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Research Paper

Stability with respect to total restrained domination in bipartite graphs

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Abstract. In a graph $G = (V, E)$ with no isolated vertices, a subset D of vertices is said to be a total dominating set (abbreviated TDS) if it has the property that every vertex of G is adjacent to some vertex in D. A TDS D is said to be a total restrained dominating set (abbreviated TRDS) if it has a further property that any vertex in $V - D$ is also adjacent to a vertex in $V - D$. Given the isolate-free graph G, the total restrained domination number of G, which we denote it by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of G. The minimum number of vertices of the graph G whose removal changes the total restrained domination number of G is called the total restrained domination stability number of G, and is denoted by $st_{\gamma_{tr}}(G)$. In this paper we study this variant in bipartite graphs. We show that the related decision problem related to this variant is NP-hard in bipartite graphs. We also determine the total restrained stability number in some families of graphs, including the families of trees and unicyclic graphs.

Keywords. total dominating set, total restrained dominating set, bipartite graph.
Mathematics Subject Classification (2020): 05C69.

1 Introduction

Given a graph $G = (V, E)$ of order n, the *open neighborhood* of a vertex v is defined by $N_G(v) = \{u | uv \in E(G)\}$. Also, the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree*

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of a vertex v, which we denote it by $\deg(v)$, is the cardinality of $N(v)$, that is, the number of neighbors of v in G. A vertex whose degree is one is called a *leaf* in this paper, and a vertex which has a leaf in its open neighborhood is called a *support vertex*. A support vertex having at least two leaves in its open neighborhood is called a *strong support vertex*. A graph is called a *unicyclic graph* if it is obtained from a tree by adding an edge between two non-adjacent vertices. A *bipartite* graph is a graph that we can partition its vertex set into two sets A and B such that no pair of vertices of A are adjacent and the same time no pair of vertices of B are adjacent, equivalently, for each edge we have an end-point in A and the same time an end-point in B. We denote by P_n a path of order n, by C_n a cycle of order n, by K_n a complete graph of order n, and by $K_{m,m}$ a complete bipartite graph whose partite sets have cardinalities m, n. The graph $K_{1,n}$ is called a *star*. A tree T is called a *double star* if T contains exactly two vertices that are not leaves. A *rooted tree* is a tree that distinguishes one vertex r as the *root*. For each vertex $v \neq r$ in a rooted tree, the parent of v is the neighbor of v on the unique (r, v)-path. Also, a child of v is any other neighbor of v. The *corona graph* of a graph G which is denoted by $cor(G)$ is a graph obtained from G by adding a leaf to every vertex of G. The *diameter* of a graph G, which is denoted by $diam(G)$, is the maximum distance among all pair of distinct vertices of G. A vertex subset S in a given graph G is a said to be a *dominating set* if for every vertex x which is not in S there is a vertex x' in S such that xx' is an edge. It is a *total dominating set*, if we have that $N(S) = V(G)$.

In [14], Ma, Chen and Sun introduced the concept of total restrained domination as a new domination variant. For an isolate-free graph G, a subset S of vertices is said to be a *total restrained dominating set* (or just TRDS) if the set S is a TDS and has the further property that each vertex $x \in V(G) - S$ is adjacent to a vertex $y \in V(G) - S$. The variant "*total restrained domination number*" of the given graph G, which we denote it denoted by $\gamma_{tr}(G)$, is defined as the minimum cardinality of a TRDS of G. We always refer to a TRDS of cardinality $\gamma_{tr}(G)$ in a given graph G as a $\gamma_{tr}(G)$-set. This concept (total restrained domination) was also studied by several authors, see, for example, [5, 10, 13].

In [2], Bauer et al. initiated the study of a variant, namely domination stability, related to the changing and unchanging domination number. We note that the behaviour of a graph under some modifications which usually addressed as "changing and unchanging" has been studied for several graph invariants including several domination variants, (consult for example, [1, 3, 7–9, 11, 12, 15, 16]). A related variant which is called the *domination stability* of a graph G and is denoted by $st_{\gamma}(G)$, is the minimum number of vertices whose removal changes the domination number.

