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Research Paper

An algorithm for counting the number of periodic points of a family of polynomials

Monireh Akbari[1,*], Maryam Rabii[2]

[1] Department of Mathematics, Faculty of Sciences, Shahid Rajaee Teacher Training University, Tehran, I. R. Iran

[2] Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, I. R. Iran

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Abstract. In this paper we consider the family $f_a(x) = ax^d(x-1) + x$ when $a < 0$ is a real number and $d \geq 2$ is an even integer. The function f_a has a unique positive critical point. By decreasing the parameter a, the behavior of the orbit of this critical point changes. In this paper, we consider two cases. In the first case, the orbit of the positive critical point converges to 0, and in the second case, the positive critical point is mapped to a repelling periodic point of period 2. In each case we give a recursive formula to determine the number of the periodic points of f_a. Also, we introduce an invariant set on which f_a is chaotic. We employ conjugacy map and symbolic dynamics in our investigations.

Keywords. Cantor set, chaos, conjugacy, periodic points, symbolic dynamics.
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1 Introduction

Symbolic dynamics is vastly employed in the study of discrete dynamical systems. The study of a sequence space on finite symbols often provides a useful tool for investigating

*Corresponding author (Email address: akbari@sru.ac.ir).
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complicated behaviors of an iterating system. For example, to prove chaos in the logistic family, $F_\mu(x) = \mu x(1-x)$, where $\mu > 4$, the sequence space on two symbols 0 and 1 is fundamental, where $\{0,1\}$ is equipped with the discrete topology (see [9,12]). Complex dynamics also benefits from symbolic dynamics [3,7]. In [10], the dynamics of certain rational functions with Julia sets that are Sierpinski curves are investigated by employing symbolic dynamics. To study a quadratic Julia set that is dendrite, the sequence space on three symbols $0,1$ and $\star$, where $\{0,1,\star\}$ is equipped with a topology whose basis is $\{\{0\},\{1\},\{0,\star,1\}\}$, is considered in [4]. General symbolic dynamics and some of its applications are studied in [2,11].

The dynamics of the family $f_a(x) = ax^2(x-1) + x$ is studied in [1] and it is shown that f_a is chaotic on some invariant set by employing negative Schwarzian derivative of f_a. In this paper we are going to investigate the real family $f_{a,d}(x) = ax^d(x-1) + x$, a generalization of the the family $f_a(x) = ax^2(x-1) + x$, when $d \geq 2$ is an even integer and $a < 0$ is a real number. Besides counting the number of periodic points, we study chaos in this family in some cases. In fact, for some values of the parameter a, we construct invariant sets on which $f_{a,d}$ is chaotic and outside of which $f_{a,d}$ has simple dynamics. In [14], it is shown that if the topological entropy of a function is positive, then there will be a subset on which the function is chaotic, although this subset is not presented explicitly. For simplicity, we fix d and replace $f_{a,d}$ with f_a. Since the Schwarzian derivative of f_a is positive at some points when $d > 2$, we will get help from some theorems of the complex dynamics to investigate the dynamics of f_a on these invariant sets. Also, symbolic dynamics is an essential tool in achieving our goals. Regarding the properties of the family f_a, we apply some special subsets of $\mathcal{A}^\omega$, where $\omega = \mathbb{N} \cup \{0\}$ and $\mathcal{A}$ is an infinite countable set which is equipped with a non-Hausdorff topology.

This paper is organized as follows: In Section 2, we introduce $\mathcal{A}$, an infinite countable set, and a basis for a topology on it. Then we equip $\mathcal{A}^\omega$ with the product topology. This topology is non-Hausdorff. Next, we define Σ, Σ_m and $\widehat{\Sigma}_m$, three subspaces of $\mathcal{A}^\omega$ that are Hausdorff and invariant under the shift map σ. Also, we show that the set of the periodic points of $\sigma|_{\widehat{\Sigma}_m}$ and the set of the periodic points of $\sigma|_\Sigma$ are dense in $\widehat{\Sigma}_m$ and in Σ, respectively. Moreover, we show that (σ,Σ) and $(\sigma,\widehat{\Sigma}_m)$ have a dense orbit in Σ and in $\widehat{\Sigma}_m$, respectively. In Section 3, we show that most properties of f_a, when $a < 0$ and $d \geq 2$ is an even integer, are independent of d. Also, we introduce closed invariant sets, Λ, Λ_m, and $\widehat{\Lambda}_m$, for $m \geq 1$, and study their properties. In Section 4, we show that (f_a,Λ) ((f_a,Λ_m) and $(f_a,\widehat{\Lambda}_m)$) is conjugate to (σ,Σ) ((σ,Σ_m) and $(\sigma,\widehat{\Sigma}_m)$), respectively (Theorem 4.1). These conjugacies enable us to prove that $(f_a,\widehat{\Lambda}_m)$ and (f_a,Λ) are chaotic (see Corollary 4.2). In Section 5, we count the number of periodic points of $(\sigma,\widehat{\Sigma}_m)$, (σ,Σ_m), and (σ,Σ) by a recursive formula. In this way, we are able to calculate the number of the fixed points of f_a^n in two special cases by employing the conjugacies introduced in Section 4.

We next describe our terminology and notations. Let I be an interval and $f : I \to I$ be a C^1 function. By f^n we mean $f \circ f^{n-1}$, where f^0 is the identity function. The *orbit* of $x \in I$ is $(f^n(x))_{n\geq 0}$. A point $x_0 \in I$ is called a *fixed point* of f if $f(x_0) = x_0$. A fixed point x_0 is called *non-hyperbolic* if $|f'(x_0)| = 1$. The point x_0 is called a *periodic point* of f of *period* n, if there is a natural number n such that $f^n(x_0) = x_0$. In this case, the orbit of x_0 is called a *periodic orbit*. The *basin of* this periodic orbit is $\cup_{i=0}^{n-1}\{x : \lim_{k\to\infty} f^{kn}(x) = f^i(x_0)\}$. The *immediate basin* of a periodic orbit is the union of the connected components of its basin which contain a point of the periodic orbit. The periodic orbit is a *periodic attractor* if its immediate basin contains an open set.

We use the notation (f, X), when $f : X \to X$ is a function. Let X be an infinite metric space without any isolated point. Then (f, X) is called *chaotic* on X if the set of the periodic points of f is dense in X and (f, X) has a dense orbit in X (see [5] and [9] for more details).

