Journal of Discrete Mathematics and Its Applications 10 (1) (2025) 21–42

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Review Paper

Definition and investigation of moduloid over an ordinal nexus: A review of some fuzzy concepts

Esmaeil Nahidi*, Hojjat Babaei, Abbas Hasankhani

Department of Mathematics, Kerman Branch, Islamic Azad University, Kerman, I. R. Iran

Academic Editor: Matthias Dehmer

Abstract. While ordinal numbers facilitate the comparison between two infinite addresses, no studies have so far defined and investigated the use of algebraic space structures over an ordinal nexus. Here, the notions of moduloid over ordinal nexus and homomorphism between two Γ- moduloids are defined and some relations between moduloid and ordinal nexus are investigated. Moreover, some of these concepts are fuzzified. By defining the fuzzy subnexuses over a nexus N, it is shown that if S (i.e., a nonempty subset of N) is a meet closed subset then N is finite. Accordingly, the present study provides insights into the notions of N^{∞}, moduloid and its subsets, moduloid over cyclic nexuses and its subsets, along with supremum of two addresses over ordinal nexuses.

Keywords. Γ-moduloid ordinal nexus, γ-moduloid homomorphism, fuzzy moduloid.
Mathematics Subject Classification (2020): $03E72$.

1 Introduction

The terms formex and plenix were initially presented by H. Nooshin in 1970s (H. Nooshin (1975) [13]), performing research on the convenient generation of information in order to design and analyze space structures. These structures were complex and comprised thousands of elements. With regard to the geometry, one can state that the space structure consists of many different types of symmetries. Accordingly, it is possible to simplify the generation of data using the space structures with different symmetries.

*Corresponding author (Email address: Nahidiyass@yahoo.com).
Received 03 September 2024 ; Revised 24 December 2024 ; Accepted 24 December 2024
First Publish Date: 01 March 2025

As a necessary first step to use the space structures, formex algebra concepts were initially introduced (H. Nooshin (1975) [13], H. Nooshin (1984) [14]). Different types of the geometric forms can be processed and algebraically represented by these concepts. Particularly, structural configurations have been well established on this basis (H. Nooshin and P. Disney (2000) [15], H. Nooshin and P. Disney (2001) [16], H. Nooshin (2002) [17]). In this regard, formex algebra was utilized in a user-friendly environment using Formian software (H. Nooshin and C. Yamamoto (1993) [18]). Subsequently, handling of large amounts of data required to define a space structure became possible through using the plenix, being a mathematical object or an advanced form of database (M. Haristchain and H. Nooshin (1980) [10], IN. Hee and H. Nooshin (1985) [11]). This term was rooted from the word plenus, which means full and reflects the capability of a plenix to show mathematical objects in a full spectrum. In fact, both explicit constant and generic forms of information can be represented by a plenix. In other words, any type of information can be contained in a plenix as a parametric formulation. It is worth noting that the early studies chiefly considered plenices as data structures.

Currently, a plenix has a generic database nature that can uniquely outperform any other normal database. This has been achieved in the early 2000s when M. Bolourian was able to consider the basic idea of a plenix as a mathematical object, comprising an arrangement of mathematical objects (M. Bolourian (2009) [4] and M. Bolourian [5] and H. Nooshin (2004) [6]). Notably, two numbers (a vector and a matrix), three sets and a Boolean entity can be arranged in a plenix, according to Figure 1.

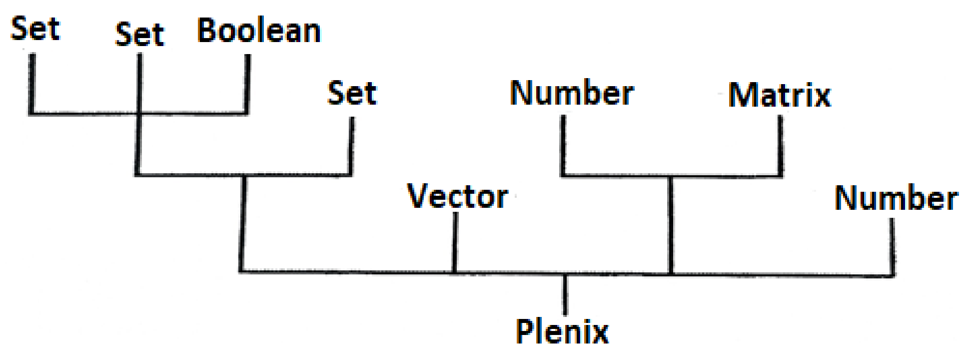

Figure 1. A schematic representation of a plenix. Generally, the representation of a plenix as a database goes beyond the normal database, involving a mathematical object that consists of an arrangement of mathematical objects. Notably, a plenix may contain an array of two numbers, a vector, a matrix, three sets, and a Boolean entity.

Efforts have been made to establish algebra on the basis of plenices, thereby meaningfully describing their relations, functions and operations, while also changing properties of the resultant algebra. Therefore, it has been possible to transform the plenix concept into an appropriate mathematical system, providing potential applications for different branches of human knowledge.

When it comes to the theory of plenices, one should also take into account the concept of an address, which plays a crucial role in the algebra properties. By representing the position of an element as a sequence of integers, an address can be simply defined in a plenix. Interestingly, it has been found that an address allows for a simple representation of the structure of a plenix. In this way, one can describe the formation of a plenix by using a set of addresses. Moreover, different reflections can be constituted on the basis of the address set in the plenix. To this end, a graphical method has been employed in order to represent both the plenix constitution and the address set. The graphical method is capable of showing a dendrogram of the plenix in a treelike graphical object, being highly effective in helping to visualize the problems. A nexus is also defined, being indicative of the mathematical object to correspond to the plenix constitution (M. Bolourian (2009) [4]). The mathematical structure can be realized in nexus algebra, as explained by Bolourian (M. Bolourian (2009) [4]).

From a general application standpoint, this idea was developed into a mathematical object, according to the literature (D. Afkhami Taba, A. Hasankhani, M. Bolourian (2012) [1], D. Afkhami, N. Ahmadkhah, A. Hasankhani (2011) [2], A. A. Estaji, T. Haghdadi, J. Farokhi(2015) [9], M. Haristchain(1980) [10], H. Hedayati and A. Asadi(2014) [12], A. Saeidi Rashkolia and A. Hasankhani(2011) [19], A. Saeidi(2009) [20], L. Torkzadeh and A. Hasankhani (2009) [21]). Later, Estaji was able to define the notion of nexuses over an ordinal (A. A. Estaji and A. As. Estaji (2015) [8]), being the generalization of a nexus. To supply address in infinite modes, it is not possible to use an address with finite number of elements. In this case, addresses defined over ordinals should be utilized. As an advantage, ordinal numbers are useful to compare two infinite addresses with each other. Nevertheless, no studies have so far defined and investigated the use of algebraic space structures over an ordinal nexus, according to the best of our knowledge.

In this paper, a moduloid structure is newly defined over an ordinal nexus. Moreover, the relationships between subnexuses and γ-moduloids are investigated. The action of a γ-moduloid homomorphism on the level of an address is also studied. Finally, some fuzzy concepts are reviewed.

2 Preliminaries

Definition 2.1. *A groupoid is defined as a set closed under a binary operation. This binary operation on an infinite set G is a function such that forms $* : G \times G$ into G that assigns a unique member such as C of G to each member (a, b) of $G \times G$, satisfying the following conditions:*

1. *The binary operation is defined on its entire domain i.e. $G \times G$*
2. *The binary operation $*$ is a well-defined function from $G \times G$ to G, assigning a single element of G to each member of $G \times G$.*
3. *The result of combining two members (a, b) under a binary operation must belong to G. In other words, the set G is closed with respect to its binary operation.*

4. *The binary operation that leads to the combination of both members of the infinite set G is usually represented by $*$ or $\circ$.*

- A semigroup G is defined as a groupoid with a binary operation $\bullet$, satisfying the associative property,

$$(a \bullet b) \bullet c = a \bullet (b \bullet c), \forall a, b, c \in G$$

- A monoid G is defined as a semigroup with an identity element.

Definition 2.2. *A semiring is defined as a set R with two operations + and $\bullet$, such that (R, +), representing a commutative monoid and $(R, \bullet)$ indicating a semigroup. The operation $\bullet$ is distributive with respect to +. In other words, one can have the following relations:*

$$a \bullet (b + c) = (a \bullet b) + (a \bullet c), \forall a, b, c \in R$$

$$(b + c) \bullet a = (b \bullet a) + (c \bullet a), \forall a, b, c \in R$$

$$0 \bullet a = a \bullet 0 = 0, \forall a \in R$$

where 0 *represents the identity element of monoid (R,+).*

Definition 2.3. *A moduloid M over the semiring R comprises a commutative groupoid (M, +), having an identity element and operation $\bullet : R \times M \longrightarrow M$. This is called scalar multiplication. As well, for all r, s in R, and a, b in M, the equations given below are valid:*

(i) $[(r + s) \bullet a = (r \bullet a) + (s \bullet a)$;

(ii) $r \bullet (a + b) = (r \bullet a) + (r \bullet b)$;

(iii) $(rs) \bullet a = r \bullet (sa)$;

(iv) $0 \bullet a = r \bullet 0 = 0$.

When $r \in R$ so that $r \bullet a = a$, M is then called unitary moduloid over R. For simplicity, on can consider $(M, +, \bullet)$ as a R-moduloid.