We aim to study a similar variation, namely, the total restrained domination stability number. For a graph G with no isolated vertices, the *total restrained domination stability*, which we denote it by $st_{\gamma_{tr}}(G)$, is defined to be the minimum number of vertices whose removal: (1) does not produce isolated vertices, (2) force the total restrained domination number to be changed. We begin in Section 2 with some basic properties and also computing this variant in complete graph, cycles and complete bipartite graphs. For the computational complexity aspect of this new variation, we show that deciding it is NP-hard when restricted to bipartite graphs in Section 3. Also we determine the total restrained domination stability for several families of graphs including the family of trees and unicyclic graphs in Section 4. We propose also a conclusion containing some suggested problems in Section 5. We make use of the following.

Proposition 1.1 ([10]). *For $n \geq 4$, $\gamma_{tr}(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor$.*

Proposition 1.2 ([4]). *For $n \geq 4$, $\gamma_{tr}(C_n) = n - 2\lfloor \frac{n}{4} \rfloor$.*

2 Preliminary results

We begin with the following propositions regarding an increasing/decreasing by more than one on the restrained domination number when one remove a vertex.

Proposition 2.1. *There are finitely many graphs G with a non-support vertex v such that the difference $\gamma_{tr}(G - v) - \gamma_{tr}(G)$ is arbitrarily large.*

Proof. Let $k \geq 2$ be an integer. Let $P_6 : v_1v_2v_3vv_4v_5$ be a path of order six. We add $k - 2$ leaves to the vertex v_1 (if $k > 2$) to form a graph G_1. For each integer $i > 1$ we add i paths P_2 and join the vertex v_1 to an end-vertex of each path P_2 and join v to the other end-vertex of each path P_2. Since any TRDS contains all leaves and all support vertices, it is now straightforward to see that for each $i \geq 1$, $\gamma_{tr}(G_i) = k + 1$ and $\gamma_{tr}(G_i - v) = |V(G_i)| - 1 = n - 1$. Thus, $\gamma_{tr}(G - v) - \gamma_{tr}(G) = n - k - 2$. In the Figure 1 we depict the graph related to $k = 5$ (the graph G_3). In this graph the black vertices make a minimum cardinality TRDS for G_3. □

Proposition 2.2. *There are finitely many graphs G with a non-support vertex v such that the difference $\gamma_{tr}(G) - \gamma_{tr}(G - v)$ is arbitrarily large.*

Proof. Let G'_n be a graph of order $n \geq 5$ with $V(G'_n) = \{y, x, z, v, x_1, ..., x_{n-4}\}$ and $E(G'_n) = \{yx, xz, zv\} \cup \{x_ix, x_iz : i = 1, 2, ..., n - 4\}$. It can be seen that $\gamma_{tr}(G'_n) = n$, while $\gamma_{tr}(G'_n - v) = 2$. Thus, $\gamma_{tr}(G'_n) - \gamma_{tr}(G'_n - v) = n - 2$. The right graph in the Figure 1 depicts the graph G'_7. □

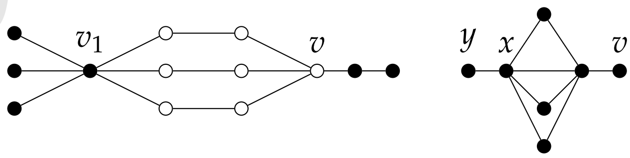

Figure 1. (left) The graphs G_3 for $k = 5$, and (right) the graph G'_7.

It is also evident that in a graph G the removing a non-support vertex v may keep the restrained domination number unchanged, that is, $\gamma_{tr}(G - v) = \gamma_{tr}(G)$. As an example, we can consider the graph K_n, for $n \geq 5$, in which $\gamma_{tr}(K_n - v) = \gamma_{tr}(K_n) = 2$ for any vertex v. We can continue this discussiones, iteratedly, by removal of further vertices. For a graph G with no isolated vertices, we define the total restrained domination stability number to be the minimum cardinality of a subset S of vertices such that: (1) $G-S$ does not produce isolated vertices and (2) $\gamma_{tr}(G-S) \neq \gamma_{tr}(G)$. We denote this new variant of G by $st_{\gamma_{tr}}(G)$.

We next determine the total restrained domination stability in some classes of graphs.

Proposition 2.3. *For $n \geq 4$, $st_{\gamma_{tr}}(K_n) = n-3$.*

Proof. Clearly note that $\gamma_{tr}(K_n) = 2$. If S is a subset of vertices with $|S| < n-3$ then $K_n - S$ has order at least four which its total restrained domination number is 2. On the other hand, if we remove $n-3$ vertices from K_n we find a K_3 with total restrained domination number 3. □

Next we determine the total restrained stability in the classes of cycles and complete bipartite graphs.

