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Research Paper

Constructing pentadiagonal matrices by partial eigen information

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Abstract. The inverse eigenvalue problem involves constructing a matrix based on its spectral information, along with providing conditions on the input data to determine the solvability of the problem. In this paper, we focus on a specific instance of the inverse eigenvalue problem, known as IEPSP, to generate symmetric pentadiagonal matrices using two pairs of eigenvalues from the desired matrix and an additional eigenvalue from each of its leading principal submatrices. Additionally, we explore a non-negative formulation of the inverse eigenvalue problem to produce a matrix that has non-negative elements. We present sufficient conditions for problem solvability, propose an algorithm, and provide several numerical examples to validate the results.

Keywords. pentadiagonal matrix, eigenvalue, eigenvector.
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1 Introduction

Consider a set of real numbers denoted by σ. The problem of finding necessary and sufficient conditions for σ to be the spectrum of a matrix is known as the inverse eigenvalue problem. Chu and Golub have done extensive work on this problem and classified various

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types of inverse eigenvalue problems, providing insights into their solutions [1].

This paper aims to focus on two types of inverse eigenvalue problems: structured inverse eigenvalue problems and partial inverse eigenvalue problems. A structured inverse eigenvalue problem involves constructing a matrix in such a way that its zero entries follow a specific pattern. Inverse eigenvalue problems have practical applications in various fields such as control theory [2], mass-spring oscillations [3,4], and finite element methods [5,6].

In this paper, we address the construction of a symmetric pentadiagonal matrix denoted as P_n as follows:

$$P_n = \begin{bmatrix} a_1 & b_1 & c_1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & c_2 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c_1 & b_2 & a_3 & b_3 & c_3 & 0 & 0 & 0 & \cdots & 0 \\ 0 & c_2 & b_3 & a_4 & b_4 & c_4 & 0 & 0 & \cdots & 0 \\ 0 & 0 & c_3 & b_4 & a_5 & b_5 & c_5 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & c_{n-5} & b_{n-4} & a_{n-3} & b_{n-3} & c_{n-3} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{n-4} & b_{n-3} & a_{n-2} & b_{n-2} & c_{n-2} \\ 0 & 0 & 0 & 0 & 0 & 0 & c_{n-3} & b_{n-2} & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_{n-2} & b_{n-1} & a_n \end{bmatrix}. \tag{1}$$

The matrix is constructed such that $b_i c_i \neq 0$. The structure of pentadiagonal matrices has various applications in discrete beam vibrations [7,8], Euler-Bernoulli beam vibrations [15], and non-Hermitian quantum mechanics [9]. Recent progress has been made in the construction of pentadiagonal matrices. Moghaddam et al. [10] investigated a generalized eigenvalue problem of the form $K_n X = \lambda M_n X$ with three pairs of eigenvalues, where both K_n and M_n are pentadiagonal matrices. Ghanbari and Mirzaei [8] proposed an algorithm to construct pentadiagonal matrices with three prescribed eigenvalues such that the first off-diagonal element of the matrix is negative, while all other elements are positive. Li et al. [15] studied the application of pentadiagonal matrices in Euler-Bernoulli beam vibrations and investigated an eigenvalue problem and an inverse eigenvalue problem. Perez et al. [13] focused on the construction of symmetric and nonsymmetric pentadiagonal matrices using input data consisting of minimum and maximum eigenvalues of each leading principal submatrix. Ghanbari and Moghadam [14] studied an inverse eigenvalue problem for symmetric pentadiagonal matrices using three sets of numbers, denoted as $(\lambda)_{i=1}^n$, $(\mu)_{i=1}^n$, and $(v)_{i=1}^n$. These sets of numbers possess the interlacing property, such that:

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \lambda_n < \mu_n,$$
$$\mu_1 < v_1 < \mu_2 < v_2 < \cdots < \mu_n < v_n.$$

The spectral data of the problem is defined as follows: $(\lambda)_{i=1}^n$ represents the eigenvalues of a pentadiagonal matrix called A, $(\mu)_{i=1}^n$ represents the eigenvalues of a matrix denoted as A^*, and $(v)_{i=1}^n$ represents the eigenvalues of a matrix referred to as A^{**}. The matrix A^*

is similar to A except for the element at position $(1,1)$, and the matrix A^{**} is similar to A^* except for the element at position $(2,2)$. Their algorithm involves computing the eigenvector and implementing the Lanczos algorithm, resulting in a time complexity of $O(n^3)$.

In this paper, we investigate an inverse eigenvalue problem called IEPSP, which focuses on constructing a matrix P_n with two pairs of eigenvalues from the required matrix and one eigenvalue from each of its leading principal submatrices. The definition of IEPSP is as follows:

IEPSP: Given a set

$$\{\lambda_1,\lambda_2,\cdots,\lambda_{n-1}\} \cup \{\lambda_n^{(1)},\lambda_n^{(2)}\} \tag{2}$$

of real numbers and two vectors

$$x = [x_1,\cdots,x_n]^T, \tag{3}$$

and

$$y = [y_1,\cdots,y_n]^T \tag{4}$$

with real entries, construct a symmetric pentadiagonal matrix P_n of size $n \times n$ such that $P_n x = \lambda_n^{(1)} x$, $P_n y = \lambda_n^{(2)} y$, and λ_i for $i = 1,2,\cdots,n-1$ is an eigenvalue of the ith leading principal submatrix of P_n. Inverse eigenvalue problems with non-negative spectra are an important class of inverse eigenvalue problems aiming to construct matrices with non-negative entries using spectral information. Chu and Golub [1] have introduced various types of problems in this category. Recent studies in this area include Nazari et al. [16], who identified conditions under which a set of positive numbers can be the eigenvalues of a symmetric matrix.

