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Research Paper

Some rank six geometries for the smallest Fischer sporadic simple group

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Abstract. This paper introduces some new rank six geometries associated with the Fischer sporadic simple group Fi_{22}. These geometries are both residually connected and firm, with Fi_{22} acts as a flag-transitive automorphism group on them. The previously known geometries for Fischer sporadic simple group Fi_{22} have rank at most four. Therefore, we investigate improvements to the lower bound of the maximum rank of the residually connected and firm geometries on which the group Fi_{22} acts as a flag-transitive automorphism group. Moreover, we demonstrate that the independent generating set of Fischer sporadic simple group Fi_{22} has a size of at least seven.

Keywords. sporadic simple group, coset geometry, independent generating set.
Mathematics Subject Classification (2020): 05*C*09, 05*C*90.

1 Introduction

For many years, researchers have been intrigued by the construction of geometric diagrams of high rank that are acted upon transitively by sporadic simple groups. Noteworthy geometries related to the sporadic simple groups McL, J_2, HS and Suz were documented in previous studies [6, 10–12]. Among these, the Fischer sporadic simple group Fi_{22}, stands

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out as the 16th smallest in the family of sporadic simple groups, of order 64561751654400. There exists a limited number of documented geometric frameworks where Fi_{22} acts as a flag-transitive automorphism group. For instance, Ronan and Stroth analyzed two rank three geometries in [16], while Buekenhout introduced a rank four incidence geometry for Fi_{22} in [5].

In this paper, we present a construction method of six geometries that are either primitive or weakly primitive, firm and residually connected all with flag-transitive action by the Fischer group Fi_{22}. Some of these geometries are notable for their good diagrams. Furthermore, we show that the size of the largest independent generating set for Fi_{22} is at least 7.

The structure of the paper is as follows: Section 2 outlines the necessary notations and preliminary concepts relating to coset geometry and independent generating sets. Section 3 reviews the existing geometries associated with the Fischer sporadic simple group Fi_{22}, documented in past literature. Section 4 introduces the rank six geometries for the Fischer group, and in Section 5, we present an independent generating set of size 7 for Fi_{22}.

2 Notation and preliminaries

Throughout this paper, our notations are standard. We refer the reader to Atlas notation for the structure of the finite simple groups [8]. Familiarity with the fundamentals of incidence geometry, as outlined in [3, 14, 15], is assumed. All computational tasks were performed using Magma [2]. Many of the ideas presented here arise from [9, 17].

Let G be a group. Let I denote a finite set and let $\{G_i\}_{i\in I}$ be a family of subgroups of the group G. We define the *pre-geometry* $\Gamma = \Gamma(G; \{G_i\}_{i\in I})$ such that the set X of *elements* of Γ consists of all cosets gG_i for $g \in G$ and $i \in I$. An *incidence relation* $*$ on X is established by, $g_1G_i * g_2G_j$ if and only if $g_1G_i \cap g_2G_j$ is non-empty in G.

The *type function* t assigns types in Γ by defining $t(gG_i) = i$. The type of a subset Y of X is the set $t(Y)$ and its *rank* corresponds to the cardinal of $t(Y)$. The rank of Γ is given by $|I|$. The subgroup $B = \cap_{i\in I} G_i$ of the pre-geometry is called *Borel subgroup*.

A *flag* is characterized as a collection of mutually incident elements from X and a *chamber* of Γ is a flag of type I. Elements of type i are referred to as *i-element*. The group G acts on Γ as its automorphism group through left translations, maintaining the type of each element. The subgroups G_i for $i \in I$ are called *maximal parabolic subgroups* of Γ. We refer to Γ as a *(coset) geometry* if every flag within Γ is included in at least one chamber. A geometry Γ is termed *flag-transitive* if G acts transitively across all chambers of Γ. Let Γ be a flag-transitive geometry and F be a flag of it. The *residue* of a flag-transitive geometry F is defined as:

$$\Gamma_F = \Gamma(\cap_{j\in t(F)} G_j; (G_i \cap (\cap_{j\in t(F)} G_j))_{i\in I\setminus t(F)}).$$

A geometry Γ is termed *firm* if every flag of rank $|I| - 1$ is contained in at least two chambers. We say Γ is *residually connected* if the incident graph for each residue of rank at least 2 is connected. Γ is considered *primitive* when the group G acts primitively on all i-elements Γ for every $i \in I$. Additionally, Γ is called *weakly primitive* if there exists at least one $i \in I$ such that G acts primitively on the i-elements of Γ.