An interval $J \subseteq I$ is called *wandering* if all its iterates $J, f(J), \cdots$ are disjoint and $\left(f^n(J)\right)_{n\geq 0}$ does not tend to a periodic orbit. A *homterval* is an interval on which f^n is monotone for all $n \geq 0$.

The set $A \subseteq X$ is called *invariant* under f if $f(A) = A$.

2 A sequence space on infinite symbols

In this section we introduce $\mathcal{A}$, a set of infinite symbols and put a topology on it. Then we define a sequence space on the symbols of $\mathcal{A}$. Let

$$\mathcal{A} = \mathbb{N} \cup \{L, R, E, W, \star\}.$$

We equip $\mathcal{A}$ with a topology given by the basis

$$\mathcal{B} = \{\{n\}, \{L\}, \{R\}, \{L, \star, R\}, \{E, 2n, 2n+2, 2n+4, \cdots\}, \{W, 2n-1, 2n+1, \cdots\} : n \in \mathbb{N}\}.$$

We set $\omega = \mathbb{N} \cup \{0\}$ and consider $\mathcal{A}^\omega$ with the product topology. It is obvious that $\mathcal{A}^\omega$ is a non-Hausdorff space. Next we define the shift map $\sigma : \mathcal{A}^\omega \to \mathcal{A}^\omega$ by $\sigma(s_0, s_1, s_2, \cdots) = (s_1, s_2, \cdots)$ which is continuous under the product topology.

In the following, we introduce three Hausdorff sub-spaces of $\mathcal{A}^\omega$ and describe their properties. We employ these sub-spaces to determine the dynamics of f_a in the special cases.

In a sequence $\mathbf{s} \in \mathcal{A}^\omega$, a bar over a group of symbols indicates that the group is repeated infinitely.

Definition 1. Let $\Sigma \subseteq \mathcal{A}^\omega$ consists of all $\mathbf{s} = (s_0, s_1, s_2, \cdots)$ such that each pair s_i, s_{i+1} where $i \geq 0$, satisfies the following conditions:

i. If $s_i = k$, then $s_{i+1} = k - 1$, for $k \geq 2$.

ii. If $s_i = \star$, then $s_{i+1} = E$.

iii. If $s_i \in \{L, R\}$, then $s_{i+1} \in \{\star, L, R, 2, 4, 6, \cdots\}$.

iv. If $s_i = 1$, then $s_{i+1} \in \{\star, L, R\}$.

v. If $s_i = W$, then $s_{i+1} = E$. Also, if $s_i = E$, then $s_{i+1} = W$.

Definition 2. For each $m \geq 1$, $\Sigma_m \subseteq \Sigma$ consists of all sequences $\mathbf{s} = (s_0, s_1, s_2, \cdots)$ such that for all $i \geq 0$, $s_i \neq \star$ and if $s_i \in \{L, R\}$, then $s_{i+1} \in \{L, R, 2, 4, \cdots, 2m-2\}$.

Note that in the case $m = 1$, if $s_i \in \{L, R\}$, then $s_{i+1} \in \{L, R\}$.

Definition 3. Let $\widehat{\Sigma}_m$ be a subset of Σ_m consisting all $\mathbf{s} = (s_0, s_1, s_2, \cdots)$ such that for $i \geq 0$, $s_i \notin \{E, W\}$ and also if $s_i \in \mathbb{N}$, then $s_i \leq 2m - 2$, where $m \geq 2$. We denote the set of all sequences on two symbols L and R by $\widehat{\Sigma}_1$.

Proposition 2.1. *The subspaces Σ, Σ_m and $\widehat{\Sigma}_m$ are Hausdorff and invariant under the shift map σ.*

Proof. We only prove that Σ is Hausdorff. The proof of the invariance is straightforward.

Let $\mathbf{s} = (s_0, s_1, s_2, \cdots)$, $\mathbf{t} = (t_0, t_1, t_2, \cdots) \in \Sigma$ and $s_i \neq t_i$ for some $i \geq 0$. Since $\mathcal{B} \setminus \{\{L, \star, R\}\}$ is a basis for the subspace topology on $\mathcal{A} \setminus \{\star\}$ which is Hausdorff, we consider only the case that $s_i = \star$ or $t_i = \star$. Suppose that $s_i = \star$. If $t_i \in \mathbb{N} \cup \{E, W\}$, set $U_i = \{L, \star, R\}$ and $V_i = \mathbb{N} \cup \{E, W\}$. Next define $U = \prod_{j \geq 0} U_j$ and $V = \prod_{j \geq 0} V_j$, where $U_j = V_j = \mathcal{A}$ if $j \neq i$, and if $t_i = L$ or R, then $t_{i+1} \in \{\star, L, R, 2n\}$ for some n, in this case, set $V_{i+1} = \{\star, L, R, 2n\}$ and $U_{i+1} = \{E, 2n+2, 2n+4, \cdots\}$. It is clear that $U_{i+1} \cap V_{i+1} = \emptyset$. Now, we define $U = \prod_{j \geq 0} U_j$ and $V = \prod_{j \geq 0} V_j$, where $U_j = V_j = \mathcal{A}$ if $j \neq i+1$.

In each case, $U = \prod_{l \geq 0} U_l$ containing $\mathbf{s}$ and $V = \prod_{l \geq 0} V_l$ containing $\mathbf{t}$ are two disjoint open subsets of $\mathcal{A}^\omega$. □

In the following, we investigate the density of the periodic points and the existence of a dense orbit in Σ and $\widehat{\Sigma}_m$ under the shift map σ. First, consider the following lemma.

Lemma 2.2. *Let n be a non-negative integer, $0 \leq i \leq n+1$, $k_i \in \mathbb{N}$ and*

$$U_i = \begin{cases} \{2k \in \mathbb{N} : k \geq k_i\} & \text{if } i \text{ is an even number,} \\ \{2k-1 \in \mathbb{N} : k \geq k_i\} & \text{if } i \text{ is an odd number.} \end{cases}$$

Then there is some $l \in \mathbb{N}$ such that $2l \geq n+2$,

$$(2l, 2l-1, 2l-2, \cdots, 2l-n) \in \prod_{i=0}^{n} U_i,$$

and

$$(2l-1, 2l-2, \cdots, 2l-n-1) \in \prod_{i=1}^{n+1} U_i.$$

Proof. Let $l_0 = \max\{k_i :\ 0 \leq i \leq n+1\}$ and $l = l_0 + n$. Then $2l - i \in U_i$. □

Proposition 2.3.

(1) The set of the periodic points of $\sigma|_{\widehat{\Sigma}_m}$ is dense in $\widehat{\Sigma}_m$.