Definition 2.4. *An address is defined as a sequence of $\mathbb{N}^* = \mathbb{N} \bigcup \{0\}$, so that $a_k = 0$ implies $a_i = 0$, for all $i \geq k$, where, $\mathbb{N}^*$ is the set of all non-negative integers, that is, $\mathbb{N}^* = \mathbb{N} \bigcup \{0\}$.*

Definition 2.5. *A typical finite address is represented by: $(a_1, a_2, ..., a_n, 0, 0, ...)$ where a_i and n belongs to $\mathbb{N}$. Hereafter, this address is denoted by:$(a_1, a_2, ..., a_n)$ and called a finite address.*

Definition 2.6. *The sequence of all zeros is called the empty address denoted by ( ).*

Definition 2.7. *A nexus N is a nonempty set of addresses where a finite address is written as follows:*

$$(a_1, a_2, ..., a_{n-1}, a_n) \in N \Rightarrow (a_1, a_2, ..., a_{n-1}, t) \in N, (\forall t)(0 \leq t \leq a_n) \ (i)$$

For an infinite address, one can write the following expression:

$$\{a_i\}_{i=1}^{\infty}, a_i \in N \Rightarrow \forall n \in N, (a_1, a_2, ..., a_n) \in N \ (ii)$$

Note that condition (ii) does imply condition (i).
For example, the set $N = \{(\ ), (1), (2), (1,1), (1,2), (2,1), (2,2), (2,3), (2,4)\}$ is a nexus.

Definition 2.8. *Let N be a nexus. A nonempty subset S of N is called a subnexus of N, indicating that S itself is a nexus.*

Definition 2.9. *Let $\theta \neq A \subseteq N$. Accordingly, the smallest subnexus of N containing A is called the subnexus of N generated by A and is denoted by $< A >$. If $A = \{a_1, a_2, ..., a_n\}$, then, instead of $< A >$, one can write $\langle a_1, a_2, ..., a_n \rangle$. If A contains only one element a, then the subnexus $< A >$ is called a cyclic subnexus of N. It is clear that $\{(\ )\}$ and N are the obvious subnexus of the nexus N.*

Definition 2.10. *If $a = (a_1, a_2, ..., a_n), a_n \neq 0$, for some $n \in \mathbb{N}$, then a is said to be of level n. The level of a is denoted by $l(a)$.*
- If a is an infinite sequence of N, then a is said to be of level ∞.
- If $a = (\ )$, then a is said to be of level 0 (zero).

Definition 2.11. *The highest level of M elements is referred to as the rise of M and is represented by rise(M). Especially, the highest level of the elements of a nexus N can be referred to as the rise of a nexus N and represented by rise(N).*

Definition 2.12. *Let N be a nexus and let $a = (a_1, a_2, ..., a_n)$ be an address of N. The first term a_1 is said to be the stem of a and is denoted by stem(a).*

Definition 2.13. *Let $a = \{a_i\}, i \in \mathbb{N}$ and $b = \{b_i\}, i \in \mathbb{N}$ be two addresses. Then $a \leq b$ if $l(a) = 0$ or if one of the cases given below is satisfied:*
Case 1. If $l(a) = 1$, that is $w = (a_1), \forall a_1 \in N$, and $a_1 \leq b_1$.
Case 2. If $1 \leq l(a) \leq \infty$, then $l(a) \leq l(b)$ and $a_{l(a)} \leq b_{l(b)}$ and for any $1 \leq i \leq l(w)$, $a_i = b_i$.
Case 3. If $l(a) = \infty$, then $a = b$.

Definition 2.14. *Let N be a nexus and let $a = (a_1, a_2, ...)$ and $b = (b_1, b_2, ...)$ be two addresses of N. Now, the operation + is defined on N as follows:*
If there exists k so that:

$$(a_1 \vee b_1, a_2 \vee b_2, ..., a_k \vee b_k) \in N$$

But

$$(a_1 \vee b_1, a_2 \vee b_2, ..., a_{k+1} \vee b_{k+1}) \notin N$$

Then

$$a + b = (a_1 \vee b_1, a_2 \vee b_2, ..., a_k \vee b_k)$$

In this case, it can be stated that $index(a, b) = k$ if there is no such a k, then

$$a + b = (a_1 \vee b_1, a_2 \vee b_2, ...).$$

In this case, it can be stated that $index(a, b) = \infty$. Nevertheless, it should be noted that $(a_1 \vee b_1) \in N$ is always true.

Remark 2.15. *We have:*
(i) Generally, + is not associative.
(ii) It is possible that $a \leq b$ but $a + c \geq b + c$, for some a, b and c in a nexus.

Consider the following examples:

Example 2.16. *Suppose that,*

$$N = \{(\ ),(1),(2),(3),(1,1),(1,2),(2,1),(3,1)\}$$

Now, consider the addresses

$$a = (1,2)\ \ ,\ \ b = (2,1)\ \ ,\ \ c = (3,1).$$

Then

$$(a+b)+c = ((1,2)+(2,1))+(3,1) = (2)+(3,1) = (3,1)$$

On the other hand,

$$a+(b+c) = (1,2)+(3,1) = (3).$$

Example 2.17. *Consider the nexus N whose generators are the addresses $(1,2,3,9)$ and $(2,3,4,8)$, namely,*

$$N =< (1,2,3,9),(2,3,4,8) >$$

Suppose that,

$$a = (1,2,3,8), \quad b = (1,2,3,9), \quad c = (2,3,4,7).$$

As can be seen: $a \leq b$ but,

$$a+c = (2,3,4,8)\ \ and\ \ b+c = (2,3,4).$$

Therefore,

$$a+c \geq b+c$$

Definition 2.18. *Let $\mathbb{N}^{\infty} = \mathbb{N} \bigcup \{0,\infty\}$, N be a nexus. The scalar multiplication*

$$\circ : \mathbb{N}^{\infty} \times N \longrightarrow N$$

Is defined on N as follows:

$$r \circ a = \begin{cases} (a_1,a_2,...,a_r)\ 0 \leq r \leq l(a) \\ (a_1,a_2,...,a_n)\ l(a) \leq r, r \rangle 0 \\ 0\ r = 0 \\ a\ r = \infty \end{cases}$$

For all $r \in \mathbb{N}^{\infty}$, $a \in N$.

Remark 2.19. *Let N be nexus. Then $(N,+,\circ)$ is unitary moduloid over $(\mathbb{N}^{\infty},\vee,\wedge,0)$.*

Definition 2.20. *Let N be a $\mathbb{N}^{\infty}$-moduloid, S be a non-empty subset of N and $0 \in S$. Then S is called $\mathbb{N}^{\infty}$-moduloid of N, if $(S,+,0)$ is a moduloid over $(\mathbb{N}^{\infty},\vee,\wedge,0)$. The set of all $\mathbb{N}^{\infty}$-moduloid of N is denoted by $SUB_M^{\mathbb{N}^{\infty}}(N)$.*

Definition 2.21. *Let S be a nonempty subset of a nexus N. Then $S \in SUB_M^{\mathbb{N}^{\infty}}(N)$ if and only if*

1) $r \circ a \in S$, $\forall r \in \mathbb{N}^{\infty}$, $\forall a \in S$,

2) $a + b \in S$, $\forall a,b \in S$

Remark 2.22. *We have:*
(i) In general, a subnexus of a nexuse is not a $\mathbb{N}^{\infty}$-moduloid.
(ii) In general, a $\mathbb{N}^{\infty}$-moduloid of a nexus is not a subnexus.

Example 2.23. *Consider the nexus*

$$N = \{(\ ),(1),(2),(1,1),(1,2),(1,3),(2,1),(2,2)\}$$

and

$$S = \{(\ ),(1),(2),(1,1),(1,2)\},$$

S is a subnexus of N but if N is considered as a moduloid, then S is not a $\mathbb{N}^{\infty}$-moduloid of N, because $(1,2)$ and (2) belong to S, but

$$(1,2) + (2) = (2,2) \notin S$$

Clearly, each subnexus S of N is closed under dot product, that is,

$$r \circ w \in S,\ \forall r \in N,\ w \in S.$$

Example 2.24. *Consider the nexus $N = <(3,2),(2,2)>$ and the subset $S = \{(\ ),(1),(2),(3),(3,2)\}$ of N.*

It is easy to check whether, S is a $\mathbb{N}^{\infty}$-submoduloid of N (closed under addition and dot product), but it is not a subnexus of N because S does not contain the address $(3,1)$.

Definition 2.25. *Let N be a nexus, $a \in N$. The cyclic subnexus $< a >$ is a $\mathbb{N}^{\infty}$-submoduloid of N. In particular, if N is a cyclic nexus, then every $\mathbb{N}^{\infty}$-submoduloid of N is a $\mathbb{N}^{\infty}$-submoduloid. Generally, it should be noted that every $\mathbb{N}^{\infty}$-submoduloid of cyclic nexus N is not a subnexus of N.*

Example 2.26. *Consider the cyclic nexus*

$$<(2,3,2)>= \{0,(1),(2),(2,1),(2,2),(2,3),(2,3,1),(2,3,2)\}$$

the subset $\{(2),(2,3),(2,3,2)\}$ is a $\mathbb{N}^{\infty}$-submoduloid of $<(2,3,2)>$ but it is not a subnexus.