For a subset $J \subset I$, the *J-truncation* of Γ is referred to as ${}^J\Gamma = \Gamma(G; \{G_i\}_{i \in J})$. The group of type-preserving automorphisms of Γ is denoted as $\mathrm{Aut}(\Gamma)$, while the overall automorphism group is denoted as $\mathrm{Cor}(\Gamma)$. The *diagram* representation of a residually connected, firm and flag-transitive geometry Γ consists of a graph with added structural features. For guidance on diagram construction, see references [4,5,15].

In the following, we describe a definition in abstract group theory that can be used in constructing coset geometry. It is based on the concept of an independent set within a group. More details can be found in [7]. Let $S = (s_i : i \in I)$ represent a set of elements from group G. For any $J \subseteq I$, define $G_J = \langle s_i : i \notin J \rangle$. We denote $G_{\{i\}}$ simply as G_i. The set S is classified as *independent* if $s_i \notin G_i$ for all $i \in I$. A family of elements generating G is independent if and only if it's a *minimal generating set* meaning that no subset of it can generates G.

3 Previous geometries of the Fischer sporadic simple group Fi_{22}

In the existing literature, there are a few incidence geometries exist where the Fischer sporadic simple group Fi_{22} acts as a flag-transitive automorphism group, with the highest rank being four. The geometries presented in this paper do not include these previously studied geometries as truncations.

In [16], Ronan and Stroth explored two rank three geometries related to Fi_{22} that the diagram illustrated in Figure 1 with the notation of [16].

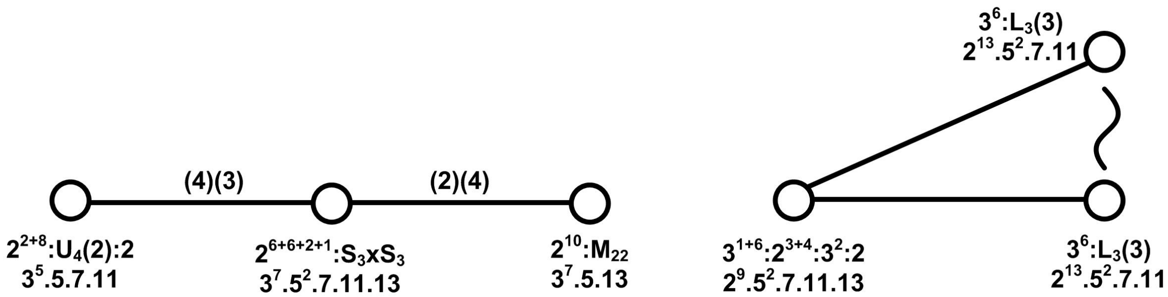

Figure 1. The diagrams of rank three geometries of Fi_{22} from [16]

Buekenhout gave in [5] one rank four incidence geometry for Fi_{22}. This is the number 46 geometry in [5]. The diagram of this geometry is depicted in Figure 2.

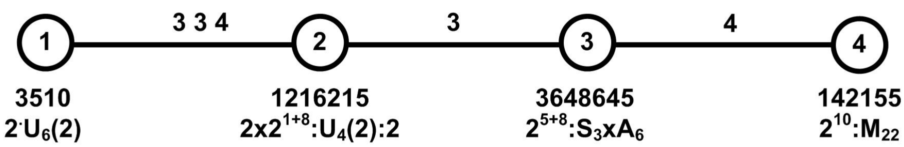

Figure 2. The diagram of rank four geometry of Fi_{22} from [5]

4 The Rank six geometries of the Fischer group Fi_{22}

The Fischer sporadic simple group Fi_{22} has order 64561751654400. The group Fi_{22} in conjugation has fourteen maximal subgroups [8]. In this section, from [1], we consider the permutation representation of Fi_{22} on 3510 right cosets of the maximal subgroup isomorphic to $2{\cdot}\mathrm{U}_6(2)$.