(2) The set of the periodic points of $\sigma|_{\Sigma}$ is dense in Σ.

Proof. Let $U = \prod_{i\geq 0} U_i$ be a basis element of $\mathcal{A}^\omega$. Therefore, there is some $k \geq 1$ such that $U_i = \mathcal{A}$ for $i \geq k$.

(1) Let $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \widehat{\Sigma}_m \cap U$. In the case $m \geq 2$, we consider all the possible symbols for s_k.

(i) $s_k = j$, where $2 \leq j \leq 2m-2$.

(ii) $s_k = L$, or R,or 1.

If s_0 is not an odd integer, in case (i) we set $\mathbf{t} = (\overline{s_0, s_1, \cdots, s_k, j-1, \cdots, 2, 1, R})$ and in case (ii) we set $\mathbf{t} = (\overline{s_0, s_1, \cdots, s_k, R})$. Then $\mathbf{t} \in U \cap \widehat{\Sigma}_m$ is a periodic point. Now suppose that $s_0 = 2l-1$, for some $1 \leq l \leq m-1$. Then $\mathbf{t} = (\overline{s_0, s_1, \cdots, s_k, j-1, \cdots, 2, 1, R, 2l})$ in case (i) and $\mathbf{t} = (\overline{s_0, s_1, \cdots, s_k, R, 2l})$ in case (ii) are periodic points contained in $U \cap \widehat{\Sigma}_m$.

By taking $\mathbf{t} = (\overline{s_0, s_1, \cdots, s_k})$ the proposition holds for $m = 1$.

(2) Suppose that $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \Sigma \cap U$. Again, we consider two cases.

(i) Let $s_i \neq \star$ for all $i \geq 0$. Then there are some integer m and some sequence $\mathbf{t} = (t_0, t_1, \cdots) \in \widehat{\Sigma}_m$ such that $s_i = t_i$ for $i = 0, 1, \cdots, k-1$. Therefore, $\mathbf{t} \in U$. By part (1), there is a periodic point of $\sigma|_{\widehat{\Sigma}_m}$ in $U \cap \widehat{\Sigma}_m$. Thus, U contains a periodic point of $\sigma|_\Sigma$.

(ii) Let $s_i = \star$ for some $i \geq 0$. If $i \geq k-1$, set

$$\mathbf{t} = \begin{cases} (\overline{L}) & \text{if } s_0 = \star, \\ (\overline{s_0, s_1, \cdots, s_{i-1}, L, 2l_0}) & \text{if } s_0 = 2l_0 - 1 \text{ for some } l_0, \\ (\overline{s_0, s_1, \cdots, s_{i-1}, L}) & \text{otherwise.} \end{cases}$$

If $i \leq k-2$, then by Lemma 2.2, we choose l such that $2l \geq k-i-1$ and

$$(2l, 2l-1, \cdots, 2l-(k-i-2)) \in U_{i+1} \times U_{i+2} \times \cdots \times U_{k-1}.$$

Then we set

$$\mathbf{t} = \begin{cases} (\overline{L, 2l, 2l-1, \cdots, 1}) & \text{if } s_0 = \star, \\ (\overline{s_0, s_1, \cdots, s_{i-1}, L, 2l, 2l-1, \cdots, 1, L, 2l_0}) & \text{if } s_0 = 2l_0 - 1 \text{ for some } l_0, \\ (\overline{s_0, s_1, \cdots, s_{i-1}, L, 2l, 2l-1, \cdots, 1, L}) & \text{otherwise.} \end{cases}$$

In each case the periodic point $\mathbf{t}$ belongs to $U \cap \Sigma$. □

Remark 1. One can see that the set of the periodic points of σ in Σ_m is not dense in Σ_m.

Proposition 2.4. *The shift map σ has a dense orbit in Σ and a dense orbit in $\widehat{\Sigma}_m$.*

Proof. Let $g_0 = L$, $g'_0 = R$, $g_i = 2i, 2i-1, \cdots, 2, 1, L$, and $g'_i = 2i, 2i-1, \cdots, 2, 1, R$ for $i \geq 1$. Let $\mathcal{D} = \cup_{i\geq 0}\{g_i, g'_i\}$. For $k \geq 1$, we call $a_1, a_2, \cdots, a_k$ a k-string when $a_j \in \mathcal{D}$. For each $k \geq 1$ we list all the possible k-strings in a sequence $(b_{kj})_{j\geq 1}$ and consider the infinite array

$$\begin{array}{cccccc} b_{11} & b_{12} & b_{13} & \cdots & b_{1s} & \cdots \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2s} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \\ b_{r1} & b_{r2} & b_{r3} & \cdots & b_{rs} & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \end{array}$$

Then we arrange all the elements of $\{b_{kj}\}_{k,j\geq 1}$ in a sequence as follows:

$$\mathbf{s}^\star = (\underbrace{b_{11}}, \underbrace{b_{12}, b_{21}}, \underbrace{b_{13}, b_{22}, b_{31}}, \cdots, \underbrace{b_{1(r+s-1)}, b_{2(r+s-2)}, \cdots, b_{rs}, \cdots, b_{(r+s-1)1}}, \cdots).$$

The orbit of $\mathbf{s}^\star$ under σ is dense in Σ because for every $\mathbf{s} = (s_0, s_1, s_2, \cdots) \in \Sigma$ and every basis element $U = \prod_{i\geq 0} U_i$ containing $\mathbf{s}$, where for some $k \geq 1$, $U_i = \mathcal{A}$ for $i \geq k$, there exists n such that $\sigma^n(\mathbf{s}^\star)$ begins with b_{rs} for some $s \geq 1$, where r and b_{rs} are defined as follows.