In this paper, we will also study non-negative IEPSP, which involves solving IEPSP in such a way that the required matrix will have non-negative entries.

The structure of the paper is as follows: Section 2 provides preliminary definitions and results. Section 3 presents the main results and the proposed algorithm. Section 4 includes a numerical example for validating the proposed algorithm. Finally, Section 5 concludes the paper.

2 Preliminary Investigation

In this section, we provide essential definitions and preliminary results. Throughout the paper, we denote the jth leading principal submatrix of P_n as P_j, and its characteristic polynomial as $\phi_j(\lambda) = \det(\lambda I_j - P_j)$, where $j = 1,2,\cdots,n$. Additionally, we use $\sigma(A)$ to represent the eigenvalues of matrix A. The following lemma presents the recursive relation of $\phi_j(\lambda)$.

Lemma 2.1. *[11] The recursive relation of $\phi_j(\lambda)$ is given by*

$$\begin{aligned}\phi_j(\lambda) = &\left((\lambda - a_j) - \frac{r_{j-2}}{b_{j-2}^2}\right)\phi_{j-1}(\lambda)\\ &- \left(b_{j-1}^2 - \frac{(\lambda - a_{j-1})r_{j-2}}{b_{j-2}^2}\right)\phi_{j-2}(\lambda)\\ &- \left(c_{j-2}^2(\lambda - a_{j-1}) - r_{j-2}\right)\phi_{j-3}(\lambda)\\ &+ c_{j-3}^2\left(c_{j-2}^2 - \frac{(\lambda - a_{j-2})r_{j-2}}{b_{j-2}^2}\right)\phi_{j-4}(\lambda)\\ &+ \left(\frac{c_{j-3}^2 c_{j-4}^2 r_{j-2}}{b_{j-2}^2}\right)\phi_{j-5}(\lambda),\end{aligned}$$

where $\phi_{-1}(\lambda) = 0$, $\phi_0(\lambda) = 1$, $\phi_1(\lambda) = a_1$, $\phi_2(\lambda) = (\lambda - a_1)(\lambda - a_2) - b_1^2$, and $r_i = -b_i b_{i+1} c_i$, for $i = 1, 2, \cdots, n-2$.

A symmetric tridiagonal matrix of order n is defined as follows:

$$T_n = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_2 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \beta_2 & \alpha_3 & \beta_3 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \beta_3 & \alpha_4 & \beta_4 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \beta_4 & \alpha_5 & \beta_5 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta_{n-4} & \alpha_{n-3} & \beta_{n-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \beta_{n-3} & \alpha_{n-2} & \beta_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{n-2} & \alpha_{n-1} & \beta_{n-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta_{n-1} & \alpha_n \end{bmatrix}, \tag{5}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ and $\beta_i \neq 0$. Let us denote the jth leading principal submatrix of T_n by T_j. As it is stated in Lemma 2.2, it is well known that T_j and T_{j+1}, $1 \leq j \leq n-1$, do not have a common eigenvalue.

Lemma 2.2. *No two successive leading principal submatrices of a real symmetric tridiagonal matrix share the same eigenvalue.*

Considering Lemma 2.2 and the fact that any pentadiagonal matrix can be expressed as the product of two tridiagonal matrices [12], Lemma 2.3 demonstrates that no two successive leading principal submatrices of a specific class of pentadiagonal matrices share a common eigenvalue.

Lemma 2.3. *For any pentadiagonal matrix P_n that can be expressed as the square of a real symmetric tridiagonal matrix, no two successive leading principal submatrices of P_n have a common eigenvalue.*

Proof. Let T_n be a real symmetric tridiagonal matrix such that $T_n^2 = P_n$. By considering the equality $\det(\lambda I_n - P_n) = \det(\lambda I_n - T_n)^2$, if there exists a number x such that $\det(xI_i - P_i) = \det(xI_{i+1} - P_{i+1}) = 0$, then we can immediately conclude $\det(xI_i - T_i) = \det(xI_{i+1} - T_{i+1}) = 0$, which contradicts Lemma 2.2. □

In the following section, the main results and the solution to the IEPSP are presented.

3 Solution to IEPSP

In this section, we focus on the main results. Lemma 3.2 demonstrates that in a pentadiagonal matrix, each component x_i of an eigenvector x is a linear combination of the components x_1 and x_2. To establish this relationship, we utilize two auxiliary matrices named M_i and N_i of size $(i-2) \times (i-2)$, where $i \geq 3$. The matrix M_i is constructed as follows:

1 Delete the second row and the $(i-1)$th column from the matrix $\lambda I_i - P_i$.

2 Remove the last row and column from the resulting matrix in step 1.

For instance, the matrix M_{10} is given by

$$M_{10} = \begin{bmatrix} \lambda - a_1 & -b_1 & -c_1 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & -b_2 & \lambda - a_3 & -b_3 & -c_3 & 0 & 0 & 0 \\ 0 & -c_2 & -b_3 & \lambda - a_4 & -b_4 & -c_4 & 0 & 0 \\ 0 & 0 & -c_3 & -b_4 & \lambda - a_5 & -b_5 & -c_5 & 0 \\ 0 & 0 & 0 & -c_4 & -b_5 & \lambda - a_6 & -b_6 & -c_6 \\ 0 & 0 & 0 & 0 & -c_5 & -b_6 & \lambda - a_7 & -b_7 \\ 0 & 0 & 0 & 0 & 0 & -c_6 & -b_7 & \lambda - a_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_7 & -b_8 \end{bmatrix}.$$

The matrix N_i is constructed as follows:

1 Delete the first row and the $(i-1)$th column from the matrix $\lambda I_i - P_i$.

2 Remove the last row and column from the resulting matrix in step 1.

For example, the matrix N_{10} is given by

$$N_{10} = \begin{bmatrix} -b_1 & \lambda - a_2 & -b_2 & -c_2 & 0 & 0 & 0 & 0 \\ -c_1 & -b_2 & \lambda - a_3 & -b_3 & -c_3 & 0 & 0 & 0 \\ 0 & -c_2 & -b_3 & \lambda - a_4 & -b_4 & -c_4 & 0 & 0 \\ 0 & 0 & -c_3 & -b_4 & \lambda - a_5 & -b_5 & -c_5 & 0 \\ 0 & 0 & 0 & -c_4 & -b_5 & \lambda - a_6 & -b_6 & -c_6 \\ 0 & 0 & 0 & 0 & -c_5 & -b_6 & \lambda - a_7 & -b_7 \\ 0 & 0 & 0 & 0 & 0 & -c_6 & -b_7 & \lambda - a_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_7 & -b_8 \end{bmatrix}.$$

The following lemma presents the recursive relation for determinant of the matrices M_i and N_i.

Lemma 3.1. *The recursive relation for the determinants of matrices M_i and N_i is given by:*

$$\det(M_i) = (\lambda - a_{i-2})c_{i-3}\det(M_{i-2}) - c_{i-4}^2 c_{i-3}c_{i-5}\det(M_{i-4}) \\ - b_{i-3}c_{i-3}c_{i-4}\det(M_{i-3}) - b_{i-2}\det(M_{i-1}),$$

and

$$\det(N_i) = (\lambda - a_{i-2})c_{i-3}\det(N_{i-2}) - c_{i-4}^2 c_{i-3}c_{i-5}\det(N_{i-4}) \\ - b_{i-3}c_{i-3}c_{i-4}\det(N_{i-3}) - b_{i-2}\det(N_{i-1}).$$

Proof. The proof follows by expanding the determinants along the last row of matrices M_i and N_i. □

Let (λ, x) be an eigenpair of P_n. From the equation $P_n x = \lambda x$, we can derive:

$$c_{i-2}x_i = (\lambda - a_{i-2})x_{i-2} - c_{i-4}x_{i-4} - b_{i-3}x_{i-3} - b_{i-2}x_{i-1} \quad \text{for } i = 3, 4, \cdots, n, \tag{6}$$

where $b_j = c_j = x_j = 0$ for $j \leq 0$. It is shown in the Lemma 3.2 that each component $x_i, i \geq 3$, of the eigenvector x can be expressed as a linear combination of x_1 and x_2.

Lemma 3.2. *If $x = (x_1, x_2, \cdots, x_n)^T$ and (λ, x) is an eigenpair of P_n, then $|x_1| + |x_2| > 0$. Furthermore, one can obtain each component x_j as follows:*

$$x_j = \frac{\det(M_j)x_1 + \det(N_j)x_2}{\prod_{i=1}^{j-2} c_i}, \quad j = 3, 4, \cdots, n. \tag{7}$$

Proof. The proof is done by induction on the x_j's. For the base case, i.e., x_3, we have:

$$M_3 = (\lambda - a_1), \\ N_3 = -b_1,$$

and by equation (6), we have:

$$x_3 = \frac{(\lambda - a_1)x_1 - b_1 x_2}{c_1} = \frac{\det(M_3)x_1 + \det(N_3)x_2}{c_1}.$$

Now, assuming the induction holds for $x_j, j = 3, 4, \cdots, i-1$, we prove it for x_i. Using equation (6), we have:

$$\begin{aligned} c_{i-2}x_i = &\frac{\lambda - a_{i-2}}{\prod_{j=1}^{i-4} c_j}(\det(M_{i-2})x_1 + \det(N_{i-2})x_2) \\ &- \frac{c_{i-4}}{\prod_{j=1}^{i-6} c_j}(\det(M_{i-4})x_1 + \det(N_{i-4})x_2) \\ &- \frac{b_{i-3}}{\prod_{j=1}^{i-5} c_j}(\det(M_{i-3})x_1 + \det(N_{i-3})x_2) \\ &- \frac{b_{i-2}}{\prod_{j=1}^{i-3} c_j}(\det(M_{i-1})x_1 + \det(N_{i-1})x_2). \end{aligned}$$

We rewrite it as follows:

$$
\begin{aligned}
c_{i-2}x_i = &\frac{\lambda - a_{i-2}}{\prod_{j=1}^{i-4} c_j}\det(M_{i-2})x_1 + \frac{\lambda - a_{i-2}}{\prod_{j=1}^{i-4} c_j}\det(N_{i-2})x_2 \\
&- \frac{c_{i-4}}{\prod_{j=1}^{i-6} c_j}\det(M_{i-4})x_1 - \frac{c_{i-4}}{\prod_{j=1}^{i-6} c_j}\det(N_{i-4})x_2 \\
&- \frac{b_{i-3}}{\prod_{j=1}^{i-5} c_j}\det(M_{i-3})x_1 - \frac{b_{i-3}}{\prod_{j=1}^{i-5} c_j}\det(N_{i-3})x_2 \\
&- \frac{b_{i-2}}{\prod_{j=1}^{i-3} c_j}\det(M_{i-1})x_1 - \frac{b_{i-2}}{\prod_{j=1}^{i-3} c_j}\det(N_{i-1})x_2.
\end{aligned} \tag{8}
$$