3 Moduloid over an Ordinal Nexus

Definition 3.1. *The order relation on the hypothetical set S uses the symbol* $\langle''"$ *and has the following two properties:*

1) if $x, y \in S \Rightarrow x\langle y$ *or* $x = y$ *or* $x\rangle y$

2) if $x, y, z \in S, x\langle y, y\langle z \Rightarrow x\langle z$

Definition 3.2. *An ordered set is a set as S on which the order relation is defined.*
For example, $(Z^+, \leq)$*is an order set because:* $\forall x, y \in Z^+, x \leq y$ *or* $y \leq x$.

Definition 3.3. *An ordered set A where for every* $x, y \in A$ *either* $x \leq y$ *or* $y \leq x$ *is said to be linearly ordered or totally ordered.*

Definition 3.4. *An ordered set A is said to be well-ordered if and only if whenever B is a non-empty subset of A, then B contains a minimum element.*

- Every well-ordered set is linearly ordered.

Definition 3.5. *Let*$(X, \leq)$ *be a well-ordered set, and* $a \in X$. *By determining the segment* X_a *of X it is meant that set* $X_a = \{ x \in X | \, x\langle a \}$.

Definition 3.6. *An ordinal number is a well-ordered set a where for all* $x \in \alpha, \alpha_x = x$.

- The collection of all ordinal numbers constitutes a proper class that is denoted by D.

- Let α *be an ordinal. If* $a \in \alpha$, *then* α_a *is an ordinal. Also, if* $Y \subseteq \alpha$ *is an ordinal, then* $Y = \alpha_x$, *for some* $a \in \alpha$.

- If α *and* β *are ordinals, then* $\alpha \bigcap \beta$ *is an ordinal.*

- Every well-ordered set is isomorphic to a unique ordinal.

Definition 3.7. *Consider* $(D, *)$, *function* $\varphi : D \longrightarrow D$ *is isomorphic if*

$$\forall \alpha, \beta \in D, \; \varphi(\alpha * \beta) = \varphi(\alpha) * \varphi(\beta).$$

Definition 3.8. *It is common in contemporary set theory to reserve lower-case Greek letters* $\alpha, \beta, \ldots$ *to denote ordinals.*

- It is also customary to denote the order relation between ordinals by $\alpha\langle\beta$*instead of the two equivalent forms* $\alpha \subseteq \beta$, $\alpha \in \beta$ *though the latter is also fairly common.*

- If α *is an ordinal, then by definition we will have* $\alpha = \{ \beta \in \Delta | \, \beta\langle\alpha \}$, *That is, an ordinal represents the set of all smaller ordinals.*

- In general, if a is an ordinal, the next ordinal will be $\alpha \bigcup \{\alpha\}$. *It is customary to denote the first ordinal after* α *by* $\alpha + 1$, *which is called the (ordinal) successor of* α. *Thus* $\alpha + 1 = \alpha \bigcap \{\alpha\}$. *If* $\beta = \alpha + 1$, *then we define* $\beta - 1 = \alpha$.

- An ordinal number greater than 0, *which is not the successor of any other ordinal, is said to be a limit ordinal.*

- An ordinal that is the successor of another ordinal is called a successor ordinal or non-limit ordinal.

- If $\alpha, \beta \in D$, *then either* $\alpha < \beta$, $\beta < \alpha$ *or* $\alpha = \beta$

- If A is a set of ordinals, then $\bigcup A$ *is an ordinal.*

- If γ *is an ordinal, then* $(\gamma, \vee, \wedge, 0)$ *is a semiring where* 0 *is the least element of* γ.

- For undefined terms and notations.

Definition 3.9. *Let $\gamma,\delta \in D$, $\delta \geq 1$ and $\gamma \geq 1$. An address over γ is a function $a : \delta \longrightarrow \gamma$, such that $a(i) = 0$ implies that $a(j) = 0$ for all $j \geq i$.*
- The set of all addresses over γ is denoted by $A(\gamma)$.
- $a(0)$ is called the stem of a. Note that the stem of ( ) is 0.
- Let $a : \delta \to \gamma$be an address over γ. If for every $i \in \delta$, $a(i) = 0$, then it is called the empty address and denoted by ( ).
- If a is a non-empty address, then there exists a unique element $a \in \delta + 1$ such that for every $i \in \alpha$, $a(i) \neq 0$ and for every $a \leq i \in \delta$, $a(i) = 0$. We denote this address by $(a_i)_{i\in\alpha}$ where $a_i = a(i)$ for every $i \in \alpha$.

Remark 3.10. *Let $a = (a_i)_{i\in\alpha}$. For $\beta \leq \alpha$, $a|_\beta$, means $a|_\beta(i) = a(i), \forall i \in \beta$ and $a|_\beta(i) = 0, \forall i \in \gamma - \beta$. In other words $a|_\beta = (a_i)_{i\in\beta}$. Note that , $a|_0 = 0 = (\ )$.*

Definition 3.11. *Let $a : \delta \to \gamma$and $b : \beta \to \eta$be addresses and $\delta \leq \beta$. It can be said $a = b$, if for every $i \in \delta$, $a_i = b_i$; and for every $i \in \beta - \gamma$, $b_i = 0$. In other words, there exists a unique element $\lambda \in D$ such that $a = (a_i)_{i\in\lambda} = b$.*

Definition 3.12. *Let $a : \delta \longrightarrow \gamma$ and $b : \beta \longrightarrow \eta$ be two addresses, then supremum a, b are defined as follows:*

$$a \vee b : \eta \longrightarrow \gamma$$

$$(a \vee b)(i) = \begin{cases} a(i) \vee b(i) \ i \in \delta \\ b(i) \ i \in \eta - \delta \end{cases}$$

Definition 3.13. *The level of $a \in A(\gamma)$ is said to be:*

1) $0\ a = 0$

2) $\beta\ (\ ) \neq a = (a_i)_{i\in\beta}$

The level of a is denoted by $l(a)$.

Example 3.14. *Let $a = (a_i)_{i\in 2\omega}$. The level of a is 2ω, $l(a) = 2\omega$.*

Definition 3.15. *Let a, b be two elements of $A(\gamma)$. Then we say $a \leq b$ if $l(a) = 0$ or one of the following cases satisfies for $a = (a_i)_{i\in\beta}$and $b = (b_i)_{i\in\delta}$:*
1. If $\beta = 1$ $a_0 \leq b_0$.
2. If $\beta \geq 2$ is a non-limit ordinal, then $a|_{\beta-1} = b|_{\beta-1}$and $a_{\beta-1} \leq b_{\beta-1}$.
3. If β is a limit ordinal, then $a = b|_\beta$.

Example 3.16. *Let $a = (a_i)_{i\in\omega}$ and $b = (b_i)_{i\in 2\omega}$. We define $a_i = i + 1$, for all $i \in 2\omega$. Therefore, $a \leq b$. Also, let $a = (a_i)_{i\in 5}$ and $b = (b_i)_{i\in 8}$ such that, $a_i = i + 1$, $b_i = i + 1$, for all $i \in 4 = \{0,1,2,3\}$ and $a_4 = 3\langle 4 = b_4$. Thus, $a \leq b$.*

Definition 3.17. *Let $( ) \neq a = (a_i)_{i\in\beta}$ be an element of $A(\gamma)$. For every $\delta \in \beta$ and $0 \leq j \leq a_\delta$, we put $a^{(\delta,j)} : \beta + 1 \longrightarrow \gamma$ so that, for every $i \in \beta + 1$,*

$$a_i^{(\delta,j)} = \begin{cases} a_j \; i \in \delta \\ j \; i = \delta \\ 0 \; otherwise \end{cases}$$

In other words, $a_i^{(\delta,j)} = (a_i)_{i\in\delta+1}$, where, $a_\delta = j$, $a_\lambda = a_i$ for all $\lambda \in \delta$.

Remark 3.18. *If δ in non-limit ordinal, then, $a^{(\delta,0)} = a^{(\delta-1,a_{\delta-1})} = (a_i)_{i\in\delta}$. Clearly,$a^{(0,0)} = ( )$.*

Example 3.19. *Let $a = (a_i)_{i\in 2\omega}$ then $a^{(\omega,j)} : \omega + 1 \longrightarrow \gamma$ so that for every $i \in \omega$, $a^{(\omega,j)}(i) = a(i)$,$a^{(\omega,j)}(\omega) = j$*

Definition 3.20. *A nexus N over γ is a set of addresses with the following properties:*
1) $\theta \neq N \subseteq A(\gamma)$.
2) If $( ) \neq a = (a_i)_{i\in\beta} \in N$, then for every $\delta \in \beta$ and $0 \leq j \leq a_\delta$, $a^{(\delta,j)} \in N$
- Note that, this definition is basically a generalization of Definition 2.3. In fact, every nexus is a nexus over ω.
- A nexus defined over an ordinal is called an ordinal nexus.

Example 3.21. *Let $a = (a_i)_{i\in\omega}$ so that, for all $i \in \omega$, $a_i = 1$, therefore $N = \{ a|_\lambda \, | \lambda \in \omega \}$ is a nexus over ω. In other words, $N = \{(1),(1,1),(1,1,1,...)\}$.*

Example 3.22. *Let $a = (a_i)_{i\in 2\omega} \in A(\gamma)$. Then $N = \{ a^{(\delta,j)} \Big| \forall\delta \in 2\omega, 0 \leq j \leq a_\delta\}$ is a nexus.*

Theorem 3.23. *Every non-empty address inducing a nexus, is denoted by $\langle a\rangle^\gamma$. We have:*
(i) If β is a limit ordinal, then $\langle a\rangle^\gamma = \{ a^{(s,j)} \Big| \delta \in \beta, 0 \leq j \leq a_\delta\} \bigcup \{a\}$.
(ii) If β is a non-limit ordinal, then $\langle a\rangle^\gamma = \{ a^{(s,j)} \Big| \delta \in \beta, 0 \leq j \leq a_\delta\}$.
(iii) $\langle a\rangle^\gamma = \{ b \in N | b \leq a\}$.