- If $s_i \notin \{E, W, \star\}$ for each $i \geq 0$, then there exists some $r \leq k+2$ such that $b_{rs} = s_0, s_1, \cdots, s_k$ when s_0 is not an odd integer, and $b_{rs} = 2j, s_0, s_1, \cdots, s_k$ otherwise.
- If $\mathbf{s} = (\overline{E, W})$, then by Lemma 2.2, there is some l such that $(2l, 2l-1, \cdots, 2l-k+1) \in U_0 \times U_1 \times \cdots \times U_{k-1}$. Then, $r = 1$ and $b_{rs} = 2l, 2l-1, \cdots, 1, R$.
- If $\mathbf{s} = (\overline{W, E})$, then by Lemma 2.2, there is some l such that $(2l-1, 2l-2, \cdots, 2l-k) \in U_0 \times U_1 \times \cdots \times U_{k-1}$. Then, $r = 1$ and $b_{rs} = 2l, 2l-1, \cdots, 1, R$.
- If $s_i = \star$ for some $i \geq k-1$ and if $i \geq 1$, then there is some $r \leq i+2$ such that $b_{rs} = s_0, s_1, \cdots s_{i-1}, R$ if s_0 is not an odd integer and $b_{rs} = 2j, s_0, s_1, \cdots s_{i-1}, R$ if $s_0 = 2j-1$ for some j. If $i = 0$, then $r = 1$ and $b_{rs} = R$.
- If $s_i = \star$ for some $i \leq k-2$, then by Lemma 2.2, there is some l such that $(2l, 2l-1, \cdots, 2l-k+i+2) \in U_{i+1} \times U_{i+2} \times \cdots \times U_{k-1}$. In the case $i \geq 1$, there is some $r \leq i+2$ such that $b_{rs} = s_0, s_1, \cdots, s_{i-1}, R, 2l, 2l-1, \cdots, 1, R$ if s_0 is not an odd integer and $b_{rs} = 2j, s_0, s_1, \cdots, s_{i-1}, R, 2l, 2l-1, \cdots, 1, R$ if $s_0 = 2j-1$ for some j. If $i = 0$, then $r = 2$ and $b_{rs} = R, 2l, 2l-1, \cdots, 1, R$.

Thus, in all of the cases, $\sigma^n(\mathbf{s}^\star) \in U \cap \Sigma$ or $\sigma^{n+1}(\mathbf{s}^\star) \in U \cap \Sigma$.

To find a dense orbit in $\widehat{\Sigma}_m$, let $\mathcal{D}_m = \cup_{i=0}^{m-1}\{g_i, g_i'\}$ and assume that $\mathbf{t}^\star \in \widehat{\Sigma}_m$ is constructed by successively listing all possible 1-strings, then 2-strings, then 3-strings, and so on. Note that in this case the set of r-strings is finite. The proof that the orbit of $\mathbf{t}^\star$ is dense is straightforward. $\square$

3 Invariant sets

In this section we state some common features of the family $f_a(x) = ax^d(x-1) + x$ when a is negative and $d \geq 2$ is an even integer. Next, we study some sets on which f_a is invariant. First, note that:

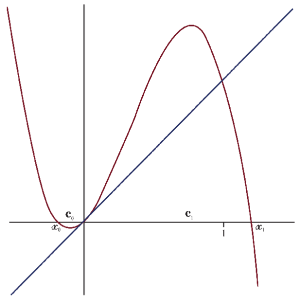

Figure 1. The graph of $f_{-4.5}$

1. f_a has only two fixed points, 0 and 1. The point 0 is non-hyperbolic.
2. $f_a(x) = 0$ has only two non-zero solutions x_0 and x_1. Moreover, $x_0 < 0 < 1 < x_1$.
3. $f_a'(x) = 0$ has only two solutions c_0 and c_1 where c_0 is a local minimum point and c_1 is a local maximum point of f_a. Also, $x_0 < c_0 < 0 < c_1 < x_1$ (see Fig. 1).
4. For $y \in (-\infty, x_0) \cup (x_1, \infty)$ the equation $f_a(x) = y$ has a unique solution in $(-\infty, x_0) \cup (x_1, \infty)$ since $f_a : (-\infty, x_0) \cup (x_1, \infty) \to \mathbb{R} \setminus \{0\}$ is decreasing.
5. There is a repelling 2-cycle $\{p_0, p_1\}$ such that $|f_a^n(x)|$ tends to ∞ for each $x \notin [p_0, p_1]$.
6. There are a sequence of open intervals $(J_n)_{n\geq 0}$ and a sequence of closed intervals $(I_n)_{n\geq 0}$ such that (see Fig. 2).
   6.1.
   $$\cdots I_{2n+1} \sqsubseteq J_{2n} \sqsubseteq I_{2n-1} \sqsubseteq \cdots \sqsubseteq J_0 \sqsubseteq I_0 \sqsubseteq \cdots \sqsubseteq J_{2n-1} \sqsubseteq I_{2n} \sqsubseteq J_{2n+1} \cdots,$$
   (the notation $I \sqsubseteq J$ for two intervals I and J means that the right endpoint of I coincides with the left endpoint of J)
   6.2. $c_0 \in J_0$, $c_1 \in I_0$,
   6.3. for every $n \geq 0$, the endpoints of I_n and J_n are eventually mapped to 0,
   6.4. $f_a(J_n) = J_{n-1}$ and $f_a(I_n) = I_{n-1}$,
   6.5. for every n the orbit of any point of the interval J_n converges to 0,
   6.6. $(p_0, p_1) = \left(\cup_{n\geq 0} I_n\right) \cup \left(\cup_{n\geq 0} J_n\right)$.

Verification of these properties is the same as the proof of Lemma 1.2 in [1] and we do not present it here. Note that the points and the intervals depend on a and d, for example we should write $c_0(a, d)$, $J_n(a, d)$, $I_n(a, d)$ and etc. However, for simplicity, we have omitted them.

These properties show that the interesting dynamics of f_a happens in the interval $[p_0, p_1]$. In fact, the orbits of the points of the interval I_i can converge to 0, converge to infinity, or

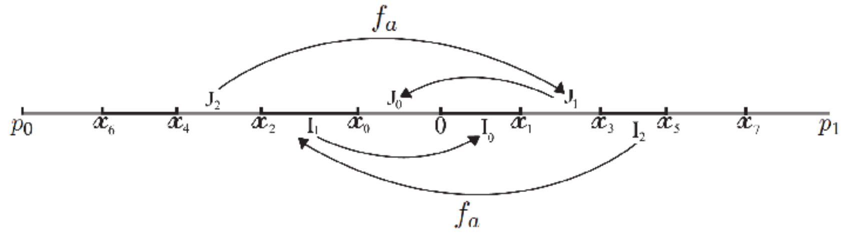

Figure 2. The images of I_i's and J_i's under f_a

always stay in $\cup_{i\geq 0} I_i$. Therefore, the behavior of the points whose orbits stay in $\cup_{i\geq 0} I_i$ under f_a is complicated. Note that if $f_a(c_1) \leq x_1$, then $\cup_{i\geq 0}^{\infty} I_i$ is invariant under f_a and the orbit of every point of $\cup_{i\geq 0}^{\infty} I_i$ meets $I_0 = [0, x_1]$ and afterward does not leave I_0. In this case the behavior of $f_a|_{I_0}$ is similar to the behavior of $F_\mu|_{[0,1]}$ when $\mu \leq 4$. Therefore, from now on, we suppose that $f_a(c_1) > x_1$. In this paper, we consider the following cases.