Equation (8) yields:

$$
\begin{aligned}
x_i = &\frac{(\lambda - a_{i-2})c_{i-3}}{\prod_{j=1}^{i-2} c_j}\det(M_{i-2})x_1 + \frac{(\lambda - a_{i-2})c_{i-3}}{\prod_{j=1}^{i-2} c_j}\det(N_{i-2})x_2 \\
&- \frac{c_{i-4}^2 c_{i-3}c_{i-5}}{\prod_{j=1}^{i-2} c_j}\det(M_{i-4})x_1 - \frac{c_{i-4}^2 c_{i-3}c_{i-5}}{\prod_{j=1}^{i-2} c_j}\det(N_{i-4})x_2 \\
&- \frac{b_{i-3}c_{i-3}c_{i-4}}{\prod_{j=1}^{i-2} c_j}\det(M_{i-3})x_1 - \frac{b_{i-3}c_{i-3}c_{i-4}}{\prod_{j=1}^{i-2} c_j}\det(N_{i-3})x_2 \\
&- \frac{b_{i-2}}{\prod_{j=1}^{i-2} c_j}\det(M_{i-1})x_1 - \frac{b_{i-2}}{\prod_{j=1}^{i-2} c_j}\det(N_{i-1})x_2.
\end{aligned} \tag{9}
$$

Based on the recursive relation of M_i and N_i in Lemma 3.1, it can be observed that (9) yields:

$$
x_i = \frac{\det(M_i)x_1 + \det(N_i)x_2}{\prod_{j=1}^{i-2} c_j}, \quad i = 3,4,\ldots,n.
$$

Now, if $x_1 = x_2 = 0$, then the eigenvector x will be zero, which contradicts the fact that x is an eigenvector. Therefore, $|x_1| + |x_2| > 0$, and the proof is complete. □

In the following theorem, the solution to the IEPSP and the sufficient conditions for its solvability are presented.

Theorem 3.3. *Let $\{\lambda_1, \cdots, \lambda_{n-1}, \lambda_n^{(1)}, \lambda_n^{(2)}\}$ be a set of numbers, and let $x = (x_1, x_2, \cdots, x_n)^T$ and $y = (y_1, y_2, \cdots, y_n)^T$ be vectors. The IEPSP has the following solution:*

$$
a_1 = \lambda_1, \quad a_2 = \frac{b_1^2}{a_1}, \tag{10}
$$

$$
\begin{aligned}
a_i = \frac{1}{\phi_{i-1}(\lambda_i)} \Bigg[& \left(\lambda_i - \frac{r_{i-2}}{b_{i-2}^2} \right) \phi_{i-1}(\lambda_i) \\
& - \left(b_{i-1}^2 - \frac{(\lambda_i - a_{i-1}) r_{i-2}}{b_{i-2}^2} \right) \phi_{i-2}(\lambda_i) \\
& - \left(c_{i-2}^2 (\lambda_i - a_{i-1}) - r_{i-2} \right) \phi_{i-3}(\lambda_i) \\
& + c_{i-3}^2 \left(c_{i-2}^2 - \frac{(\lambda_i - a_{i-2}) r_{i-2}}{b_{i-2}^2} \right) \phi_{i-4}(\lambda_i) \\
& + \left(\frac{c_{i-3}^2 c_{i-4}^2 r_{i-2}}{b_{i-2}^2} \right) \phi_{i-5}(\lambda_i) \Bigg],
\end{aligned} \tag{11}
$$

$$
b_i = \frac{\det\left(\begin{bmatrix} \lambda_n^{(1)} x_i - c_{i-2} x_{i-2} - b_{i-1} x_{i-1} - a_i x_i & x_{i+2} \\ \lambda_n^{(2)} y_i - c_{i-2} y_{i-2} - b_{i-1} y_{i-1} - a_i y_i & y_{i+2} \end{bmatrix} \right)}{\det\left(\begin{bmatrix} x_{i+1} & x_{i+2} \\ y_{i+1} & y_{i+2} \end{bmatrix} \right)}, i = 1, \cdots, n-2, \tag{12}
$$

$$
c_i = \frac{\det\left(\begin{bmatrix} x_{i+1} & \lambda_n^{(1)} x_i - c_{i-2} x_{i-2} - b_{i-1} x_{i-1} - a_i x_i \\ y_{i+1} & \lambda_n^{(2)} y_i - c_{i-2} y_{i-2} - b_{i-1} y_{i-1} - a_i y_i \end{bmatrix} \right)}{\det\left(\begin{bmatrix} x_{i+1} & x_{i+2} \\ y_{i+1} & y_{i+2} \end{bmatrix} \right)}, i = 1, \cdots, n-2, \tag{13}
$$

subject to the conditions:

$$
\lambda_i \neq \lambda_{i+1}, \quad i = 1, 2, \cdots, n-1, \tag{14}
$$

$$
x_i y_{i+1} \neq x_{i+1} y_i, \quad i = 1, 2, \cdots, n-1, \tag{15}
$$

$$
x_i \neq 0, \quad y_i \neq 0, \quad i = 1, 2, \cdots, n, \tag{16}
$$

$$
\lambda_i \notin \sigma(P_{i-1}), \tag{17}
$$

$$
\lambda_1 \neq 0.
$$

Proof. We consider the set $\{\lambda_1, \cdots, \lambda_{n-1}, \lambda_n^{(1)}, \lambda_n^{(2)}\}$ and the vectors $x = (x_1, x_2, \cdots, x_n)^T$ and $y = (y_1, y_2, \cdots, y_n)^T$ as the definitions (2), (3), and (4) of the IEPSP, respectively. Let the set $\{\lambda_1, \cdots, \lambda_{n-1}, \lambda_n^{(1)}, \lambda_n^{(2)}\}$ satisfy condition (14), and let the vectors $x = (x_1, x_2, \cdots, x_n)^T$ and $y = (y_1, y_2, \cdots, y_n)^T$ satisfy conditions (15) and (16). Now, we will prove the existence of a pentadiagonal matrix P_n with the described properties. We know that $\lambda_j \in \sigma(P_j)$, thus