Proof. (i): Let $\theta \neq a = (a_i)_{i\in\beta} \in A(\gamma)$ and Let β be a limit ordinal. We show that $N = \{ a^{(s,j)} \Big| \delta \in \beta, 0 \leq j \leq a_\delta\} \bigcup \{a\}$ is a nexus over γ. Let $a^{(s,j)} \in N$ and $a^{(s,j)} = b = (b_i)_{i\in\delta+1}$ so that, $\forall\lambda \in \delta, b_i = a_i, b_\delta = j$, We show that, $\forall\delta' \in \delta + 1, (\forall j)(0 \leq j \leq b_{\delta'}), b^{(\delta',j')} \in N$ we have two cases:
case 1. $\delta' = \delta$, then

$$b^{(\delta',j')}(\lambda) = \begin{cases} b_\lambda \; \lambda \in \delta' = \delta \\ j' \; \lambda = \delta' = \delta \\ 0 \; otherwise \end{cases} = \begin{cases} a_\lambda \; \lambda \in \delta \\ j' \; \lambda = \delta \\ 0 \; otherwise \end{cases} = a^{(\delta,j')}(\lambda)$$

because $\delta \in \beta, 0 \leq j \leq a_\delta$. Hence, $b^{(\delta',j')} \in N$.
case 2. $\delta' \in \delta$ then

$$b^{(\delta',j')}(\lambda) = \begin{cases} b_\lambda \; \lambda \in \delta' \\ j' \; \lambda = \delta' \\ 0 \; otherwise \end{cases} = \begin{cases} a_\lambda \; \lambda \in \delta' \\ j' \; \lambda = \delta' \\ 0 \; otherwise \end{cases} = a^{(\delta,j')}(\lambda)$$

Hence, $b^{(\delta',j')} \in N$. Therefore N is a nexus.

(ii): If β is a non-limit ordinal, then according to Remark 3.18, $N = \{ a^{(s,j)} \Big| \delta \in \beta, 0 \leq j \leq a_\delta \}$ is a nexus.

(iii): It is obvious that $\langle a \rangle^\gamma \subseteq \{ b \in N | b \leq a \}$. Conversely, if $b \leq a$ and $b = (b_i)_{i\in\beta}$ we will have two cases,

case 1. If β is a non-limit ordinal, then $b|_{\beta-1} = a|_{\beta-1}$ and $b_{\beta-1} \leq a_{\beta-1}1$ 00-1: Thus, for all $i \in \gamma$

$$b_i = \begin{cases} a_i \; i \in \beta - 1 \\ b_{\beta-1} \; i = \beta - 1 \;\; 0 \\ 0 \; otherwise \end{cases}$$

Hence, for all $i \in \gamma$, $b_i = a_i^{(\beta-1,b_{\beta-1})}$ therefore $b \in \langle a \rangle^\gamma$.

case 2. If $\beta$is a limit ordinal, then $b = a|_\beta$. Thus, for all $i \in \gamma$

$$b_i = \begin{cases} a_i \; i \in \beta \\ 0 \; otherwise \end{cases}$$

Hence, for all $i \in \gamma$,$b_i = a_i^{(\beta,0)}$; therefore $b \in \langle a \rangle^\gamma$. □

Remark 3.24. *Let N be the set of addresses over γ. Then N is a nexus over γif and only if $\theta \neq N \subseteq A(\gamma)$ and for every $(a,b) \in N \times A(\gamma), b \leq a$, implies that $b \in N$.*

Definition 3.25. *Let N be a nexus over γ and $\theta \neq M \subseteq N$. M is called a subnexus of N, if M itself is a nexus over γ. The set of all subnexuses of N is denoted by $Sub(N)$. It is clear that ( ), N represents the trivial subnexuses of the nexus N.*

Remark 3.26. *If N is a nexus over γ and $\{M_i\}_{i\in I} \subseteq Sub(N)$, then $\bigcup_{i\in I} M_i \in Sub(N)$ and $\bigcap_{i\in I} M_i \in Sub(N)$.*

Definition 3.27. *Let N be a nexus over γ and $X \subseteq N$. The smallest subnexus of N containing X is called the subnexus generated by X and denoted by $< X >$. If $|X| = 1$, then $< X >$ is called a cyclic subnexus of N.*

4 Relation between Subnexuses and γ-Moduloid

Henceforth, γ is an ordinal number and N is a nexus over γ.

Definition 4.1. *Let $a = (a_i)_{i\in\beta}$ and $b = (b_i)_{i\in\delta}$ be two addresses of N. The operation $\oplus$ is defined as follows:*
If there exists $\eta \in \gamma$ so that $(a \vee b)|_\eta \in N$ $(a \vee b)|_{\eta+1} \notin N$, then $a \oplus b =: (a \vee b)|_\eta \in N$.
In this case, one may write $index_N^{(a,b)} = \eta$. If there is no such a η, then $index_N^{(a,b)} = \gamma$. In other words, we write $a \oplus b =: a \vee b$. In this case, β or δ is equal to γ.
However, note that always $(a_0 \vee b_0) \in N$, where a_0 and b_0 are the first terms of a and b respectively.

Remark 4.2. *For all non-empty addresses $a,b \in N$*

1) $a \oplus b = b \oplus a$

2) $a \oplus 0 = 0$

3) $l(a \oplus b) \leq l(a) \vee l(b)$

Example 4.3. *Let $a^i : \{0\} \to \gamma$ by $a^i(0) = i$ for all $i = 1,2,3$ and for $4 \leq i \leq 7$, $a^i : \{0,1\} \to \gamma$ by $a^4(j) = 1$, $a^5(j) = j+1$, $a^6(0) = 2$, $a^6(1) = 1$, $a^7(0) = 3$, and let $a^7(1) = 1$. Therefore, $a^5 \vee a^6(1) = 2$, $a^5 \vee a^6(0) = 2$, and $a^5 \oplus a^6 = a^5 \vee a^6 : \{0,1\} \to \gamma$, $N = \{(\ )\} \bigcup \{a^i | i = 1,2,...,7\}$. Also, $(a^5 \oplus a^6) = a^2$; therefore, $(a^5 \vee a^6)|_1 = a^2 \in N$, but $(a^5 \vee a^6)\big|_2 \notin N$, $l(a^5 \oplus a^6) = 1 \leq 2 = l(a^5) \vee l(a^6)$.*

Remark 4.4. *Let N be a nexus over γ, $\lambda \in \omega$ and $a = (a_i)_{i \in \lambda} \in N$. Clearly, we have*

$$a = (a_0, a_1, a_2, ..., a_{\lambda-1}).$$

For example, let $a^0, a^1, a^2, ..., a^7$be as defined in Example 4.3, then $a^0 = (a_i)_{i \in 0} = (\ )$, $a^1 = (a_i)_{i \in 1} = (1)$, $a^2 = (a_i)_{i \in 1} = (2)$, $a^3 = (a_i)_{i \in 1} = (3)$, $a^4 = (a_i)_{i \in 2} = (1,1)$, $a^5 = (a_i)_{i \in 2} = (1,2)$, $a^6 = (a_i)_{i \in 2} = (1,2)$,$a^7 = (a_i)_{i \in 2} = (3,1)$.

Lemma 4.5. *Let a and b be two addresses in N.*
(i) If $a \leq b$then $a \oplus b = b$.
(ii) In a cyclic nexus N, since every two addresses are comparable.
Then the summation of two addresses will be equal to the greater summaand.

Proof. (i). Suppose that $a = (a_i)_{i \in \beta}$. Since $a \leq b$, we have two cases:
Case 1: β is a non-limit ordinal, so $a|_{\beta-1} = b|_{\beta-1}$ and $a_{\beta-1} \leq b_{\beta-1}$, thus $(a \vee b)|_\beta = b|_\beta$.
Case 2: β is a limit ordinal, then $a|_\beta = b|_\beta = \alpha$, so $(a \vee b)|_\beta = b$ thus $a \oplus b = b$.
(ii). If N is cyclic, then $a \oplus b = a$ or $a \oplus b = b$. ☐

Definition 4.6. *The scalar multiplication (dot product) $\bullet$ is defined on as follows:*

$$\begin{aligned} &\bullet : \gamma \times N \longrightarrow N \\ &(a,b) \to a|_{\alpha \wedge \beta} = (a_i)_{i \in \alpha \wedge \beta} \end{aligned}$$

Where $a = (a_i)_{i \in \beta}$

Remark 4.7. *Let $a \in N$ and $\alpha \in \gamma$, then $l(\alpha \bullet a) \leq l(a)$.*

Lemma 4.8. *Let $a = (a_i)_{i \in \beta} \in N, \delta \in D$ and $\delta \leq \beta$. Then $\alpha \bullet (a|_\delta) = a|_{\alpha \wedge \beta}$*

Proof. According to Remark 3.10. $a|_\delta = (a_i)_{i \in \delta}$. Hence $\alpha \bullet (a|_\delta) = \alpha \bullet (a_i)_{i \in \delta} = (a_i)_{i \in \alpha \wedge \delta} = a|_{\alpha \wedge \beta}$. ☐

Theorem 4.9. $(N, \oplus, \bullet)$ *is moduloid over* $(\gamma, \vee, \wedge, 0)$*. For simplicity, Nis called, a γ-moduloid.*
We show that, the following properties are valid:

(i). $(\alpha \wedge \beta) \bullet a = \alpha \bullet (\beta \bullet a)$

(ii). $\alpha \bullet (a \oplus b) = (\alpha \bullet a) \oplus (\alpha \bullet b)$

(iii). $(\alpha \vee \beta) \bullet a = (\alpha \bullet a) \oplus (\alpha \bullet b)$

(iv). $0 \bullet a = a \bullet 0 = 0$

For all $\alpha, \beta \in \gamma$ and $a, b \in N$.