In this permutation representation involving 3510 points, each point's subgroup stabilizer is the maximal subgroup isomorphic to $2{\cdot}\mathrm{U}_6(2)$. Consider $\Delta=\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ as a set of points. We define $S_{\{\Delta\}}$ and $S_{[\Delta]}$ as the set-wise and point-wise subgroup stabilizers of Δ, respectively. We denote by S_α the subgroup stabilizer of point α.

Leemans and Rodrigues presented an algorithm in [13] for classifying all rank two primitive geometries within a finite group G. They used that algorithm to identify all primitive rank two geometries for the eleven smallest sporadic simple groups. In this section, we extend their algorithm to construct one rank six geometry, denoted by Γ_1, for the Fischer sporadic simple group Fi_{22}. The sketch of our algorithm is described in the following.

Let G be a finite permutation group that acts transitively on a set Ω. For constructing a geometry of rank r we use a procedure as follows. We start with a firm and flag-transitive geometry of rank $r-1$ denoted as $\Gamma(G;\{K_1,K_2,\ldots,K_{r-1}\})$. We define $H=\cap_{i=1}^{r-1}K_i$. Let H be non-trivial and has orbit lengths $l_1,l_2,\ldots,l_n$ on Ω. For each $1\le i\le n$, let H_i represent the stabilizer of the representative from the orbit of length l_i such that l_i is different from 1. Next, for each H_i we consider $\Gamma=\Gamma(G;\{K_1,K_2,\ldots,K_{r-1},H_i\})$. If this geometry is residually connected, firm and flag-transitive then we retain the set $\{K_1,K_2,\ldots,K_{r-1},H_i\}$.

For constructing the first geometry from the Fischer sporadic simple group Fi_{22}, we use the described procedure with starting geometry $\Gamma(\mathrm{Fi}_{22};\{K_1,K_2\})$ such that K_i for $i\in\{1,2\}$ is isomorphic to the maximal subgroup $2{\cdot}\mathrm{U}_6(2)$.

The proof of all the following propositions is concluded with the aid of Magma [2].

Proposition 4.1. *Let $F=\{S_1,S_{76},S_{568},S_{1439},S_{1883},S_{3082}\}$. Set $\Gamma_1=\Gamma(\mathrm{Fi}_{22};F)$. Then Γ_1 is a primitive, residually connected and firm coset geometry such that Fi_{22} acts flag-transitively on it. Furthermore, the following properties hold:*

(1) all the maximal parabolic subgroups of Γ_1 are isomorphic to $2{\cdot}\mathrm{U}_6(2)$;

(2) the Borel subgroup of Γ_1 has order 2;

(3) $\mathrm{Aut}(\Gamma_1)\cong\mathrm{Fi}_{22}:2$ and $\mathrm{Cor}(\Gamma_1)\cong(\mathrm{Fi}_{22}{:}2)\times 2^2$;

(4) the diagram of Γ_1 is depicted in Figure 3. In this diagram b is "4 2 4" and a is "3 2 3".

The diagram of Γ_1 is a complete graph, so that is not interesting. We construct geometry Γ_2 from Γ_1 by replacing a maximal parabolic subgroup. The procedure is as follows.