$\phi_1(\lambda_1) = 0$ and $\phi_2(\lambda_2) = 0$, which in turn imply $a_1 = \lambda_1$ and $a_2 = \dfrac{b_1^2}{a_1}$. Again, by using $\phi_j(\lambda_j) = 0$ for $j = 3,4,\cdots,n$, we obtain the a_j as given in (11). By condition (17), $\phi_{j-1}(\lambda_j)$ is nonzero. Now, by solving the following system of equations, we can compute the values of the unknowns b_i, c_i for $i = 1,2,\cdots,n-2$:

$$\begin{cases} b_i x_{i+1} + c_i x_{i+2} = \lambda_n^{(1)} x_i - c_{i-2} x_{i-2} - b_{i-1} x_{i-1} - a_i x_i, \\ b_i y_{i+1} + c_i y_{i+2} = \lambda_n^{(2)} y_i - c_{i-2} y_{i-2} - b_{i-1} y_{i-1} - a_i y_i. \end{cases} \tag{18}$$

The system of equations (18) has a unique solution in terms of b_i and c_i if and only if condition (15) holds, and the unique solution is given by equations (12) and (13). The element b_{n-1} is also obtained using the following equation:

$$b_{n-1} = \frac{(\lambda_n^{(1)} - a_{n-1}) x_{n-1} - b_{n-2} x_{n-3} - c_{n-3} x_{n-4}}{x_n}. \tag{19}$$

In the following, we will prove that the obtained b_i and c_i are nonzero to satisfy the conditions of the matrix P_n. According to equation (7), we can express c_j as follows:

$$c_{j-2} = \frac{\det(M_j) x_1 + \det(N_j) x_2}{x_j \prod_{i=1}^{j-3} c_i}, \quad j = 3,4,\cdots,n.$$

Now, if condition (16) holds, the c_j's will be nonzero. To prove the nonzeroness of b_j, we first define the expression:

$$\begin{aligned} \chi_j(\lambda) = {} & \left(b_{j-2}^2(\lambda - a_j) - r_{j-2}\right)\phi_{j-1}(\lambda) \\ & - \left(b_{j-2}^2 b_{j-1}^2 - (\lambda - a_{j-1}) r_{j-2}\right)\phi_{j-2}(\lambda) \\ & - b_{j-2}^2\left(c_{j-2}^2(\lambda - a_{j-1}) - r_{j-2}\right)\phi_{j-3}(\lambda) \\ & + c_{j-3}^2\left(b_{j-2}^2 c_{j-2}^2 - (\lambda - a_{j-2}) r_{j-2}\right)\phi_{j-4}(\lambda) \\ & + \left(c_{j-3}^2 c_{j-4}^2 r_{j-2}\right)\phi_{j-5}(\lambda), \end{aligned}$$

and rewrite recursive equation of $\phi_j(\lambda)$ as follows:

$$\phi_j(\lambda) = \frac{\chi_j(\lambda)}{b_{j-2}^2}. \tag{20}$$

According to (20), the zeros of $\phi_j(\lambda)$ and $\chi_j(\lambda)$ are the same. It was assumed in (17) that $\phi_j(\lambda_{j+1}) \neq 0$, which implies $\chi_j(\lambda_{j+1}) \neq 0$. If $b_{j-2} = 0$, then $\phi_j(\lambda) b_{j-2}^2 = \chi_j(\lambda)$ for $\lambda = \lambda_{j+1}$ would lead to a contradiction, as the left side is zero while the right side is nonzero. Therefore, b_js must be nonzero.

□

Remark 3.4. *Considering that equation (19) is obtained from the equation $P_n x = \lambda_n^{(1)} x$, its value can also be computed based on the equation $P_n y = \lambda_n^{(2)} y$. Therefore, the IEPSP problem has at most two solutions.*

In the following theorem, sufficient conditions for the non-negativity solvability of IEPSP are provided.

Theorem 3.5. *The IEPSP has a non-negative solution if the conditions of Theorem 3.3 are satisfied, and additionally:*

$$\lambda_1 > 0, \tag{21}$$

and

$$\begin{aligned}&((\lambda_n^{(1)} + \lambda_n^{(2)} - a_i)L(x_i,y_i) - b_{i-1}L(x_{i-1},y_{i-1}) - c_{i-2}L(x_{i-2},y_{i-2})\\ &+ (\lambda_n^{(2)} y_{i+2}x_i - \lambda_n^{(1)} y_i x_{i+2}))L(x_{i+1},y_{i+1}) \geq 0, i = 1,\cdots,n-1,\end{aligned} \tag{22}$$

and

$$\begin{aligned}&((\lambda_n^{(1)} + \lambda_n^{(2)} - a_i)H(x_i,y_i) - b_{i-1}H(x_{i-1},y_{i-1}) - c_{i-2}H(x_{i-2},y_{i-2})\\ &+ (\lambda_n^{(2)} y_i x_{i+2} - \lambda_n^{(1)} y_{i+2}x_i))H(x_{i+2},y_{i+2}) \geq 0, i = 1,\cdots,n-2,\end{aligned} \tag{23}$$

where

$$L(x_k,y_k) = \det\left(\begin{bmatrix} y_k & x_k \\ y_{i+2} & x_{i+2}\end{bmatrix}\right), H(x_k,y_k) = \det\left(\begin{bmatrix} x_k & y_k \\ x_{i+1} & y_{i+1}\end{bmatrix}\right),$$