Proof. (i). Let $\alpha, \beta \in \gamma, a = (a_i)_{i \in \delta}$then $\alpha \bullet (\beta \bullet a) = \alpha \bullet (a|_{\beta \wedge \delta}) = a|_{\alpha \wedge (\beta \wedge \delta)} = a|_{(\alpha \wedge \beta) \wedge \delta} = (\alpha \wedge \beta) \bullet a$

(ii). Let $a = (a_i)_{i \in \beta}$ and $b = (b_i)_{i \in \delta}$ is be two elements of N. Without loss of generality, suppose that $\beta \leq \delta$. If $index_N^{(a,b)} = \eta$, then $\alpha \bullet (a \oplus b) = \alpha \bullet (a \vee b)|_\eta = (a \vee b)|_{\alpha \wedge \eta}$
On the other hand $\alpha \bullet a = a|_{\alpha \wedge \beta}$ and $\alpha \bullet b = b|_{\alpha \wedge \delta}$. Now, consider the following two cases:
Case 1: Let $\alpha \leq \beta \leq \delta$. If $\alpha \leq \eta$, then

$$\alpha \bullet a \oplus \alpha \bullet b = a|_{\alpha \wedge \beta} \oplus b|_{\alpha \wedge \delta} = a|_\alpha \oplus b|_\alpha\,(\sin ce \alpha \leq \eta) = (a \vee b)|_\alpha = (a \vee b)|_{\alpha \wedge \eta}$$

Now, if $\alpha \rangle \eta$, then

$$\alpha \bullet a \oplus \alpha \bullet b = a|_\alpha \oplus b|_\alpha\,(\sin ce \alpha \rangle \eta) = (a \vee b)_\eta (\sin ce index_N^{(a,b)} = \eta) = (a \vee b)_{\alpha \wedge \eta}$$

thus $\alpha \bullet a \oplus \alpha \bullet b = \alpha \bullet (a \oplus b)$.
Case 2: Let $\beta \leq \alpha \leq \gamma$, then

$$\alpha \bullet a = a|_{\alpha \wedge \beta} = a|_\beta = a$$

$$\alpha \bullet b = b|_{\alpha \wedge \delta} = b|_\alpha$$

If $\alpha \leq \eta$ that is, $\beta \leq \alpha \leq \eta$. Since $index_N^{(a,b)} = \eta$, then $\alpha \bullet a \oplus \alpha \bullet b = (a \oplus b)|_\alpha = (a \vee b)|_\alpha$.
On the other hand

$$\alpha \bullet (a \oplus b) = (a \vee b)|_{\eta \wedge \alpha}\,(\alpha \leq \eta) = (a \vee b)|_\alpha = (a \vee b) j$$

therefore, $\alpha \bullet (a \oplus b) = \alpha \bullet a \oplus \alpha \bullet b$.
Now, if $\alpha \rangle \eta$, then $\alpha \bullet a = a|_{\alpha \wedge \beta} = a|_\beta$ and $\alpha \bullet b = b|_{\alpha \wedge \delta} = b|_\alpha$. Therefore,

$$\alpha \bullet a \oplus \alpha \bullet b = (a \oplus b)|_\alpha = (a \vee b)|_\eta = (a \vee b)|_{\alpha \vee \eta} = \alpha(a \oplus b).$$

Case 3. If $\beta \leq \delta \leq \alpha$, then $\alpha \bullet a \oplus \alpha \bullet b = a|_{\alpha \wedge \beta} \oplus b|_{\alpha \wedge \delta} = a|_\beta \oplus b|_\delta = (a \vee b)|_\eta = (a \vee b)|_{\alpha \wedge \eta} = \alpha \bullet (a \oplus b)|_\eta = \alpha \bullet (a \oplus b)$
If $index_N^{(a,b)} = \gamma$ then $\beta = \gamma$or $\delta = \gamma$, so, $\alpha \bullet (a \oplus b) = \alpha \bullet (a \vee b) = \alpha \bullet ((a \vee b)|_\gamma) = (a \vee b)|_{\gamma \wedge \alpha} = (a \vee b)|_\alpha$
On the other hand, without loss of generality, suppose that, $\beta = \gamma$, Hence $\alpha \bullet a = a|_\alpha$, $\alpha \bullet b = b|_{\alpha \wedge \delta}$, we have two cases:

(1) If $\alpha \leq \delta$, then $\alpha \bullet a \oplus \alpha \bullet b = a|_\alpha \oplus b|_\alpha = (a \vee b)|_\alpha$.

(2) If $\alpha\rangle\delta$, then $\alpha \bullet a \oplus \alpha \bullet b = a|_{\alpha} \oplus b|_{\delta} = (a \vee b)|_{\alpha}$.
(iii). Let $\alpha, \beta \in \gamma$ and $\beta\langle\alpha$, $a = (a_i)_{i\in\delta} \in N$, then, $(\alpha \vee \beta) \bullet a = \alpha \bullet a = a|_{\alpha\wedge\delta}$.
On the other hand $\alpha \bullet a = a|_{\alpha\wedge\delta}$ and $\beta \bullet a = a|_{\beta\wedge\delta}$. Since $\delta \wedge \beta \leq \delta \wedge \alpha$, so, $\alpha \bullet a \oplus \beta \bullet a = a|_{\delta\wedge\gamma} = (\alpha \vee \beta) \bullet a$.
(iv). Let $a = (a_i)_{i\in\beta}$ be an elements of N, then $0 \bullet a = 0 \bullet (a|_0) = a|_0 = 0$ □

Example 4.10. *we define $\bullet := (\omega + 1) \times N \longrightarrow N$ as follows*

$$\gamma \bullet a = \begin{cases} a|_{\lambda}\ l(a)\rangle\lambda\rangle 0 \\ a\ l(a) \leq \lambda\rangle 0 \\ 0\ \lambda = 0 \\ a\ \lambda = \omega \end{cases}$$

For all $\gamma \in \omega + 1$ and $a \in N$. So, every $\mathbb{N}^{\infty}$-moduloid is a unitary $\omega + 1$-moduloid.
Henceforth, we write αa instead of $\alpha \bullet a$, for all $\alpha \in \gamma$and $a \in N$.

Remark 4.11. *If γ is a non-limit ordinal and N is a moduloid over γ. Then*

$$(\gamma - 1)a = a, \forall a \in N$$

Hence, in this case, Nis a unitary γ-moduloid.

Definition 4.12. *Let $(N, \oplus, \bullet, 0)$ be a moduloid over γ and let S be a nonempty subset of N and $(\ ) = 0 \in S$. Then S is called submoduloid of N, if $(S, \oplus, 0)$ is a γ- moduloid over $(\gamma, \vee, \wedge, 0)$. The set of all submoduloids of N is denoted by $SUB_M^{\gamma}(N)$.*

Remark 4.13. *In general, for γ-submoduloids M, T of N, MUT is not γ -moduloids of N.*

Example 4.14. *Consider γ-moduloid N of Example 4.3, we defined $M = \{0, a^1, a^5\}$ and $T = \{0, a^2, a^6\}$. Therefore, M and N are γ- submoduloids, but, $M \bigcup T = \{0, a^1, a^2, a^5, a^6\}$ is not γ-submoduloid becausea$^5 \oplus a^6 = a^7 \notin M \bigcup N$.*

Example 4.15. *Consider the nexus*

$$N = \{(0),(1),(2),(3),(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2)\}$$

and submoduloids,

$$A = \{(\ ),(1),(1,1),(1,2)\} \text{ and } B = \{(\ ),(1),(2),(2,1)\}.$$

So, $AUB = \{0,(1),(2),(2),(2,1),(1,2)\}$ is not γ-moduloid of N because, $(1,2) \in M \bigcup N$, $(2,1) \in M \bigcup N$ but $(1,2) \oplus (2,1) = (2,2) \notin M \bigcup N$.

Remark 4.16. *If N is a γ-moduloid and $\{M_i\}_{i\in I} \subseteq SUB_M^{\gamma}(N)$, then $\bigcap_{i\in I} M_i \in SUB_M^{\gamma}(N)$.*

Proof. It is evident. □

Definition 4.17. *If N is a nexus and λ is an ordinal number, then, by definition of dot product* $\bullet.\lambda \bullet N = \{\lambda a | a \in N\} = \{a \in N | l(a) \leq \lambda\} = l(\lambda)$.
Therefore, according to the above theorem, $\lambda \bullet N$ will be a γ-submoduloid of N, for every $\lambda \in \gamma$ and it is called λ-cut of N.