(1) $f_a(c_1) \in J_{2m-1}$ for some $m \geq 1$.

(2) $f_a(c_1) = p_1$.

We denote

$$\{x \in [p_0, p_1] : \ f_a^n(x) \in \cup_{i\geq 0} I_i \cup \{p_0, p_1\} \text{ for all } n \geq 0\}$$

by Λ_m if $f_a(c_1) \in J_{2m-1}$ for some $m \geq 1$ and by Λ if $f_a(c_1) = p_1$. Note that the value of a determines whether $f_a(c_1) \in J_{2m-1}$ for some m, or $f_a(c_1) = p_1$. In fact, for each $a < 0$ and even $d \geq 2$, only one of the Λ_m's and Λ may be defined. Thus, Λ_m and Λ depend on a and d but for simplicity in writing we omit them from these notations.

Also, we define $\widehat{\Lambda}_m \subseteq \Lambda_m$ as follows.

$$\widehat{\Lambda}_m = \{x \in \Lambda_m : \ f_a^n(x) \in \cup_{i=0}^{2m-2} I_i \text{ for all } n \geq 0\}.$$

It is clear that $\Lambda \cap J_i = \emptyset$ and $\Lambda_m \cap J_i = \emptyset$ for all $i \geq 0$. Also, we have $f_a(\Lambda) \subseteq \Lambda$, $f_a(\Lambda_m) \subseteq \Lambda_m$, and $f_a(\widehat{\Lambda}_m) \subseteq \widehat{\Lambda}_m$. Note that the critical point c_1 belongs to Λ but it does not belong to Λ_m and $\widehat{\Lambda}_m$. Moreover, $x \in \Lambda_m \setminus \{p_0, p_1\}$ is a periodic point of $f_a|_{\Lambda_m}$ if and only if x is a periodic point of $f_a|_{\widehat{\Lambda}_m}$. In the following, we present more properties of these sets.

Proposition 3.1. *The sets Λ, Λ_m, and $\widehat{\Lambda}_m$ are closed subsets of $[p_0, p_1]$ and are invariant under f_a.*

Proof. The set Λ is a closed subset of $[p_0, p_1]$ since $\Lambda = [p_0, p_1] \setminus \cup_{n\geq 0} f_a^{-n}(J_0)$. To prove the invariance of Λ under f_a, note that $f_a(p_0) = p_1$, $f_a(p_1) = p_0$, $f_a(I_{i+1}) = I_i$, and $f_a(I_0) \supseteq \cup_{i\geq 0} I_{2i}$. Hence, for $y \in \Lambda \cap (\cup_{i\geq 0} I_i)$ there is some $x \in \cup_{i\geq 0} I_i$ such that $f_a(x) = y$. The point x is in Λ, otherwise $f_a^n(x) \in J_0$ for some $n \geq 0$ which is a contradiction.

Similar arguments show that Λ_m and $\widehat{\Lambda}_m$ are closed subsets of $[p_0, p_1]$ and are invariant under f_a. ☐

Note that the theory of complex dynamics guarantees that the function $f_a(x) = ax^d(x-1) + x$ does not have any periodic attractor besides 0, when the orbit of critical point c_1 converges to the fixed point 0 or $f_a(c_1) = p_1$ (see [6, Theorem 9.3.1.], [13, Theorem 10.15.]). Therefore, no non-zero point of Λ and Λ_m could be in the basin of a periodic attractor. Thus, we can conclude the following proposition.

Proposition 3.2. *The sets Λ, Λ_m, and $\widehat{\Lambda}_m$ are totally disconnected.*

Proof. If the interval L is a subset of Λ_m or Λ, then this interval does not contain the critical points of f_a and also their preimages, because the critical point $c_0 \in J_0$, in the case $f_a(c_1) \in J_{2m-1}$, the orbit of c_1 finally enters to J_0, and in the case $f_a(c_1) = p_1$, each neighborhood of $f_a(c_1)$ contains infinitely many points of J_n's. Thus, f_a^n is monotone on L for all $n \geq 1$. This means that L is a homterval. We know that a homterval is a wandering interval or every point of it is contained in the basin of a periodic orbit (see [8, Chapter II, Lemma 3.1]). Since f_a does not have any wandering interval (see [8, Chapter II, Theorem 6.2]) and 0 is the only periodic attractor of f_a, then Λ and Λ_m are totally disconnected. Also, $\widehat{\Lambda}_m$ is totally disconnected because $\widehat{\Lambda}_m \subseteq \Lambda_m$. □

4 Conjugacy

In this section, by introducing some conjugacy maps, we show that the restriction of the shift map on the introduced subspaces in Section 2 can describe some dynamical behaviors of f_a on the invariant sets defined in Section 3.

Theorem 4.1.

(1) If $f_a(c_1) = p_1$, then (f_a, Λ) is conjugate to (σ, Σ).

(2) If $f_a(c_1) \in J_{2m-1}$ for some $m \geq 1$, then (f_a, Λ_m) is conjugate to (σ, Σ_m) and $(f_a, \widehat{\Lambda}_m)$ is conjugate to $(\sigma, \widehat{\Sigma}_m)$.

Proof. (1) Let $x \in \Lambda$. Define $h(x) = (s_0, s_1, s_2, \cdots)$, where

$$s_n = \begin{cases} j & \text{if } f_a^n(x) \in I_j \text{ for } j \in \mathbb{N}, \\ L & \text{if } f_a^n(x) \in [0, c_1) = I_L, \\ R & \text{if } f_a^n(x) \in (c_1, x_1] = I_R, \\ E & \text{if } f_a^n(x) = p_1, \\ W & \text{if } f_a^n(x) = p_0, \\ \star & \text{if } f_a^n(x) = c_1. \end{cases}$$

a. h is one-to-one. Note that $h(x) = (\overline{W, E})$ ($h(x) = (\overline{E, W})$, $h(x) = (\star, \overline{E, W})$, respectively) if and only if $x = p_0$ ($x = p_1$, $x = c_1$, respectively). Therefore, we assume that $x, y \in \Lambda \setminus \{p_0, p_1, c_1\}$, $x < y$, and $h(x) = h(y) = (s_0, s_1, s_2, \cdots)$. If $s_i \neq \star$ for $i \geq 0$, then $f_a^n(x), f_a^n(y) \in I_{s_n}$ for each $n \geq 0$. Since f_a is monotonic on I_{s_n}, $f_a^n([x,y])$ is a subinterval of I_{s_n}, for all $n \geq 0$, with the end points $f_a^n(x)$ and $f_a^n(y)$. Thus, all points of $[x,y]$ belong to Λ. This contradicts Proposition 3.2. Now, suppose that $h(x) = h(y) = (s_0, s_1, s_2, \cdots, s_{i-1}, \star, \overline{E,W})$, for some $i \geq 1$. Then $f_a^{i-1}(x), f_a^{i-1}(y) \in I_{s_{i-1}}$ and $f_a^i(x) = f_a^i(y) = c_1$. This is a contradiction since f_a is monotone on $I_{s_{i-1}}$.

b. h is onto. We know that $h(p_1) = (\overline{E,W})$, $h(p_0) = (\overline{W,E})$, and $h(c_1) = (\star, \overline{E,W})$.