and

$$\begin{aligned}\phi_{i-1}(\lambda_i)\Bigg[&\det\left(\begin{bmatrix} 1 & \lambda_i - a_{i-1} \\ c_{i-2} & r_{i-2}\end{bmatrix}\right)\phi_{i-3}(\lambda_i) b_{i-2}^2\\ &- \det\left(\begin{bmatrix} 1 & \lambda_i - a_{i-1} \\ \phi_{i-2}(\lambda_i) & \phi_{i-1}(\lambda_i)\end{bmatrix}\right) r_{i-2}\\ &+ \det\left(\begin{bmatrix} \lambda_i & b_{i-1}^2 \\ \phi_{i-2}(\lambda_i) & \phi_{i-1}(\lambda_i)\end{bmatrix}\right) b_{i-2}^2\\ &+ \det\left(\begin{bmatrix} b_{i-2}^2 & \lambda_i - a_{i-2} \\ r_{i-2} & c_{i-2}^2\end{bmatrix}\right)\phi_{i-4}(\lambda_i) c_{i-3}^2\\ &+ c_{i-3}^2 c_{i-4}^2 r_{i-2}\phi_{i-5}(\lambda_i)\Bigg] > 0\end{aligned} \tag{24}$$

Proof. Given that $a_1 = \lambda_1$, it is obvious that condition (21) must hold. Let's assume that condition (22) is satisfied. This condition can be expressed as follows:

$$\begin{aligned}&((\lambda_n^{(1)} + \lambda_n^{(2)} - a_i)L(x_i,y_i) - b_{i-1}L(x_{i-1},y_{i-1}) - c_{i-2}L(x_{i-2},y_{i-2})\\ &+ (\lambda_n^{(2)} y_{i+2}x_i - \lambda_n^{(1)} y_i x_{i+2}))L(x_{i+1},y_{i+1})\\ &= (a_i(x_{i+2}y_i - x_i y_{i+2}) + b_{i-1}(x_{i+2}y_{i-1} - x_{i-1}y_{i+2}) + c_{i-2}(x_{i+2}y_{i-2} - x_{i-2}y_{i+2}) +\\ &\lambda_n^{(1)} x_i y_{i+2} - \lambda_n^{(2)} x_{i+2}y_i)L(x_{i+1},y_{i+1})\\ &= \det\left(\begin{bmatrix} \lambda_n^{(1)} x_i - c_{i-2}x_{i-2} - b_{i-1}x_{i-1} - a_i x_i & x_{i+2} \\ \lambda_n^{(2)} y_i - c_{i-2}y_{i-2} - b_{i-1}y_{i-1} - a_i y_i & y_{i+2}\end{bmatrix}\right)\det\left(\begin{bmatrix} x_{i+1} & x_{i+2} \\ y_{i+1} & y_{i+2}\end{bmatrix}\right).\end{aligned}$$

Considering the equation (12), it is obvious that if the last expression holds, the values of b_i will be non-negative. Equation (23) implies the following result:

$$
\begin{aligned}
&((\lambda_n^{(1)} + \lambda_n^{(2)} - a_i)M(x_i, y_i) - b_{i-1}M(x_{i-1}, y_{i-1}) - c_{i-2}M(x_{i-2}, y_{i-2}) \\
&+ (\lambda_n^{(2)} y_i x_{i+2} - \lambda_n^{(1)} y_{i+2} x_i))M(x_{i+2}, y_{i+2}) \\
&= (a_i(x_i y_{i+1} - x_{i+1} y_i) + b_{i-1}(x_{i-1} y_{i+1} - x_{i+1} y_{i-1}) + c_{i-2}(x_{i-2} y_{i+1} - x_{i+1} y_{i-2}) \\
&- \lambda_n^{(1)} x_i y_{i+1} + \lambda_n^{(2)} x_{i+1} y_i)M(x_{i+2}, y_{i+2}) \\
&= \det\left(\begin{bmatrix} x_{i+1} & \lambda_n^{(1)} x_i - c_{i-2}x_{i-2} - b_{i-1}x_{i-1} - a_i x \\ y_{i+1} & \lambda_n^{(2)} y_i - c_{i-2}y_{i-2} - b_{i-1}y_{i-1} - a_i y_i \end{bmatrix}\right) \det\left(\begin{bmatrix} x_{i+1} & x_{i+2} \\ y_{i+1} & y_{i+2} \end{bmatrix}\right).
\end{aligned}
$$

Considering the equation (13), it is evident that if the last expression holds, the values of c_i will be non-negative. The expression of (24) can also be written as follows:

$$
\begin{aligned}
&(\lambda_j b_{j-2} - r_{j-2})\phi_{j-1}(\lambda_j) \\
&- (b_{j-1}^2 b_{j-2}^2 - (\lambda_j - a_{j-1})r_{j-2})\phi_{j-2}(\lambda_j) \\
&- b_{j-2}^2(c_{j-2}^2(\lambda_j - a_{j-1}) - r_{j-2})\phi_{j-3}(\lambda_j) \\
&+ c_{j-3}^2(c_{j-2}^2 b_{j-2}^2 - (\lambda_j - a_{j-2})r_{j-2})\phi_{j-4}(\lambda_j) \\
&+ c_{j-3}^2 c_{j-4}^2 r_{j-2}\phi_{j-5}(\lambda_j).
\end{aligned}
$$

Based on equation (11), this expression is equal to

$$\phi_{j-1}^2(\lambda_j) b_{j-2}^2 a_j. \tag{25}$$

Since the terms $\phi_{j-1}(\lambda_j)$ and b_{j-2} appear squared in equation (25), it is evident that if the expression (24) is non-negative, then a_j for $j \geq 3$ are also non-negative. Furthermore, since $a_2 = b_1^2/a_1$, if the above conditions hold, a_2 is also non-negative, completing the proof. □

Now, using Theorem 3.3, we obtain the following algorithm. The algorithm takes inputs $\Lambda = \lambda_1, \cdots, \lambda_{n-1}, \lambda_n^{(1)}, \lambda_n^{(2)}$, $x = x_1, \cdots, x_n$, and $y = y_1, \cdots, y_n$, and generates the elements of P_n.