Observation 2.4. *For a cycle C_n we have, $st_{\gamma_{tr}}(C_n) = 1$.*

Proof. Removing a vertex from C_n leaves a path P_n. Now by Propositions 1.1 and 1.2 it can be that $\gamma_{tr}(P_{n-1}) < \gamma_{tr}(P_n)$ if $n \equiv 3 \pmod 4$, and $\gamma_{tr}(P_{n-1}) > \gamma_{tr}(P_n)$ otherwise. □

Proposition 2.5. *For $2 \leq m \leq n$ we have, $st_{\gamma_{tr}}(K_{m,n}) = m-1$.*

Proof. It can be seen that $\gamma_{tr}(K_{m,n}) = 2 = \gamma_{tr}(K_{r,s})$ for all $s \geq r \geq 2$. Thus the removal of at most $m-2$ vertices of $K_{m,n}$ does not change the total restrained domination number. On the other hand removal all but one vertices of a partite set leaves a star with total restrained domination number greater than two. Thus, $st_{\gamma_{tr}}(K_{m,n}) = m-1$. □

3 NP-hardness

The following is the decision problem relating to our new variant.

TRDSP

Instance: An isolate-free graph G with the given total restrained domination number $\gamma_{tr}(G)$.
Question: Is it true that $st_{\gamma_{tr}}(G) > 0$?

To study the above decision problem we use a well-known problem namely the 3-SAT problem which is proved to be NP-complete, (see [6]). Let $U = \{u_1, ..., u_n\}$ is the set of boolean variables. A literal is either a variable of U or the negation of a variable of U. In the 3-SAT, a clause is a disjunction of three literals, that is, it contains 3 distinct occurrences of a variable u_i or its complement $\bar{u_i}$. The decision proble of 3-SAT is stated as follows:

3-SAT

Instance: A collection $C = \{C_1, C_2, \cdots, C_m\}$, where we call them clauses, over a finite set U of variables, such that $|C_j| = 3$ for $j = 1, 2, ..., m$.
Question: Does there exists a truth assignment for U that satisfies all clauses of the collection C?

Theorem 3.1. *TRDSP is NP-hard even when restricted to bipartite graphs.*

Proof. To prove, we use a transformation from the known 3-SAT problem. Assume that there is given an instance of the 3-SAT, including a set $U = \{u_1, u_2, \ldots, u_n\}$ of literals together with a set $C = \{c_1, c_2, \ldots, c_m\}$ of clauses. In what follows, we construct a graph G as follows. For each literal u_i, let G_i be a gadget with $V(G_i) = \{u_i, \bar{u}_i, a_i, a'_i, b_i, b'_i, e_i, e'_i\}$ and

$$E(G_i) = \{u_ia_i, u_ia'_i, u_ib_i, \bar{u}_ia_i, \bar{u}_ia'_i, \bar{u}_ib'_i, b'_ie_i, b'_ie'_i, b_ie_i, b_ie'_i, a_ie_i, a'_ie'_i, u_ib'_i, a_ie'_i, a'_ie_i, \bar{u}_ib_i\}.$$

Figure 2 shows the gadget G_i.

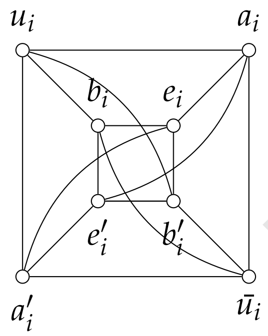

Figure 2. The gadget G_i.

Cooresponing to each clause C_j we consider now a clause-vertex, namely c_j, and then join c_j to the literal in C_j. Then add a gadjet G_0, where $V(G_0) = \{w_1, w_2, w_3, w_4, \bar{w}_1, \bar{w}_2, \bar{w}_3, \bar{w}_4\}$ and $E(G_0) = \{w_1w_2, w_2w_3, w_3w_4, w_4w_1, \bar{w}_1\bar{w}_2, \bar{w}_2\bar{w}_3, \bar{w}_3\bar{w}_4, \bar{w}_4\bar{w}_1, w_1\bar{w}_1, w_2\bar{w}_2, w_3\bar{w}_3, w_4\bar{w}_4, w_1\bar{w}_3, w_2\bar{w}_4, w_3\bar{w}_1, w_4\bar{w}_2\}$.