Remark 4.18. *Let N be a γ-moduloid and let M be γ-submoduloid of N. Then for every $\lambda \in \gamma$, $\lambda \bullet M$ is a γ-submoduloid of N.*

Theorem 4.19. *Let N be a γ-moduloid and $\alpha \in \gamma$. Consider the subset*

$$N_\alpha = \{a \in N | stem(a) = \alpha\} = \{a \in N | a(0) = \alpha\} \bigcup \{(\ )\}$$

The subset N_α is a γ-submoduloid of N and it is called α-stem.

Proof. Let $a, b \in N_\alpha$. Thus, $a = (a_i)_{i\in\beta}$, $a(0) = \alpha$ and $b = (b_i)_{i\in\delta}$, $b(0) = \alpha$. By definition of γ-moduloid summation, $(a \oplus b)(0) = a(0) \vee b(0) = \alpha \vee \alpha = \alpha$. Therefore, $(a \oplus b) \in N_\alpha$. Now, suppose that $a \in N_\alpha$, and $\lambda \in \gamma$. Hence, by definition of dot product, $(\lambda a)(0) = \alpha$ for $\lambda \neq 0$ and $(\lambda a)(0) = 0 = (\ ) \in N_\alpha$, for $\lambda = 0$. Thus $\lambda a \in N_\alpha$. Hence, N_α is γ-moduloid of N. □

5 γ-Muduloid Homomorphism

In this section M and N are γ-moduloids.

Definition 5.1. *Let N and M be two γ-moduloids and let $f : N \longrightarrow M$ be a function. Then, f is called γ-moduloid homomorphism if:*
(i) $f(a \oplus b) = f(a) \oplus f(b)$, $\forall a, b \in N$
(ii) $f(\lambda a) = \lambda f(a) \forall a \in N$, $\forall \lambda \in \gamma$
Moreover, if is injective (surjective), (bijective) then f is said to be a γ-moduloid monomorphism (epimorphism), (isomorphism). The kernel of f is defined by $f^{-1}(\{0\})$ and denoted by $Kerf$.

Lemma 5.2. *Let a be an address in γ-moduloid N and $a^{(\delta_1, j_1)} = a^{(\delta_2, j_2)}, j_1 \neq 0 \neq j_2$. Then $\delta_1 = \delta_2$and $j_1 = j_2$.6*

Proof. If $\delta_1 \neq \delta_2$, then we have two cases:
Case 1. $\delta_1 \in \delta_2$. So, $a^{(\delta_1, j_1)}(\delta_1) = a^{(\delta_2, j_2)}(\delta_2)$ implies $j_2 = 0$, which is a contradiction.
Case 2. $\delta_2 \in \delta_1$. Similar to Case l, $j_1 = 0$, which is also a contradiction.
Hence, $\delta_1 = \delta_2$. Now, $a^{(\delta_1, j_1)}(\delta_1) = a^{(\delta_2, j_2)}(\delta_2)$and therefore $j_1 = j_2$. □

Lemma 5.3. *Let a be on address in γ-moduloid. Then,*

$$a^{(\delta_1, j_{\delta_1})} \oplus a^{(\delta_2, j_{\delta_2})} = a^{(\delta_1 \vee \delta_2, j_{\delta_1 \vee \delta_2})}$$

Proof. Without loss of generality, suppose that $\delta_1 \in \delta_2$, then

$$a^{(\delta_1,j_{\delta_1})} \oplus a^{(\delta_2,j_{\delta_2})} = \begin{cases} a_i \; i \in \delta_1 \\ j_{\delta_1} \; i = \delta_1 \\ 0 \; otherwise \end{cases} \oplus \begin{cases} a_i \; i \in \delta_2 \\ j_{\delta_2} \; i = \delta_2 \\ 0 \; otherwise \end{cases} =$$

$$\begin{cases} a_i \; i \in \delta_1 \\ j_{\delta_1} \vee a_{\delta_2} \; i = \delta_1 \\ a_i \; i \in \delta_2 - \delta_1 \\ j_{\delta_2} \; i = \delta_2 \\ 0 \; otherwise \end{cases} = \begin{cases} a_i \; i \in \delta_2 \\ j_{\delta_2} \; i = \delta_2 \\ 0 \; otherwise \end{cases} = \begin{cases} a_i \; i \in \delta_1 \vee \delta_2 \\ j_{\delta_1 \vee \delta_2} \; i = \delta \\ 0 \; otherwise \end{cases} = a^{(\delta_1 \vee \delta_2, j_{\delta_1 \vee \delta_2})}$$

□

Example 5.4. *Le $a = (a_i)_{i\in\omega}$ so that, for all $i \in \omega$, $a_i = a(i) = 3$ and $b = (b_i)_{i\in\omega}$ such that, for all $i \in \omega$, $b_i = b(i) = 1$. We defined $N = \langle a\rangle^\gamma$, $M = \langle b\rangle^\gamma$. According to Remark 3.18, $N = \{ a^{(\delta,j)} \big| \delta \in \omega, 0\langle j \leq a_\delta\} \bigcup \{(\ )\}$ and $M = \{ b^{(\delta,j)} \big| \delta \in \omega, 0\langle j \leq b_\delta\} \bigcup \{(\ )\}$ and according to Theorems 3.23 and 4.9, M and N are γ-moduloids. On the other hand, we defined $f : N \longrightarrow M$ by $f(a^{(\delta,j)}) = b^{(\delta,1)}$, $f(a^{(0,0)}) = b^{(0,0)}$ for all $\delta \in \beta$, $0\langle j \leq a_\delta$. By Lemma 5.2, f is well-defined and by Lemma 5.3, we have $f(a^{(\delta_1,j_{\delta_1})} \oplus a^{(\delta_2,j_{\delta_2})}) = f(a^{(\delta_1\vee\delta_2,j_{\delta_1\vee\delta_2})}) = b^{(\delta_1\vee\delta_2,1)} = b^{(\delta_1,1)} \oplus b^{(\delta_2,1)}$. For all $\lambda \in \gamma$, $f(\lambda a^{(\delta,j)}) = f(a^{(\delta,j)}\big|_{\lambda\wedge\delta+1}) = b^{(\delta,j)}\big|_{\lambda\wedge\delta+1} = \lambda b^{(\delta,1)} = \lambda f(a^{(\delta,j)})$*
Therefore, f is γ-moduloid homomorphism and $\ker f = \{0\} = a^{(0,0)}$*. Note that, f is not one to one.*

Theorem 5.5. *Let $f : N \longrightarrow M$ be a γ-moduloid homomorphism. Then*
(i) $f(0) = 0$
(ii) If $a, b \in N, a \leq b$, then $f(a) \oplus f(b) = f(b)$.
(iii) If $a \in N, l(a) = \alpha$, then $l(f(a)) \leq \alpha$ and $f(a|_\lambda) = f(a), (\forall\lambda)(l(f(a)) \leq \lambda \leq l(a)$.
(iv) If $l(f(a)) = l(a)$ then $f(a|_\lambda) = b|_\lambda, (\forall\lambda)(\lambda \leq l(a))$.

Proof. (ii) : If $a \leq b$, then $a \oplus b = b$. Therefore, $f(a \oplus b) = f(a) \oplus f(b) = f(b)$
(iii) : If $f(a) = 0$, the result is obvious. Let $a = (a_i)_{i\in\alpha}$, and let $b = f(a) = (b_i)_{i\in\beta} \neq 0$, since $f(\alpha a) = \alpha f(a)$.
Therefore, $\alpha b = \alpha f(a) = f(\alpha a) = f(a) = b$, so, $l(f(a)) = l(b) \leq \alpha$. Therefore

$$f(a|_\lambda) = f(\lambda a) = \lambda f(a) = f(a), (\forall\lambda)(l(f(a)) \leq \lambda \leq l(a))$$

(iv) : For all $\lambda \leq l(a) = l(f(a), f(a|_\lambda) = f(\lambda a) = \lambda f(a) = f(a)|_\lambda$.

□

Theorem 5.6. *Let $f : N \longrightarrow M$ be a γ-moduloid homomorphism and rise$(N) = \alpha$ and rise$(M) = \beta$. Then:*
(i) If f is γ-moduloid monomorphism then $\alpha \leq \beta$.
(ii) If f is γ-moduloid epimorphism then $\alpha \geq \beta$.

Proof. (i) : Let $a = (a_i)_{i\in\alpha}$ be an address in N and $f(a) = b = (b_i)_{i\in\theta}$. If $\beta\langle\alpha$, then $\theta \leq \beta\langle\alpha$, and $f(a|_\lambda) = f(a) = b, (\forall\lambda)(\theta \leq \lambda \leq \alpha)$. Therefore, f is not γ-moduloid monomorphism, which is a contradiction.
(ii): Let f be γ-moduloid epimorphism and $\alpha \leq \beta$. Therefore, there is an address $a = (a_i)_{i\in\alpha}$ in M such that $f(a) = b = (b_i)_{i\in\theta}$ where, $\theta \leq \alpha$.