Algorithm 1 IEPSP Algorithm

input $\Lambda = \{\lambda_1, \cdots, \lambda_{n-1}, \lambda_n^{(1)}, \lambda_n^{(2)}\}$, $x = \{x_1, \cdots, x_n\}$, and $y = \{y_1, \cdots, y_n\}$

1: $a_1 = \lambda_1$, $c_{-1} = c_0 = b_{-1} = b_0 = x_{-1} = x_0 = y_{-1} = y_0 = 0$

2: **For** $j = 1$ **to** $j = n-2$

3: $$b_j = \frac{\det\left(\begin{bmatrix} \lambda_n^{(1)} x_j - c_{j-2}x_{j-2} - b_{j-1}x_{j-1} - a_j x_j & x_{j+2} \\ \lambda_n^{(2)} y_j - c_{j-2}y_{j-2} - b_{j-1}y_{j-1} - a_j y_j & y_{j+2} \end{bmatrix}\right)}{\det\left(\begin{bmatrix} x_{j+1} & x_{j+2} \\ y_{j+1} & y_{j+2} \end{bmatrix}\right)}$$

4: $$c_j = \frac{\det\left(\begin{bmatrix} x_{j+1} & \lambda_n^{(1)} x_j - c_{j-2}x_{j-2} - b_{j-1}x_{j-1} - a_j x_j \\ y_{j+1} & \lambda_n^{(2)} y_j - c_{j-2}y_{j-2} - b_{j-1}y_{j-1} - a_j y_j \end{bmatrix}\right)}{\det\left(\begin{bmatrix} x_{j+1} & x_{j+2} \\ y_{j+1} & y_{j+2} \end{bmatrix}\right)}$$

5:

$$\begin{aligned} a_{j+1} = \frac{1}{\phi_j(\lambda_{j+1})}\Bigg[&\left(\lambda_{j+1} - \frac{r_{j-1}}{b_{j-1}^2}\right)\phi_j(\lambda_{j+1}) \\ &- \left(b_j^2 - \frac{(\lambda_{j+1} - a_j)r_{j-1}}{b_{j-1}^2}\right)\phi_{j-1}(\lambda_{j+1}) \\ &- \left(c_{j-1}^2(\lambda_{j+1} - a_j) - r_{j-1}\right)\phi_{j-2}(\lambda_{j+1}) \\ &+ c_{j-2}^2\left(c_{j-1}^2 - \frac{(\lambda_{j+1} - a_{j-1})r_{j-1}}{b_{j-1}^2}\right)\phi_{j-3}(\lambda_{j+1}) \\ &+ \left(\frac{c_{j-2}^2 c_{j-3}^2 r_{j-1}}{b_{j-1}^2}\right)\phi_{j-4}(\lambda_{j+1})\Bigg]. \end{aligned}$$

6: **Repeat**

7: $b_{n-1} = \dfrac{(\lambda_n^{(1)} - a_{n-1})x_{n-1} - b_{n-2}x_{n-3} - c_{n-3}x_{n-4}}{x_n}$

8:

$$\begin{aligned} a_n = \frac{1}{\phi_{n-1}(\lambda_n^{(1)})}\Bigg[&\left(\lambda_n^{(1)} - \frac{r_{n-2}}{b_{n-2}^2}\right)\phi_{n-1}(\lambda_n^{(1)}) \\ &- \left(b_{n-1}^2 - \frac{(\lambda_n^{(1)} - a_{n-1})r_{n-2}}{b_{n-2}^2}\right)\phi_{n-2}(\lambda_n^{(1)}) \\ &- \left(c_{n-2}^2(\lambda_n^{(1)} - a_{n-1}) - r_{n-2}\right)\phi_{n-3}(\lambda_n^{(1)}) \\ &+ c_{n-3}^2\left(c_{n-2}^2 - \frac{(\lambda_n^{(1)} - a_{n-2})r_{n-2}}{b_{n-2}^2}\right)\phi_{n-4}(\lambda_n^{(1)}) \\ &+ \left(\frac{c_{n-3}^2 c_{n-4}^2 r_{n-2}}{b_{n-2}^2}\right)\phi_{n-5}(\lambda_n^{(1)})\Bigg] \end{aligned}$$

Considering the computation of the determinant of a symmetric matrix in the algorithm, its time complexity is given by

$$\sum_{i=1}^{n-2} O(i^3) \leq \sum_{i=1}^{n} O(n^3) = O(n^4).$$

4 Experimental Results

Example 4.1. *The Algorithm 1 is utilized to construct a pentadiagonal matrix with the following elements:*

$$a_i = 1 + e^{\sqrt{\frac{i}{10}}},$$
$$b_i = \sin(i) - \frac{i}{10},$$
$$c_i = \frac{\sin(i)}{i}.$$