Then join the vertex w_1 to c_j, for all $j = 1, 2, ..., m$. Note that all graphs G_i (for all $i = 0, 1, ..., n$) are isomorphic. We observe that the constructed graph G is bipartite. In addition the graph G can be built in polynomial time, and so this is a polynimoal transformation. Now, let S be an arbitrary $\gamma_{tr}(G)$-set. Definitely S contains at least two vertices of each gadjet G_i for $i = 0, 1, 2, ..., n$. This means that $\gamma_{tr}(G) \geq 2n + 2$. Now, the set $\{w_1, \bar{w}_3, a_i, e'_i : i = 1, 2, ..., n\}$ is a TRDS for G which implies that $\gamma_{tr}(G) \leq 2n + 2$. We deduce that $\gamma_{tr}(G) = 2n + 2$.

We next show that the collection C has a satisfying truth-assignment t if and only if we have $st_{\gamma_{tr}}(G) > 1$. For this, assume first that t is a satisfying truth-assignment for C. We aim to show $st_{\gamma_{tr}}(G) > 1$. For this aim, we show that for any non-support-vertex $v \in V(G)$, $\gamma_{tr}(G - v) = 2n + 2 = \gamma_{tr}(G)$. Let $G' = G - v$ and S be a $\gamma_{tr}(G')$-set. We proceed according to the possibilities where $v \in \{c_1, ..., c_m\}$, $v \in V(G_i)$ for some $i \in \{1, 2, ..., n\}$, or $v \in V(G_0)$.

- Assume that $v \in \{c_1, ..., c_m\}$. Clearly no vertex in a gadjet G_i ($i = 0, 1, ..., n$) can dominate all vertices. Thus $|S \cap V(G_i)| \geq 2$, for $i = 0, 1, ..., n$. This implies that $|S| \geq 2n + 2$. Now it is evident that the set $\{w_1, \bar{w}_3, a_i, e'_i : i = 1, 2, ..., n\}$ is a TRDS for G'. So, we obtain that $\gamma_{tr}(G') \leq 2n + 2$. Consequently, $\gamma_{tr}(G') = 2n + 2 = \gamma_{tr}(G)$.

- Assume that $v \in V(G_i)$, where $i \in \{1, 2, ..., n\}$. W.l.o.g., we may assume that $i = 1$. Notice that no vertex in a gadjet G_j ($j \neq 1$) can dominate all verttices, and so $|S \cap V(G_j)| \geq 2$, for $j = 0, 2, 3, ..., n$. It is also clear that $S \cap V(G_1 - v) \neq \emptyset$. If $S \cap \{c_1, ..., c_m\} \neq \emptyset$, then we obtain $|S| \geq 2n+2$, as desired. Thus we may assume that $S \cap \{c_1,...,c_m\} = \emptyset$. We notice that any vertex of the graph $G_1 - v$ can dominate at most 4 vertices of $G_1 - v$. Thus we obtain that $S \cap V(G_1 - v)| \geq 2$. Hence, $\gamma_{tr}(G') \geq 2n+2$. Now for the other hand, we notice that if $v \notin \{a_1, e'_1\}$, then $\{w_1, \bar{w}_3, a_i, e'_i : i = 1,2,...,n\}$ is a TRDS for $G - v$ which leads to the inequality $\gamma_{tr}(G - v) \leq 2n+2$. Thus for the next we may assume that $v \in \{a_1, e'_1\}$. This time the set $\{e_1, a'_1\} \cup \{w_1, \bar{w}_3, a_i, e'_i : i = 2,...,n\}$ is a TRDS for $G - v$ which leads to $\gamma_{tr}(G') \leq 2n+2$. Consequently, $\gamma_{tr}(G') = 2n+2 = \gamma_{tr}(G)$.

