$ adjacent to s along the longest path connecting s and t, we deduce that $|n_x - n_y| \geq r + 1$.

In all the above cases, two edges with different Szeged contributions are identified, indicating that $C_{Sz}(G) \neq 1$. □

4 Conclusions

In this study, twelve distinct families of tricyclic graphs were systematically classified according to their Sz-complexity. Among these, only two families were found to contain graphs with Sz-complexity one. Furthermore, earlier investigations have examined unicyclic and bicyclic graphs through the lens of the vector $(C_{Sz}, C_W) = (1, C_W)$. By extending this classification framework to include three additional tricyclic families, a complete categorization of all tricyclic graphs with respect to this vector becomes attainable.

Funding

This research received no external funding.

Data Availability

Data sharing is not applicable to this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this article.

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Citation: Z. Vaziri, On the characterization of tricyclic graphs with Szeged complexity one, J. Disc. Math. Appl. 10(4) (2025) 393–401.

doi **https://doi.org/10.22061/jdma.2025.12502.1161**