The inputs to the algorithm consist of the eigenvalues:

$$\begin{aligned} \lambda_1 &= 2.371942, \quad \lambda_2 = 3.215605, \quad \lambda_3 = 1.932602, \quad \lambda_4 = 1.626050, \\ \lambda_5 &= 4.041036, \quad \lambda_6 = 1.875368, \quad \lambda_7 = 1.834516, \\ \lambda_8^{(1)} &= 0.880711, \quad \lambda_8^{(2)} = 1.639473, \end{aligned}$$

and the eigenvectors:

$$\begin{aligned} x &= \{-0.04820, 0.17503, -0.06880, -0.46205, -0.63351, -0.53618, -0.24449, -0.01383\}, \\ y &= \{0.83156, -0.38116, -0.38798, 0.02416, -0.05747, -0.07996, -0.04881, -0.00322\}, \end{aligned}$$

where x and y correspond to $\lambda_8^{(1)}$ and $\lambda_8^{(2)}$, respectively. After executing the algorithm, the resulting matrix P_8 is as follows:

$$\begin{bmatrix} 2.37194 & 0.74147 & 0.84147 & 0 & 0 & 0 & 0 & 0 \\ 0.74147 & 2.5639 & 0.70929 & 0.45464 & 0 & 0 & 0 & 0 \\ 0.84147 & 0.70929 & 2.7293 & -0.1588 & 0.04704 & 0 & 0 & 0 \\ 0 & 0.45464 & -0.15888 & 2.88223 & -1.1568 & -0.189201 & 0 & 0 \\ 0 & 0 & 0.04704 & -1.1568 & 3.02811 & -1.45892 & -0.19178 & 0 \\ 0 & 0 & 0 & -0.18920 & -1.45892 & 3.16972 & -0.87941 & -0.04656 \\ 0 & 0 & 0 & 0 & -0.19178 & -0.87941 & 3.30864 & -0.04301 \\ 0 & 0 & 0 & 0 & 0 & -0.04656 & -0.04301 & 3.44593 \end{bmatrix}.$$

To validate the obtained matrix, we calculate its spectral information.

$\sigma_1 = \{\mathbf{2.371942}\}$,
$\sigma_2 = \{1.720285, \mathbf{3.215605}\}$,
$\sigma_3 = \{4.092001, 1.640597, \mathbf{1.932602}\}$,
$\sigma_4 = \{4.112669, 3.045519, \mathbf{1.626050}, 1.763188\}$,
$\sigma_5 = \{4.222864, \mathbf{4.041036}, 1.498982, 1.666840, 2.145818\}$,
$\sigma_6 = \{4.839105, 4.108931, 3.256661, 1.021721, 1.643471, \mathbf{1.875368}\}$,
$\sigma_7 = \{4.941708, 4.119229, 3.812395, 0.881202, 2.825357, 1.639492, \mathbf{1.834516}\}$,
$\sigma_8 = \{\mathbf{0.880711}, \mathbf{1.639473}, 1.834331, 4.941927, 2.822326, 4.119230, 3.812971, 3.448865\}$.

The following equality can be obtained easily:

$$P_8 x = (0.880711)x$$
$$P_8 y = (1.639473)y.$$

Example 4.2. *In this example, we construct matrix P_8 with the following eigenvalues:*

$\lambda_1 = -17$, $\lambda_2 = 16$, $\lambda_3 = 18$, $\lambda_4 = 2$, $\lambda_5 = 13$, $\lambda_6 = 14$, $\lambda_7 = -10$, $\lambda_8^{(1)} = 17$, $\lambda_8^{(2)} = 13$,

and eigenvectors:

$$x = \{-4, 4, 1, 8, -5, 11, 8, 8\},$$
$$y = \{-3, -5, 14, 8, 17, -3, 18, -16\},$$

After executing the algorithm, the resulting matrix P_8 is as follows:

$$\begin{bmatrix} -17 & -29.7377 & -17.0492 & 0 & 0 & 0 & 0 & 0 \\ -29.7377 & -10.7979 & -15.4187 & 0.9574 & 0 & 0 & 0 & 0 \\ -17.0492 & -15.4187 & 9.4485 & -2.1312 & -3.6159 & 0 & 0 & 0 \\ 0 & 0.9574 & -2.1312 & 2.3435 & 9.6819 & 14.9057 & 0 & 0 \\ 0 & 0 & -3.6159 & 9.6819 & 9.5975 & -10.1054 & 0.03846 & 0 \\ 0 & 0 & 0 & 14.9057 & -10.1054 & -3.5097 & 3.3730 & 3.6061 \\ 0 & 0 & 0 & 0 & 0.03846 & 3.3730 & -10.2317 & -26.7272 \\ 0 & 0 & 0 & 0 & 0 & 3.6061 & -26.7272 & -9.2532 \end{bmatrix}.$$

To validate the obtained matrix, we calculate its spectral information.

$$
\begin{aligned}
&\sigma_1 = \{\mathbf{-17}\},\\
&\sigma_2 = \{-43.7979, \mathbf{16}\},\\
&\sigma_3 = \{-51.3471, 15.9992, \mathbf{18}\},\\
&\sigma_4 = \{-51.3497, 16.0268, 18.3163, \mathbf{2}\},\\
&\sigma_5 = \{-51.4943, -4.3782, 21.3144, 16.0298, \mathbf{13}\},\\
&\sigma_6 = \{-52.1232, -21.95079, 21.3149, 16.1038, 12.9895, \mathbf{14}\},\\
&\sigma_7 = \{-53.7421, -22.4184, \mathbf{-10}, 21.3146, 16.1128, 12.9911, 14.2273\},\\
&\sigma_8 = \{-52.3756, -37.4838, -21.1869, 21.3150, \mathbf{13}, 14.2250, 16.1123, \mathbf{17}\}.
\end{aligned}
$$

5 Conclusion

In this paper, the inverse eigenvalue problem of pentadiagonal matrices is addressed by utilizing two pairs of eigenvalues from the required matrix and one eigenvalue from each of its leading principal submatrices. The sufficient conditions for the solvability of the problem are established, and a numerical algorithm for constructing the matrix is obtained. The problem is also studied for the case where the elements of the constructed matrix are non-negative.

The obtained solution is based on a linear combination relationship among the elements of the eigenvectors of the pentadiagonal matrix, derived in Lemma 3.2. Finally, several numerical examples are presented to validate the results.

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Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declare that they have no conflict of interest.

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