First, let $(t_0, t_1, t_2, \cdots) \in \Sigma$, and $t_i \in \mathbb{N} \cup \{L, R\}$. Let $\widehat{I}_{t_0,t_1,\cdots,t_n} = \overline{I}_{t_0} \cap f_a^{-1}(\overline{I}_{t_1}) \cap \cdots \cap f_a^{-n}(\overline{I}_{t_n})$, where $\overline{I}_{t_i}$ is the closure of I_{t_i}. Note that if $t_n \in \mathbb{N}$, then $\overline{I}_{t_n} = I_{t_n}$, but if $t_n = L$ or R, then $\overline{I}_{t_n} = I_{t_n} \cup \{c_1\}$. By induction on n, we show that $\widehat{I}_{t_0,t_1,\cdots,t_{n+1}}$ is a closed subinterval of $\overline{I}_{t_0}$. It is clear that $\widehat{I}_{t_0} = \overline{I}_{t_0}$ is a closed interval. By induction hypothesis $\widehat{I}_{t_1,t_2,\cdots,t_{n+1}}$ is a closed subinterval of $\overline{I}_{t_1}$. Next we consider three cases.

(1) Let $t_0 = L$ or R. Then $t_1 \in \{L, R, 2, 4, \cdots\}$. In this case $\widehat{I}_{t_1,t_2,\cdots,t_{n+1}} \subseteq \overline{I}_{t_1} \subseteq [0, p_1]$ and $f_a : \overline{I}_{t_0} \to [0, p_1]$ is monotone.

(2) Let $t_0 = 1$. Then $t_1 = L$ or R, $\widehat{I}_{t_1,t_2,\cdots,t_{n+1}} \subseteq \overline{I}_{t_1} \subseteq [0, x_1]$, and $f_a : \overline{I}_{t_0} \to [0, x_1]$ is monotone.

(3) Let $t_0 = k \geq 2$. Then $t_1 = k-1$, $\widehat{I}_{t_1,t_2,\cdots,t_{n+1}} \subseteq \overline{I}_{t_1}$, and $f_a : \overline{I}_{t_0} \to \overline{I}_{t_1}$ is monotone.

Thus, in all these cases, $\widehat{I}_{t_0,t_1,\cdots,t_{n+1}} = \overline{I}_{t_0} \cap f_a^{-1}(\widehat{I}_{t_1,t_2,\cdots,t_{n+1}})$ is a closed subinterval of $\overline{I}_{t_0}$.

Therefore, $(\widehat{I}_{t_0,t_1,t_2,\cdots,t_n})_{n \geq 0}$ is a sequence of nested closed intervals. Hence, there is some $x \in \cap_{n \geq 0} \widehat{I}_{t_0,t_1,\cdots,t_n}$, consequently $f_a^n(x) \in \overline{I}_{t_n}$ for each $n \geq 0$. If $f_a^n(x) = c_1$ for some n, then $f_a^{n+1}(x) = p_1$. This is a contradiction since $p_1 \notin \overline{I}_{t_{n+1}}$. Therefore, $x \in I_{t_0,t_1,\cdots,t_n} = I_{t_0} \cap f_a^{-1}(I_{t_1}) \cap \cdots \cap f_a^{-n}(I_{t_n})$ and $h(x) = (t_0, t_1, t_2, \cdots)$.

Next, suppose that $(t_0, t_1, \cdots, t_{i-1}, \star, \overline{E,W}) \in \Sigma$, for some $i \geq 1$. Then $t_{i-k} \in \{k, k-1, \cdots, 2, 1, L, R\}$, for $1 \leq k \leq i$. To prove the ontoness of h, we use induction on k. For $k = 1$, $t_{i-1} \in \{1, L, R\}$. Since, the functions

$$f : I_L \to [0, p_1), \tag{1}$$

$$f : I_R \to [0, p_1), \tag{2}$$

and

$$f : I_1 \to I_L \cup I_R \cup \{c_1\} \tag{3}$$

are onto, there is $y_{i-1} \in I_{t_{i-1}}$ such that $f_a(y_{i-1}) = c_1$. Now, we suppose that there is $y_{i-k} \in I_{t_{i-k}}$ such that $f_a(y_{i-k}) = y_{i-(k-1)}$ where $k \geq 2$. If $t_{i-k} \in \mathbb{N}$, then $t_{i-(k+1)} \in \{t_{i-k}+1, L, R\}$. We have

$$f : I_{t_{i-k}+1} \to I_{t_{i-k}} \tag{4}$$

is onto and $\cup_{i \geq 0} I_{2i} \subseteq [0, p_1]$. Therefore, from (1), (2), and (4) we conclude that there is $y_{i-(k+1)} \in I_{t_{i-(k+1)}}$ such that $f_a(y_{i-(k+1)}) = y_{i-k}$. Finally, if $t_{i-k} = L$ or R, then $t_{i-(k+1)} \in \{1, L, R\}$. Again, from (1), (2), and (3) we conclude that there is $y_{i-(k+1)} \in I_{t_{i-(k+1)}}$ such that $f_a(y_{i-(k+1)}) = y_{i-k}$. Therefore, we showed that there is a finite sequence $\{y_{i-k}\}_{k=1}^{i}$ such that $y_{i-k} \in I_{t_{i-k}}$ and $f_a(y_{i-k}) = y_{i-(k-1)}$ for $1 \leq k \leq i$, where $y_i = c_1$. Thus, $h(y_0) = (t_0, t_1, \cdots, t_{i-1}, \star, \overline{E, W})$.

c. h is continuous. Suppose $x \in \Lambda$, $U = \prod_{i \geq 0} U_i$ is a basis element of $\mathcal{A}^\omega$ containing $h(x) = (t_0, t_1, t_2, \cdots)$; thus, there is some j such that $U_t = \mathcal{A}$ for $t \geq j$.