If $\theta\langle\beta$, then for all $\lambda,\theta \leq \lambda\langle\beta$, $f(a) = f(\lambda a) = \lambda f(a) = b|_\theta$ which is contradiction, since f is function. So $\theta = \beta$. Since $rise(N) = \alpha, \theta \leq \alpha,$. Therefore, $\beta \leq \alpha$. □

Definition 5.7. *Let $a = (a_i)_{i\in\beta}$ be an address of nexus N over γ. We define $q_a = \{b \in N | a = b|_\beta, b \neq a\}$ and $Q_a = \{b \in N | b \geq a\}$.*

Theorem 5.8. *Let Nand M be two γ-moduloids and let $f : N \longrightarrow M$ be a γ-moduloid homomorphism. Suppose that $a = (a_i)_{i\in\beta}$ is an address in N and $f(a) = c = (c_i)_{i\in\beta}$, where $\alpha\rangle\beta$; then:*
(i) $f(b) = c$, for all $b \in q_a$.
(ii) If $\alpha$is a limit ordinal, then: $f(b) = c$, for all $b \in Q_a$

Proof. (i) : Let b be an address in q_a. Thus, $b|_a = a$. Now, we have

$$f((\beta+1)b) = (\beta+1)f(b) \Rightarrow f(b|_{\beta+1}) = f(b)|_{\beta+1} \Rightarrow f(a|_{\beta+1}) = f(a)|_{\beta+1} \tag{1}$$

On the other hand

$$f(a) = c \Rightarrow f(a|_\lambda) = c, (\forall\lambda)(\beta \leq \lambda \leq \alpha) \Rightarrow f(a|_{\beta+1}) = c \tag{2}$$

(1) and (2) imply, $f(a|_{\beta+1}) = c \Rightarrow (f(b))_\beta = 0$.
By definition of an address, $(f(b))_\lambda = 0, \forall\lambda \geq \beta$ therefore $f(b) = c$.
(ii) : Let α be a limit ordinal and b be an address in Q_a. Thus, $b|_\alpha = a$. Now, using the part one, the proof is complete. □

Remark 5.9. *Note that, in the above theorem, if $l(a))$ is not a limit ordinal there may be $b \in Q_a$ so that $f(b) \neq c$. For example, $N = \{0,(1),(2),(2,1),(2,2)\}$ and $M = \{0,(1),(2),(2,1)\}$. Put $a = (2,1)$, $b = (2,2)$, $c = (2)$ and $\gamma = 3$. We defined, $f : N \longrightarrow M$, where $f(0) = 0$, $f(1) = f(2) = f(2,1) = (2)$, $f(2,2) = (2,1)$. It is easy to show that f is 3-homomorphism and $b \in Q_a$, but $f(b) \neq c$.*

Theorem 5.10. *Let N and M be two γ-moduloids and $f : N \longrightarrow M$ be a γ-moduloid homomorphism. Then f is monotone map, that is, $a \leq b$ implies that $f(a) \leq f(b)$.*

Proof. Let a and b be two addresses in N and let $a \leq b$. suppose that $a = (a_i)_{i\in\alpha}$, then, we have:

if α be non-limit ordinal , so, $b|_{\alpha-1} = a|_{\alpha-1}$, $b_{\alpha-1} \leq a_{\alpha-1}$. Suppose $f(a) = (c_i)_{i\in\delta}$, where $\delta \leq \alpha$. Consider two cases:

case 1. $\delta\langle\alpha$. So, $f(a) = (c_i)_{i\in\delta} = \delta f(a) = f(\delta a) = f(a|_\delta) = f(b|_\delta) = f(\delta b) = \delta f(b) = f(b)|_\delta \leq f(b))$.

case 2. $\delta = \alpha$. So, $f(b|_\alpha \oplus a) = f(b|_\alpha) \oplus f(a)$.

Since $a\langle b$, $f(b|_\alpha) = f(b|_\alpha) \oplus c$ and $f(b|_\alpha)_{\alpha-1} = f(b|_\alpha)_{\alpha-1} \vee c_{\alpha-1}$. Thus, $c_{\alpha-1} \leq f(b|_\alpha)_{\alpha-1}$. This means that, $(c_i)_{i\in\delta} \leq f(b|_\alpha) = f(\alpha b) = \alpha f(b) = f(b)|_\alpha \leq f(b)$.

□

6 Fuzzy Moduloid over an Ordinal Nexus

Definition 6.1. *A fuzzy subset f on set X is a function $f : X \longrightarrow [0,1]$. We denote by $F(X)$ the set of all fuzzy subsets of X.*

-For $f, g \in F(X)$, we say $f \subseteq g$, if and only if $f(x) \leq g(x)$ for every $x \in X$.

-Let $f \in F(X)$, and $t \in [0,1]$. Then the set $f_t = \{x \in X : f(x) \geq t\}$ is called the level subset of X w.r.t. f.

-Also we put $f_* = \{x \in X : f(x) = 1\}$. For $x \in X$ and $t \in [0,1]$.

- $x^t \in F(X)$ is called a fuzzy point, if and only if $x^t(y) = 0$ for $y \neq x$ and $x^t(x) = t$.
The fuzzy point x^t is said to belong to $f\big(F(X)\big)$, written $x^t \in f$, if and only iff$(x) \geq t$.

Remark 6.2. *The set of all fuzzy subnexuses of N is denoted by $F(N)$.*

Definition 6.3. *A functionf $: \gamma \to [0,1]$, is called a fuzzy set over on ordinal.*

Definition 6.4. *Fuzzy subset $f : \gamma \to [0,1]$, that for all $x \in \gamma$, $f(x) = 1$, we denoted by* 1.

Definition 6.5. *Fuzzy subset $f : \gamma \to [0,1]$, that for all $x \in \gamma$, $f(x) = 0$, we denoted by* 0.

Definition 6.6. *Let $\tilde{p}$ be a fuzzy subset of a set γ. For $t \in [0,1]$, the set $\tilde{p}_t = \{\alpha \in \gamma : \tilde{p}_t(\alpha) \geq t\}$ is called a level subset of $\tilde{p}_t$.*

Remark 6.7. *The set of all fuzzy subnexus of $A(\gamma)$ is denoted by $FSUB(A(\lambda))$.*

Definition 6.8. *The scalar multiplication (dot product) $\bullet$ is defined on γ as follows:*

$$\bullet : \gamma \times N \longrightarrow [0,1]$$

$$(\alpha, a) \to a|_{\alpha \wedge \beta} = \begin{cases} 1 \; \alpha, \beta \in \gamma \\ 0 \; otherwise \end{cases}$$

Remark 6.9. *Let $a \in N$ and $\alpha \in \gamma$ then $l(\alpha \bullet a) \leq l(a)$.*

Remark 6.10. *Let $x \in \gamma$ and $t \in [0,1]$. Then $\langle x^t \rangle : \gamma \to [0,1]$, defined by*

$$\langle x^t \rangle(\alpha) = \begin{cases} t \; x \in \alpha \\ 0 \; x \notin \alpha \end{cases}$$

is a fuzzy ordinal.

Proposition 6.11. *Let $\alpha, \beta \in \gamma$ and $r, t \in [0,1]$. Then $\alpha \leq \beta$, if and only if $\langle \alpha^t \rangle \leq \langle \beta^t \rangle$.*

Proof. Let $\alpha \leq \beta$. Since $\alpha \in \gamma$, implies that $\beta \in \gamma$, we can conclude that $\langle \alpha^t \rangle(x) = t$ implies that $\langle \beta^t \rangle(x) = t$. Hence, $\langle \alpha^t \rangle \leq \langle \beta^t \rangle$.
Conversely, let $\langle \alpha^t \rangle \leq \langle \beta^t \rangle$. Hence, $t = \langle \alpha^t \rangle(\alpha) \leq \langle \beta^t \rangle(\alpha) \leq t$, i.e. $\langle \beta^t \rangle(\alpha) = t$. Therefore, $\beta \in \alpha$. □

Definition 6.12. *Let $\mathbb{N}^{\infty} = \mathbb{N} \bigcup \{0, \infty\}$, γ be a ordinal and the scalar multiplication6*

$$\bullet : \mathbb{N}^{\infty} \times \gamma \to [0,1]$$

is defined on γ as follows:

$$r \bullet \alpha = \begin{cases} \alpha \; r = \infty \\ 0 \; r = 0 \end{cases}$$

for all, $r \in \mathbb{N}^{\infty}$and $\alpha \in \gamma$.

Definition 6.13. *Let N be a nexus over γ, and f be a fuzzy subset of N then:*

$$\langle f \rangle(a) = \vee_{b \in \uparrow a} f(b).$$

Proposition 6.14. *If N is a nexus over γ, and $f, g \in F(N)$, then:*

$$\langle f \rangle \bigcap \langle g \rangle \geq \langle f \bigcap g \rangle$$

Proof. For every $a \in N$,

$$\begin{aligned} (\langle f \rangle \cap \langle g \rangle)(a) &= \min\{\langle f \rangle(a), \langle g \rangle(a)\} \\ &= \min\{\vee_{b \in \uparrow a} f(b), \vee_{b \in \uparrow a} g(b)\} \geq \vee_{b \in \uparrow a} \min\{f(b), g(b)\} \\ &= \vee_{b \in \uparrow a} (f \cap g)(b) = \langle f \cap g \rangle(a) \end{aligned}$$

. □

Example 6.15. *Let $\gamma = 3$, $N = \{(\ ), (1), (2)\}$, and $f, g : N \longrightarrow [0,1]$ be functions such that*

$$f = \begin{pmatrix} (\ ) & (1) & (2) \\ 0.1 & 0.2 & 0.3 \end{pmatrix}$$

and

$$g = \begin{pmatrix} (\ ) & (1) & (2) \\ 0.3 & 0.2 & 0.1 \end{pmatrix}.$$

It is clear that $\langle f \rangle \cap \langle g \rangle \neq \langle f \cap g \rangle$.