In the permutation representation on 3510 points, the group Fi_{22} has rank three. Therefore, the Fischer sporadic simple group Fi_{22} has two orbits on all sets of distinct points of length 2. The set-wise stabilizer of the representative of these two orbits is isomorphic to

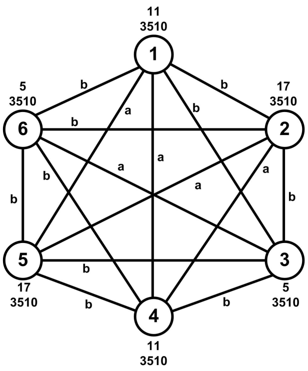

Figure 3. The diagram of Γ_1

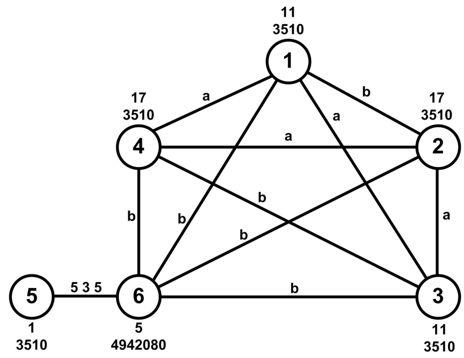

Figure 4. The diagram of Γ_2

the maximal subgroup $2 \times 2^{1+8} : \mathrm{U}_4(2) : 2$ and the subgroup $2 \times \mathrm{U}_4(3) : 2$. The subgroup $2 \times \mathrm{U}_4(3) : 2$ is included within the larger maximal subgroups $2{\cdot}\mathrm{U}_6(2)$ and $S_3 \times \mathrm{U}_4(3) : 2$.

We construct Γ_2 from Γ_1 by replacing the subgroup S_{568} with $S_{\{X_1\}}$ for $X_1 = \{3082, 3234\}$. The subgroup $S_{\{X_1\}}$ is isomorphic to $2 \times \mathrm{U}_4(3) : 2$ and is contained in S_{568}.

Proposition 4.2. *Let* $F = \{S_1, S_{76}, S_{1439}, S_{1883}, S_{3082}, S_{\{X_1\}}\}$. *Set* $\Gamma_2 = \Gamma(\mathrm{Fi}_{22}; F)$. Γ_2 *is weakly primitive, residually connected and firm coset geometry such that* Fi_{22} *acts flag-transitively on it. Furthermore, the following properties hold:*

(1) five maximal parabolic subgroups of Γ_2 *are isomorphic to* $2{\cdot}\mathrm{U}_6(2)$*;*

(2) one of the maximal parabolic subgroups of Γ_2 *is isomorphic to the group* $2 \times \mathrm{U}_4(3) : 2$*;*

(3) the Borel subgroup of Γ_2 *has order* 2*;*

(4) $\mathrm{Cor}(\Gamma_2) \cong \mathrm{Aut}(\Gamma_2) \times 2$*;*

(5) the diagram of Γ_2 *is depicted in Figure 4. In this diagram b is "*4 2 4*" and a is "*3 2 3*".*

Proof. Since the geometry Γ_2 has a large number of elements, we can not compute $\mathrm{Aut}(\Gamma_2)$ directly. But, we can determine $\mathrm{Cor}(\Gamma_2)$ respected to $\mathrm{Aut}(\Gamma_2)$. From analyzing the diagram of Γ_2, it's evident that there's a notable correlation when we exchange the types $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$, respectively. The diagram also indicates that we cannot independently swap types 1 and 3 or types 2 and 4. As a result, no additional correlations can be established, leading to the conclusion that the correlation group $\mathrm{Cor}(\Gamma_2)$ is isomorphic to $\mathrm{Aut}(\Gamma_2) \times 2$. □

For constructing geometry Γ_3, we continue the method on which was used in constructing geometry Γ_2 from Γ_1. We replace the parabolic subgroup S_1 of Γ_2 with $S_{\{X_2\}}$ for $X_2 = \{187, 1439\}$. Thus, we can state the following proposition:

Proposition 4.3. *Let* $F = \{S_{76}, S_{1439}, S_{1883}, S_{3082}, S_{\{X_1\}}, S_{\{X_2\}}\}$. *Set* $\Gamma_3 = \Gamma(\mathrm{Fi}_{22}; F)$. Γ_3 *is weakly primitive, residually connected and firm coset geometry such that* Fi_{22} *acts flag-transitively on it. Furthermore, the following properties hold:*