If $x = p_1$, then $h(x) = (\overline{E, W})$. In this case, choose $k \geq 1$ and $\delta > 0$ such that $2k > j$, $(2k, 2k-1, 2k-2, \cdots, 2k-j) \in U_0 \times U_1 \times U_2 \times \cdots \times U_j$, $I_{2k} \subseteq (p_1 - \delta, p_1]$, and $I_{2k-2} \cap (p_1 - \delta, p_1] = \emptyset$. Then $(\cup_{n \geq k} I_{2n}) \cap \Lambda = (p_1 - \delta, p_1) \cap \Lambda$. Therefore, for $y \in (p_1 - \delta, p_1) \cap \Lambda$, we have $y \in I_{2n}$ for some $n \geq k$. Hence $h(y) = (2n, 2n-1, \cdots, 2n-j, \cdots, 1, \cdots) \in U \cap \Sigma$. The proof of the continuity of h at p_0 is similar.

Next, let $t_i \in \mathbb{N} \cup \{L, R\}$ for each $i \geq 0$. Choose $\delta > 0$ such that $(x - \delta, x + \delta)$ intersects only $I_{t_0, t_1, t_2, \cdots, t_j}$ and does not intersect the other $I_{s_0, s_1, \cdots, s_j}$'s. Such δ exists, since $I_{s_0, s_1, \cdots, s_j}$'s are disjoint and the number of them is finite. Therefore $(x - \delta, x + \delta) \cap \Lambda \subseteq I_{t_0, t_1, t_2, \cdots, t_j}$. Then for each $y \in (x - \delta, x + \delta) \cap \Lambda$, the sequence $h(y)$ agrees with the sequence $h(x)$ in the first $j+1$ terms. Hence $h(y) \in U \cap \Sigma$.

Finally, we consider the case that $h(x) = (t_0, t_1, \cdots, t_{i-1}, \star, \overline{E, W})$ for some $i \geq 1$ or $h(x) = (\star, \overline{E, W})$. We may suppose $j \geq i+1$. Then $U_i = \{L, \star, R\}$ or $U_i = \mathcal{A}$. Choose k and $\epsilon > 0$ such that

$$(2k, 2k-1, 2k-2, \cdots, 2k-j+i+1) \in U_{i+1} \times U_{i+2} \times \cdots \times U_j,$$

$I_{2k} \subseteq (p_1 - \epsilon, p_1)$, and $I_{2k-2} \cap (p_1 - \epsilon, p_1) = \emptyset$. Thus, $\big(\cup_{n \geq k} I_{2n}\big) \cap \Lambda = (p_1 - \epsilon, p_1) \cap \Lambda$. Choose $\eta > 0$ such that $(c_1 - \eta, c_1 + \eta) \subseteq I_L \cup I_R \cup \{c_1\}$ and $f_a\big((c_1 - \eta, c_1 + \eta)\big) \subseteq (p_1 - \epsilon, p_1]$. Since $f_a^i(x) = c_1$, we choose $\delta > 0$ such that $(x - \delta, x + \delta) \cap \Lambda$ is a subset of $I_{t_0, t_1, \cdots, t_{i-1}}$ and $f_a^i\big((x - \delta, x + \delta)\big) \subseteq (c_1 - \eta, c_1 + \eta)$. Then for each $y \in (x - \delta, x + \delta) \cap \Lambda$, where $y \neq x$, we have $f_a^i(y) \in (c_1 - \eta, c_1 + \eta) \setminus \{c_1\}$ and consequently, $f_a^{i+1}(y) \in I_{2n}$ for some $n \geq k$. This implies that $h(y) \in U \cap \Sigma$.

d. h is a closed function. Since $h : \Lambda \to \Sigma$ is continuous, Λ is compact, and Σ is Hausdorff, we conclude that h is a closed function.

One can easily show that $h \circ f_a = \sigma \circ h$. Thus, (f_a, Λ) is conjugate to (σ, Σ) under h.

(2) The proof of Part (2) is similar to the proof of Part (1). We just use the following function. For $x \in \Lambda_m$, we define $g(x) = (s_0, s_1, s_2, \cdots)$, where

$$s_n = \begin{cases} j & \text{if } f_a^n(x) \in I_j \text{ for } j \in \mathbb{N}, \\ L & \text{if } f_a^n(x) \in [0, c_1) = I_L, \\ R & \text{if } f_a^n(x) \in (c_1, x_1] = I_R, \\ E & \text{if } f_a^n(x) = p_1, \\ W & \text{if } f_a^n(x) = p_0. \end{cases}$$

The last claim holds since $g(\widehat{\Lambda}_m) = \widehat{\Sigma}_m$. $\square$

By Propositions 2.3, 2.4, and Theorem 4.1, we have the following corollary.

Corollary 4.2. *(1) Let $f_a(c_1) = p_1$. Then (f_a, Λ) is chaotic on Λ.*

(2) Let $f_a(c_1) \in J_{2m-1}$ for some $m \geq 1$. Then $(f_a, \widehat{\Lambda}_m)$ is chaotic on $\widehat{\Lambda}_m$.

Remark 2. Since $\widehat{\Sigma}_m \subseteq \Sigma$ for every $m \in \mathbb{N}$ and (σ, Σ) is conjugate to (f_a, Λ), the set Λ has infinite nested subsets each of which is invariant and chaotic under f_a.

5 The number of the periodic points

In this section our aim is to present some algorithms for computing the number of the periodic points of period k of the systems $(\sigma, \widehat{\Sigma}_m)$, (σ, Σ_m), and (σ, Σ). These algorithms and Theorem 4.1 enable us to count the number of the periodic points of period k of $(f_a, \widehat{\Lambda}_m)$, (f_a, Λ_m), and (f_a, Λ).

Definition 4. We call a finite sequence $(s_0, s_1, \cdots, s_{k-1})$ of length k, allowable in $\widehat{\Sigma}_m$ if there is $\mathbf{s} \in \widehat{\Sigma}_m$ such that the first k entries of $\mathbf{s}$ are precisely $(s_0, s_1, s_2, \cdots, s_{k-1})$. We denote the set of all allowable finite sequences of length $k \geq 2$ that begin with a and end in b by $\mathcal{S}_{a,b}(k)$ and its cardinality by $|\mathcal{S}_{a,b}(k)|$.