- Assume that $v \in V(G_0)$. As the previous cases we have $|S \cap V(G_j)| \geq 2$, for all $j = 1,2,3,...,n$. Also it can be easily seen that $|S| \geq 2n+2$ if $S \cap \{c_1,...,c_m\} \neq \emptyset$. Thus we may assume that $S \cap \{c_1,...,c_m\} = \emptyset$. Then it is evident that $S \cap V(G_0 - v)| \geq 2$. This leads to $\gamma_{tr}(G') \geq 2n+2$. If v is neighther w_1 nor $\bar{w}_3$, then the set $\{w_1, \bar{w}_3, a_i, e'_i : i = 1,2,...,n\}$ is a TRDS for G' which leads to $\gamma_r(G') \leq 2n+2$. Next we may assume that $v \in \{w_1, \bar{w}_3\}$. Now we can form a set D of vertices as follows. For each $i = 1,2,...,n$ if $t(u_i) = T$ then we let $u_i, b'_i \in D$, and if $t(u_i) = F$ then we let $\overline{u_i}, b_i \in D$. Then $|D| = 2n$, and the set $D \cup \{w_3, \bar{w}_1\}$ is a TRDS for G' which leas to $\gamma_{tr}(G') \leq 2n+2$. Consequently, $\gamma_{tr}(G') = 2n+2 = \gamma_{tr}(G)$.

According the above possibilities, we conclude that $st_{\gamma_{tr}}(G) > 1$.

For the converse, assume that the collection C of cluases has no satisfying truth assignments. We remove the vertex w_1 from the constructed graph G and consider $G - w_1$. Assume that S be a $\gamma_{tr}(G - w_1)$-set. As what was seen in the previous part we observe that $|S \cap V(G_0 - w_1)| \geq 2$, and also for all $j = 1,2,3,...,n$, we have $|S \cap V(G_j)| \geq 2$. These mean that, $|S| = \gamma_{tr}(G - w_1) \geq 2n+2$. We show that $\gamma_{tr}(G - w_1) \neq 2n+2$. Suppose to the contrary that $\gamma_{tr}(G - w_1) = 2n+2$. We then have that for each $j = 1,2,3,...,n$, $|S \cap V(G_0 - w_1)| = |S \cap V(G_j)| = 2$. But we know that S is already a $\gamma_{tr}(G - w_1)$-set. Thus each vertex c_i for $i = 1,2,...,m$, is dominated by some vertex in $S \cap V(G_j)$, where $j \in \{1,2,...,n\}$. If there exists an integer $i \in \{1,2,...,n\}$ such that $\{u_i, \bar{u}_i\} \subseteq S$, then the edge e_i can not be dominated by the set S which is a contradiction. This implies that $|\{u_i, \bar{u}_i\} \cap S| \leq 1$, for each $i = 1,2,...,n$. We may also assume for the next that $|\{u_i, \bar{u}_i\} \cap S| = 1$ for each $i = 1,2,...,n$., because if there exists an integer i such that $\{u_i, \bar{u}_i\} \cap S = \emptyset$ then we can simply replace $S \cap V(G_i)$ by $\{u_i, b'_i\}$. Hence we assume that each vertex c_i, for $i = 1,2,...,m$, is dominated by a vertex u_j or $\bar{u}_j$ in $V(G_j)$, for some $j \in \{1,2,...,n\}$. Now we define the desired truth assignment for C as follows. Let $t_1 : U \to \{T, F\}$ be defined by the following rules:

"$t_1(u_i) = T$ if $u_i \in S$ and $t_1(u_i) = F$ if $\overline{u_i} \in S$".

Since any clause-vertex c_i, for $i = 1,2,...,m$, is adjacent to a vertex in $S \cap V(G_j)$ ($j \in \{1,2,...,n\}$) (and so dominated by it), and c_i is only adjacent to u_j or $\bar{u}_j$, it is evident that t_1 is a truth assignment for C. This is a contradiction. Thus $\gamma_{tr}(G - w_1) > 2n+2$. Consequently, $st_{\gamma_{tr}}(G) = 1$, as desired. □

4 Trees and unicyclic graphs

We first determine the total restrained stability in any tree of order at least three.