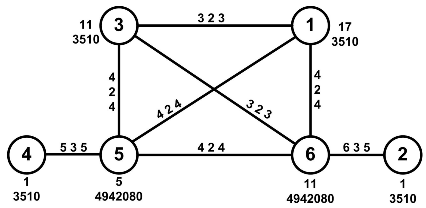

Figure 5. The diagram of Γ_3

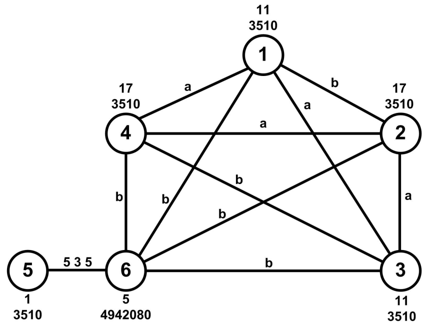

Figure 6. The diagram of Γ_4

(1) *four maximal parabolic subgroups of* Γ_3 *are isomorphic to* $2^{\cdot}\mathrm{U}_6(2)$*;*

(2) *two maximal parabolic subgroups of* Γ_3 *are isomorphic to* $2{\times}\mathrm{U}_4(3){:}2$*;*

(3) *the Borel subgroup of* Γ_3 *has order* 2*;*

(4) $\mathrm{Cor}(\Gamma_3) \cong \mathrm{Aut}(\Gamma_3)$*;*

(5) *the diagram of* Γ_3 *is depicted in Figure 5.*

Proof. Since the geometry Γ_3 has a large number of elements, we can not compute $\mathrm{Aut}(\Gamma_3)$ directly. But, we can determine $\mathrm{Cor}(\Gamma_3)$ respected to $\mathrm{Aut}(\Gamma_3)$. The diagram of Γ_3 shows us that, there is not any correlation that swaps the element types. Therefore, the correlation group of Γ_3 is isomorphic to $\mathrm{Aut}(\Gamma_3)$. □

The maximal parabolic subgroups of Γ_2 and Γ_3 which are isomorphic to $2 \times \mathrm{U}_4(3) : 2$ have orbit lengths $1^1, 2^1, 126^1, 567^1, 1134^1, 1680^1$. As noted above, a subgroup $2 \times \mathrm{U}_4(3) : 2$ is contained in the maximal subgroup $S_3 \times \mathrm{U}_4(3) : 2$. The maximal subgroup $S_3 \times \mathrm{U}_4(3) : 2$, which includes the subgroup $2 \times \mathrm{U}_4(3) : 2$, acts as the set-wise stabilizer for the union of the orbits with length 1 and 2. We construct the three following geometries from Γ_2 and Γ_3 by replacing these maximal parabolic subgroups $2 \times \mathrm{U}_4(3) : 2$ with maximal subgroup isomorphic to $S_3 \times \mathrm{U}_4(3) : 2$ that contain them. In the following proposition $X_3 = \{3082, 3234, 568\}$.

Proposition 4.4. *Let* $F = \{S_1, S_{76}, S_{1439}, S_{1883}, S_{3082}, S_{\{X_3\}}\}$*. Set* $\Gamma_4 = \Gamma(\mathrm{Fi}_{22}; F)$*.* Γ_4 *is primitive, firm and residually connected coset geometry such that* Fi_{22} *acts flag-transitively on it. Furthermore, the following properties hold:*

(1) *five maximal parabolic subgroups of* Γ_4 *are isomorphic to* $2^{\cdot}\mathrm{U}_6(2)$*;*

(2) *one of the maximal parabolic subgroups of* Γ_4 *is isomorphic to the group* $S_3 \times \mathrm{U}_4(3) : 2$*;*