Definition 6.16. *Let N be a non-trivial nexus over γ, i.e. $N \neq \{(\ )\}$. A fuzzy subnexus f of N is called a prime fuzzy nexus if:*

$$f(a \wedge b) \leq \max\{f(a), f(b)\}, \forall a, b \in N$$

Remark 6.17. *The set of all prime fuzzy subnexus of N is denoted by $PF(N)$.*

Remark 6.18. *It is clear that if $f \in PF(N)$, then $f(a \wedge b) = f(a)$ or $f(b), \forall a, b \in N$.*

Proposition 6.19. *Let N be a non-trivial nexus over γ, and f be a fuzzy subnexus of N, $N \subseteq A(\gamma)$. Then f_r is a prime fuzzy subnexus.*

Proof. Let $r \in [0,1]$, and f_r be a non-empty subset of N. If $a, b \in N$ and $a \wedge b \in f_r$, then $r \leq f(a \wedge b) \leq \max\{f(a), f(b)\}$, and which follows that $a \in f_r$ or $b \in f_r$. So f_r is a prime subnexus of N. □

Proposition 6.20. *Let $F : M \longrightarrow N$ be a homomorphism between nexus; then:*
if g is a prime fuzzy subnexus of M, then $f = gF$ is a prime fuzzy subnexus of N.

Proof. For every $a, b \in N$,

$$f(a \wedge b) = gF(a \wedge b) = g(F(a \wedge b)$$
$$= g(F(a) \wedge F(b)) =\leq \max\{g(F(a), g(F(b))\}$$

Hence, f is a prime fuzzy subnexus of N. □

Example 6.21. *let $\gamma = 3$, $N = \{(\ ), (1), (2)\}$, and $h, f, g : N \longrightarrow [0,1]$ be functions such that*

$$f = \begin{pmatrix} (\ ) & (1) & (2) \\ 0.3 & 0.2 & 0.125 \end{pmatrix}$$

and

$$g = \begin{pmatrix} (\ ) & (1) & (2) \\ 0.4 & 0.35 & 0.1 \end{pmatrix}$$

and

$$h = \begin{pmatrix} (\ ) & (1) & (2) \\ 0.3 & 0.2 & 0.1 \end{pmatrix}$$

It is clear that $h \in F(N)$ is prim, and $f, g \in F(N)$. Also $f \wedge g \subseteq h$ but $f \not\subset h$ and $g \not\subset h$.

Remark 6.22. *Let A and B be two fuzzy submodules of a fuzzy module X. Then $A + B$ is a fuzzy submodule of X.*

Definition 6.23. *Let N be an N^∞-moduloid, and let $f : N \longrightarrow [0,1]$ be a function. Then f is called an fuzzy N^∞-moduloid homomorphism if*

1) $f(a + b) = f(a) + f(b)$, $\forall a, b \in N$

2) $f(ra) = rf(a)$, $\forall a \in N$, $\forall r \in N^\infty$

Proposition 6.24. *Let $f : N \longrightarrow [0,1]$ be an fuzzy N^∞-moduloid homomorphism. Then:*

(i) if $a, b \in N$, $a \leq b$, then $f(a) + f(b) = f(b)$;

(ii) if $a \in N, l(a) = n\langle\infty$, then $l(f(a)) \leq n$. In particular, every principal element is going to 0 or a principal element by f.

Proof. (i) : If $a \leq b$, then $a + b = b$. Therefore, $f(a + b) = f(a) + f(b) = f(b)$.
(ii) : if $f(a) = 0$ we have the result let $a = (a_1, ...a_n)$ and let $b = f(a) = \{b_i\}_{i \in \mathbb{N}} \neq 0$.
Since $f(na) = nf(a)$ therefore:

$$nb = nf(a) = nf((a_1, ..., a_n)) = f(n(a_1, ..., a_n)) = f((a_1, ..., a_n)) = b = nb = ((b_1, ..., b_n)).$$

it implies that $0 = b_{n+1} = b_{n+2} = ...$, hence, $l(f(a)) \leq l(b)$. □

7 Conclusion

The generation of data can be simplified by using space structures such as formex and plenix. From a general application viewpoint, mathematical structures realized in nexus algebra have been developed into mathematical objects. The notion of nexuses has also been defined over an ordinal to be represented as an address. As an advantage, ordinal numbers facilitated the comparison between two infinite addresses.
Here, a moduloid structure over an ordinal nexus was for the first time defined. The notions of γ-moduloid nexuses, γ-submoduloid, γ-moduloid homomorphism, and level of a γ-nexus were defined and the relationships between them were investigated. In particular, it was shown that:
(i) Every cyclic subnexus $< a >$ of $N$was a γ-submoduloid of N. In particular, if $N$was a cyclic nexus, then every subnexus of $N$was a γ-submoduloid.
(ii) Every λ-cut and α-stem of any γ-moduloid were γ-submoduloid.
(iii) Every γ-moduloid homomorphism was monotone.

Acknowledgement

The hard work and effort of my dear professor and family are highly appreciated.

Funding

This research received no external funding.

Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declare that they have no conflict of interest.

References

[1] D. Afkhami Taba, A. Hasankhani, M. Bolourian, Soft nexuses, Computers and Mathematics with Applications 64 (2012) 1812–1821. https://doi.org/10.1016/j.camwa.2012.02.048
[2] D. Afkhami, N. Ahmadkhah, A. Hasankhani, A fraction of nexuses, International Journal of Algebra 5(18) (2011) 883–896. https://www.m-hikari.com/ija/ija-2011/ija-17-20-2011/afkhamiIJA17-20-2011.pdf
[3] N. K. Al-daban, semi-essential submodules and semi- uniform module, M.Sc. Thesis, University of Tikrit,(2005). https://doi.org/10.32894/kujss.2009.40796
[4] M. Bolourian, Theory of plenices, PhD. Thesis (supervised by H Nooshin), University of Surrey,

England (2009).
[5] M. Bolourian, A. Hasankhani, Moduloid over nexus, perprint.
[6] M. Bolourian, H. Nooshin, Element of theory of plenices, International Journal of Space Structures 19(4) (2004) 203–244. https://doi.org/10.1260/0266351043492690
[7] K. Devlin, The Joy of Sets: Fundamentals of Contemporary Set, Springer-Verlag, New York, (1993).
[8] A. A. Estaji, A. A. Estaji, Nexus over an ordinal, Journal of Uncertain Systems 9(3) (2015) 183–197. https://www.researchgate.net0bdd6c95816/Nexus-over-an-ordinal.pdf
[9] A. A. Estaji, T. Haghdadi, J. Farokhi, Fuzzy nexus over an ordinal. Journal of Algebraic Systems 3(1) (2015) 65–82. https://doi.org/10.22044/jas.2015.489
[10] M. Haristchain. Formex and plenix structural analysis, PhD Thesis (supervised by H Nooshin), University of Surrey, England (1980).
https://openresearch.surrey.ac.uk/esploro/outputsCorpusID:125839080
[11] I. N. A. Hee, Plenix Structural Analysis, PhD Thesis (supervised by H Nooshin), University of Surrey, (1985). https://openresearch.surrey.ac.uk/Plenix-structural-analysis/99516443402346
[12] H. Hedayati, A. Asadi, Normal, maximal and product fuzzy subnexuses of nexuses, J. Intell. Fuzzy Syst. 26(3)(2014) 1341–1348. https://doi.org/10.3233/IFS-130820
[13] H. Nooshin, Algebraic Representation and Processing of Structural Configurations, Int. J. of Computers and Structures, June (1975) 119–130. https://doi.org/10.1016/0045-7949(75)90002-4
[14] H. Nooshin, Formex Configuration Processing in Structural Engineering, (Elsevier Applied Science Publishers, London, 1984). https://api.semanticscholar.org/CorpusID:106734390
[15] H. Nooshin, P. Disney, Formex configuration processin I, International Journal of Space Structures 15(1) (2000) 1–52. https://doi.org/10.1260/0266351001494955
[16] H. Nooshin, P. Disney, Formex configuration processing II, Int. j. of space structures 16(1) (2001) 1–56. https://doi.org/10.1260/0266351011495313
[17] H. Nooshin, P. Disney, Formex Configuration Processing III, Int. J. of Space Structures 17(1) (2002) 1–50. https://doi.org/10.1260/026635102760123042
[18] H. Nooshin, P. Disney, C. Yamamoto, Formian, (Multi-Science Publishing Company, 1993).
[19] A. Saeidi Rashkolia, A. Hasankhani, Fuzzy subnexuses, Italian Journal of Pure and Applied Mathematics (28)(2011) 229–242 https://ijpam.uniud.it/online_issue/201128/22-saedi-hasankhani.pdf
[20] A. Saeidi Rashkolia, Prime fuzzy subnexuses, Proceeding of 20th Seminar on Algebra, Tarbiat Moallem University, Iran, April 22-23 (2009) 190–193.
[21] L. Torkzadeh, A. Hasankhani, Some results on prime and maximal subnexuses of a nexus Proceeding of 20th Seminar on Algebra, Tarbiat Moallem University, Iran, April 22-23 (2009) 222–224.
[22] M. M. Zahedi, Some Results on L-Fuzzy Modules, Fuzzy Sets and Systems, Vol. 55, PP. 355-363, (1993). https://doi.org/10.1016/0165-0114(93)90264-i

Citation: E. Nahidi, H. Babaei, A. Hasankhani, Definition and investigation of moduloid over an ordinal nexus: A review of some fuzzy concepts, J. Disc. Math. Appl. 10(1) (2024) 21–42.
https://doi.org/10.22061/jdma.2024.11243.1096