Proposition 4.1. *If a graph G has a strong support vertex, then $st_{\gamma_{tr}}(G) = 1$.*

Proof. Assume that a graph G has a strong support vertex u. Let u_1 and u_2 be two leaves which are adjacent to u and let S be a $\gamma_{tr}(G)$-set. Observe that S contains u and its leaf-neighbors. Now the set $S - \{u_1\}$ is a TRDS for $G - u_1$ which leads to $st_{\gamma_{tr}}(G) = 1$. □

Theorem 4.2. *$st_{\gamma_{tr}}(T) = 1$ for a tree T of order at least three.*

Proof. Let T be a tree of order $n \geq 3$ and let $diam(T) = d$. We root the tree T at a leaf x_0 of a diametrical path $P : x_0, \cdots, x_d$. For the case $d = 2$, clearly T is a star and thus $\gamma_{tr}(T) = n$ while $\gamma_{tr}(T - x_0) = n - 1$, and so $st_{\gamma_{tr}}(T) = 1$. If $d = 3$ then the tree T is a double-star for which $\gamma_{tr}(T) = n$ while $\gamma_{tr}(T - x_0) = n - 1$, and so $st_{\gamma_{tr}}(T) = 1$. Thus assume that $d \geq 4$. By Proposition 4.1 we may assume that T has no strong support vertices. Thus, $\deg(x_1) = \deg(x_{d-1}) = 2$.

Let D be a $\gamma_{tr}(T)$-set. It is evident that D contains every leaf and every support vertex of T. If $x_{d-2} \in D$, then $D - \{x_d\}$ is a TRDS for $T - x_d$. This implies that $\gamma_{tr}(T - x_d) < \gamma_{tr}(T)$, and so we obtain $st_{\gamma_{tr}}(T) = 1$. Thus we may assume that $x_{d-2} \notin D$. Since each child of x_{d-2} is a support vertex of degree two, we find that $x_{d-3} \notin D$ and so $d \geq 5$. Note that x_{d-3} is not a support vertex. Let D_1 be a $\gamma_{tr}(T - x_{d-3})$-set. Clearly D_1 contains x_{d-2}, x_{d-1} and x_d. Suppose that $|D_1| = \gamma_{tr}(T)$. If $\deg(x_{d-3}) = 2$, then $x_{d-4} \in D_1$, and if $\deg(x_{d-3}) \geq 3$, then D_1 contains all children of x_{d-3}. We deduce that $D_1 - \{x_{d-2}\}$ is a TRDS for T, a contradiction. Thus, $|D_1| \neq \gamma_{tr}(T)$, and so $st_{\gamma_{tr}}(T) = 1$. □

We next consider unicyclic graphs.

Theorem 4.3. *Let G be a unicyclic graph. Then, $st_{\gamma_{tr}}(G) \leq 2$. Furthermore, this bound is best possible.*

Proof. Let C be the (unique) cycle of a unicyclic graph G. By Observation 2.4 the result follow if $G = C$. Thus we may assume G has some further edhes and vertice, that is, $G \neq C$. If all vertices on C are support vertices, then they belong to every $\gamma_{tr}(G)$-set, and clearly $\gamma_{tr}(G - w) < \gamma_{tr}(G)$, where w is leaf adjacent to a vertex of C. This leads to $st_{\gamma_{tr}}(G) = 1$. Thus we may assume that there are some vertices on C that are not support vertices. If there exists a non-support-vertex v on the cycle C such that we have $\gamma_{tr}(G - v) \neq \gamma_{tr}(G)$, then we find that $st_{\gamma_{tr}}(G) = 1 < 2$, as desired. Accordingly, we may assume that $\gamma_{tr}(G - v) = \gamma_{tr}(G)$, for all non-support vertices v on the cycle C.

Let a vertex v be a non-support-vertex on the cycle C, and let G' be the graph obtained by removal of v. We then have $\gamma_{tr}(G') = \gamma_{tr}(G)$. Furthermore, definitely each component of G' is a tree. Assume that G'' be a component of G' that has maximum possible order. If $|V(G'')| \geq 3$, then by Theorem 4.2 we have $st_{\gamma_{tr}}(G'') = 1$. This means that there is a vertex u in the graph G'' such that $\gamma_{tr}(G'' - u) \neq \gamma_{tr}(G'')$ which leads to the inequality $\gamma_{tr}(G' - u) \neq \gamma_{tr}(G')$. Now, we find that $\gamma_{tr}(G-v-u)=\gamma_{tr}(G'-u)\neq\gamma_{tr}(G')=\gamma_{tr}(G)$, and thus $st_{\gamma_{tr}}(G)\leq 2$. Thus assume for the next that $|V(G'')|=2$, that is, G'' is a path P_2. This implies that if $\deg(v)=2$ then $G=C_3$, and if $\deg(v)\geq 3$ then G is obtained from a cycle C_3 by adding $\frac{n-3}{2}$ paths P_2 and joining a vertex of C_3 to an end-point of each P_2. If $G=C_3$ then by Observation 2.4, $st_{\gamma_{tr}}(G)=1$. Thus assume that G is obtained from a cycle $C_3: xyz$ by adding $\frac{n-3}{2}$ paths P_2 and joining x to an end-point of each P_2. Then $V(G)-\{y,z\}$ is a minimum-cardinality TRDS for G, and it is evident that $\gamma_{tr}(G)=n-2$. Therefore, $\gamma_{tr}(G-w)=n-3$, where w is a leaf of G, and so $st_{\gamma_{tr}}(G)=1$.