(3) *the Borel subgroup of* Γ_4 *is isomorphic to* 2^2*;*

(4) $\mathrm{Cor}(\Gamma_4) \cong \mathrm{Aut}(\Gamma_4) \times 2$*;*

(5) *the diagram of* Γ_4 *is depicted in Figure 6. In this diagram b is "4 2 4" and a is "3 2 3".*

Proof. Since the geometry Γ_4 has a large number of elements, we can not compute $\mathrm{Aut}(\Gamma_4)$ directly. But, we can determine $\mathrm{Cor}(\Gamma_4)$ respected to $\mathrm{Aut}(\Gamma_4)$. From analyzing the diagram of Γ_4 it's evident that there's a notable correlation when we exchange the types $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$, respectively. The diagram also indicates that we cannot independently swap types 1 and 3 or types 2 and 4. As a result, no additional correlations can be established, leading to the conclusion that the correlation group $\mathrm{Cor}(\Gamma_4)$ is isomorphic to $\mathrm{Aut}(\Gamma_4) \times 2$. □

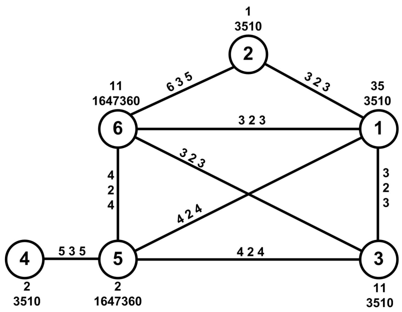

Figure 7. The diagram of Γ_5

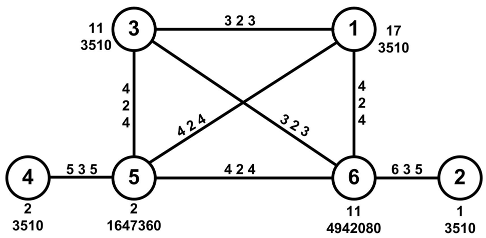

Figure 8. The diagram of Γ_6

In the following proposition, $X_4 = \{187, 1439, 2989\}$ and X_3 is the same as Proposition 4.4.

Proposition 4.5. *Consider the set* $F = \{S_{76}, S_{1439}, S_{1883}, S_{3082}, S_{\{X_3\}}, S_{\{X_4\}}\}$. *Let* $\Gamma_5 = \Gamma(\mathrm{Fi}_{22}; F)$. Γ_5 *is primitive, residually connected and firm coset geometry such that* Fi_{22} *acts flag-transitively on it. Furthermore, the following properties hold:*

(1) four maximal parabolic subgroups of Γ_5 *are isomorphic to* $2{}^{\cdot}\mathrm{U}_6(2)$;

(2) two maximal parabolic subgroups of Γ_5 *are isomorphic to* $S_3 \times \mathrm{U}_4(3){:}2$;

(3) the Borel subgroup of Γ_5 *is isomorphic to* 2^2;

(4) $\mathrm{Cor}(\Gamma_5) \cong \mathrm{Aut}(\Gamma_5)$;

(5) the diagram of Γ_5 *is depicted in Figure 7.*

Proof. Since the geometry Γ_5 has a large number of elements, we can not compute $\mathrm{Aut}(\Gamma_5)$ directly. But, we can determine $\mathrm{Cor}(\Gamma_5)$ respected to $\mathrm{Aut}(\Gamma_5)$. The diagram of Γ_5 shows us that, there is not any correlation that swaps the element types. Therefore, the correlation group of Γ_5 is isomorphic to $\mathrm{Aut}(\Gamma_5)$. □

In the following proposition, X_2 and X_3 are the same as Proposition 4.5.