To see the sharpness, for any integer $n\geq 3$, consider a corona graph $cor(C_n)$ and add a leaf to each leaf of $cor(C_n)$ to obtain a graph H_n. Figure 3 depicts the graph H_3 where the black vertices form a TRDS, and note that removal of a leaf and its support vertex decreases the total restrained domination number by two. Then it can be seen that $\gamma_{tr}(H_n)=2n$ and $st_{\gamma_{tr}}(H_n)=2$. □

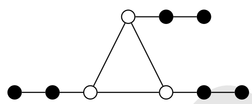

Figure 3. The graph H_3 with $st_{\gamma_{tr}}(H_3)=2$.

We remark that it is easy to find other infinite families of unicyclic graphs acheiving equality of the bound given in the Theorem 4.3. We have also the following result on graphs with total restrained stability number 2.

Proposition 4.4. *Any connected graph G of order $n\geq 3$ is an induced subgraph of a graph with total restrained stability number two.*

Proof. Given a connected graph G of order $n\geq 3$, we build a graph H containing G as follows. Assume that $V(G)=\{x_1,...,x_n\}$. For $i=1,2,...,n$ add a path $P_2: a_ib_i$ and join x_i to a_i. Then H is a connected graph of order $3n$. If $G=C_3$ then the graph H is the same graph H_3 which is depicted in Figure 3. Note that $\cup_{i=1}^{n}\{a_i,b_i\}$ is a TRDS for H, and so $\gamma_{tr}(H)\leq 2n$. Since any TRDS for H contains any leaf and any support vertex, we find that $\gamma_{tr}(H)\geq 2n$, and so $\gamma_{tr}(H)=2n$. Let $H'=H-a_1-b_1$. Then $\{x_2\}\cup\cup_{i=2}^{n}\{a_i,b_i\}$ is a TRDS for H' with cardinality less than $\gamma_{tr}(H)$. Thus $st_{\gamma_{tr}}(H)\leq 2$. We show that $st_{\gamma_{tr}}(H)=2$. Suppose $v\in V(H)$ such that $\gamma_{tr}(H-v)\neq\gamma_{tr}(H)$. Assume that $v\in V(G)$, and without loss of generality assume that $v=x_1$. Then $\cup_{i=1}^{n}\{a_i,b_i\}$ is a TRDS for $H-v$ and it is a minimum cardinality TRDS, since any TRDS contains any leaf and any support vertex. This is contradiction. Thus assume that v is a leaf. Without loss of generality assume that $v=b_1$. Then $\{x_1,a_1\}\cup\cup_{i=2}^{n}\{a_i,b_i\}$ is a TRDS for $H-v$, and as before it is a minimum cardinality TRDS, a contradiction. We deduce that $st_{\gamma_{tr}}(H)=2$. □

Corollary 4.5. *There is no forbidden-subgraph-characterization for a graph G with $st_{\gamma_{tr}}(G)=2$.*

5 Conclusion

Theorems 4.2 and 4.3 state that a bipartite graph G has total restrained stability number $i+1$, where i is the number of cycles of G. It is a good question characterizing unicyclic graphs achieving equality of this bound. Furthermore, perhaps one can prove an upper bound for the total restrained stability number in a bipartite graph in terms of the number of its even cycles and characterize those bipartite graphs achieving equality for the given bound.

As another good problem, we proved in the Theorem 3.1 that TRDSP is NP-hard on bipartite graphs. It is good to consider this problem on the planar bipartite graphs.

Finding

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Data is contained within the article.

Conflicts of Interests

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