Proposition 4.6. *Let* $F = \{S_{76}, S_{1439}, S_{1883}, S_{3082}, S_{\{X_2\}}, S_{\{X_3\}}\}$. *Set* $\Gamma_6 = \Gamma(\mathrm{Fi}_{22}; F)$. Γ_6 *is weakly primitive, residually connected and firm coset geometry such that* Fi_{22} *acts flag-transitively on it. Furthermore, the following properties hold:*

(1) four maximal parabolic subgroups of Γ_6 *are isomorphic to* $2{}^{\cdot}\mathrm{U}_6(2)$;

(2) *one of the maximal parabolic subgroups of* Γ_6 *is isomorphic to the group* $S_3 \times \mathrm{U}_4(3):2$;

(3) *one of the maximal parabolic subgroups of* Γ_6 *is isomorphic to the group* $2 \times \mathrm{U}_4(3):2$;

(4) *the Borel subgroup of* Γ_6 *is isomorphic to* 2^2;

(5) $\mathrm{Cor}(\Gamma_6) \cong \mathrm{Aut}(\Gamma_6)$;

(6) *the diagram of* Γ_6 *is depicted in Figure 8.*

Proof. Since the geometry Γ_6 has a large number of elements, we can not compute $\mathrm{Aut}(\Gamma_6)$ directly. But, we can determine $\mathrm{Cor}(\Gamma_6)$ respected to $\mathrm{Aut}(\Gamma_6)$. The diagram of Γ_6 shows us that, there is not any correlation that swaps the element types. Therefore, the correlation group of Γ_6 is isomorphic to $\mathrm{Aut}(\Gamma_6)$. □

5 Independent generating set of the Fischer sporadic simple group Fi_{22}

Here, one independent generating set for the sporadic simple group Fi_{22} is peresnted. This independent generating set is constructed from geometry Γ_3 from Section 4. For constructing this independent generating set we do as follows. Let Γ represent a geometry of rank n, characterized by its maximal parabolic subgroups P_i for $i \in \{1,2,\ldots,n\}$ and non-trivial Borel subgroup B. Set $F_i = (\cap_{j \neq i} P_j) \setminus B$. We consider the set $E = \{g_1, g_2, \ldots, g_{n+1}\}$ such that $g_i \in F_i$ for $i \in \{1,2,...,n\}$ and $g_{n+1} \in B$. For each set E, we check independency and generating properties. We used this procedure with Γ_3 and found one independent generating set.

The result of this section is concluded with the aid of Magma [2]. In the following, we present the properties of the independent generating set denoted by E.

Table 1. The sets of points.

No.	X_i
1	$X_1 = \{1,4,5,6,32\}$
2	$X_2 = \{1,12,14,18,23\}$
3	$X_3 = \{10,16,21,22,32\}$
4	$X_4 = \{1,6,10,11,24\}$
5	$X_5 = \{3,10,14,15,50\}$
6	$X_6 = \{1,143,187,568,727,994\}$
7	$X_7 = \{1780,3082\}$

Let $S_{[X_i]}$ for $i \in \{1,2,\ldots,6\}$ be the point-wise stabilizer of the set X_i that is given in Table 1. The subgroup $S_{[X_i]}$ for $i \in \{1,2,\ldots,5\}$ has order 2 and $S_{[X_6]}$ has order 3. Let g_i for $i \in \{1,2,\ldots,6\}$ be a generator of subgroup $S_{[X_i]}$. Also, let $S_{\{X_7\}}$ be the set-wise stabilizer of the set X_7. The center of subgroup $S_{\{X_7\}}$ is of order 2. Let g_7 be a generator of this subgroup. The set $E = \{g_1, g_2, \ldots, g_7\}$ forms an independent generating set for the sporadic group Fi_{22}. The element g_i for $i \in \{1,2,...,5\}$ is in conjugacy class $2A$, g_6 is a member of conjugacy class $3B$ and g_7 is in conjugacy class $2B$.

As a result, we present the next proposition.

Proposition 5.1. *The Fischer sporadic simple group* Fi_{22} *has independent generating set of size at least* 7.

6 Conclusion

This paper presents significant advancements in the study of geometries associated with the Fischer sporadic simple group Fi_{22}. It introduces six new geometries that are primitive, firm, and residually connected, all admitting flag-transitive actions by Fi_{22}. This represents a notable improvement over previously known geometries, which had a maximum rank of four.

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Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this article.

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