Note that $\mathcal{S}_{a,b}(k)$ depends on m but for simplicity we omit m. In fact, if $\mathcal{A}_1 = \{L, R\}$ and $\mathcal{A}_m = \{L, R, 1, 2, \cdots, 2m-2\}$ for $m \geq 2$, then $\widehat{\Sigma}_m \subseteq \mathcal{A}_m^{\omega}$ for $m \geq 1$. Hence, the members of $\mathcal{S}_{a,b}(k)$ do not have entries exceeding $2m-2$. In the following we study the properties of $\mathcal{S}_{a,b}(k)$ that are useful in obtaining an algorithm for calculating the number of the periodic points of $(\sigma, \widehat{\Sigma}_m)$.

By Definitions 1-3, we have the following lemma.

Lemma 5.1. *Let $m \geq 1$ be given. For each $k \geq 2$ and $b \in \mathcal{A}_m$, we have $|\mathcal{S}_{L,b}(k)| = |\mathcal{S}_{R,b}(k)|$ and $|\mathcal{S}_{L,R}(k)| = |\mathcal{S}_{R,L}(k)|$. Also, $|\mathcal{S}_{L,b}(2)| = 0$ if b is odd, otherwise $|\mathcal{S}_{L,b}(2)| = 1$.*

Lemma 5.2. *Let $k \geq 3$ and m be given. Then*

1. $|\mathcal{S}_{L,2\ell}(k)| = |\mathcal{S}_{L,2\ell+1}(k-1)| + 2|\mathcal{S}_{L,L}(k-1)|$ *for $m \geq 3$ and $1 \leq \ell \leq m-2$.*
2. $|\mathcal{S}_{L,2\ell+1}(k)| = |\mathcal{S}_{L,2\ell+2}(k-1)|$ *for $0 \leq \ell \leq m-2$ and $m \geq 2$.*
3. $|\mathcal{S}_{L,L}(k)| = |\mathcal{S}_{L,1}(k-1)| + 2|\mathcal{S}_{L,L}(k-1)|$ *for $m \geq 2$.*
4. $|\mathcal{S}_{L,2m-2}(k)| = 2|\mathcal{S}_{L,L}(k-1)|$ *for $m \geq 2$.*
5. $|\mathcal{S}_{L,L}(k)| = 2|\mathcal{S}_{L,L}(k-1)|$ *for $m = 1$.*

Proof. Note that

1. $(L, a_1, a_2, \cdots, a_{k-2}, 2\ell) \in \mathcal{S}_{L,2\ell}(k)$ if and only if $(L, a_1, a_2, \cdots, a_{k-3}, a_{k-2}) \in \mathcal{S}_{L,2\ell+1}(k-1) \cup \mathcal{S}_{L,L}(k-1) \cup \mathcal{S}_{L,R}(k-1)$.

2. $(a_0, a_1, \cdots, a_{k-2}, 2\ell - 1) \in \mathcal{S}_{L,2\ell-1}(k)$ if and only if $(a_0, a_1 \cdots, a_{k-2}) \in \mathcal{S}_{L,2\ell}(k-1)$.

3. $(a_0, a_1, \cdots, a_{k-2}, L) \in \mathcal{S}_{L,L}(k)$ if and only if $(a_0, a_1, \cdots a_{k-2}) \in \mathcal{S}_{L,L}(k-1) \cup \mathcal{S}_{L,R}(k-1) \cup \mathcal{S}_{L,1}(k-1)$.

4. $(a_0, a_1, \cdots, a_{k-2}, 2m-2) \in \mathcal{S}_{L,2m-2}(k)$ if and only if $(a_0, a_1, \cdots, a_{k-2}) \in \mathcal{S}_{L,L}(k-1) \cup \mathcal{S}_{L,R}(k-1)$.

5. For $m = 1$, the entries of members of $\mathcal{S}_{a,b}(k)$ are only L or R. Hence, $(a_0, a_1, \cdots, a_{k-2}, L) \in \mathcal{S}_{L,L}(k)$ if and only if $(a_0, a_1, \cdots, a_{k-2}) \in \mathcal{S}_{L,L}(k-1) \cup \mathcal{S}_{L,R}(k-1)$.

Hence, by Lemma 5.1, our claims hold. □

By parts 1 and 2 of Lemma 5.2 we have the following corollary.

Corollary 5.3. *Suppose that $m \geq 3$ is given. Let $1 \leq \ell \leq m-2$. Then for $k \geq 4$ we have*

$$|\mathcal{S}_{L,2\ell}(k)| = |\mathcal{S}_{L,2\ell+2}(k-2)| + 2|\mathcal{S}_{L,L}(k-1)|. \tag{5}$$

Proposition 5.4. *Suppose that $m \geq 2$ and $1 \leq \ell \leq m-1$ are given. Then for $k \geq 3$ we have*

$$|\mathcal{S}_{L,2\ell}(k)| = \begin{cases} 2\sum_{t=0}^{n-2} |\mathcal{S}_{L,L}(k-2t-1)| & \text{if } k = 2n-1 \leq 2(m-\ell), \\ 1 + 2\sum_{t=0}^{n-2} |\mathcal{S}_{L,L}(k-2t-1)| & \text{if } k = 2n \leq 2(m-\ell), \\ 2\sum_{t=0}^{m-\ell-1} |\mathcal{S}_{L,L}(k-2t-1)| & \text{if } k \geq 2(m-\ell)+1. \end{cases}$$

Proof. First, by applying (5), $m-\ell-1$ times and then using part 4 of Lemma 5.2, the proposition holds for $k \geq 2(m-\ell)+1$.

To verify the other relations, we apply (5), $n-1$ times in the case $k = 2n \leq 2(m-\ell)$ and $n-2$ times in the case $k = 2n-1 \leq 2(m-\ell)$, to obtain the following equalities:

$$|\mathcal{S}_{L,2\ell}(k)| = |\mathcal{S}_{L,2\ell+k-2}(2)| + 2\sum_{t=0}^{n-2} |\mathcal{S}_{L,L}(k-2t-1)|; \text{ if } k = 2n,$$

$$|\mathcal{S}_{L,2\ell}(k)| = |\mathcal{S}_{L,2\ell+k-3}(3)| + 2\sum_{t=0}^{n-3} |\mathcal{S}_{L,L}(k-2t-1)|; \text{ if } k = 2n-1.$$

Now, by using $|\mathcal{S}_{L,2\ell+2n-2}(2)| = 1$ and $|\mathcal{S}_{L,2\ell+2n-4}(3)| = 2|\mathcal{S}_{L,L}(2)|$, the proof of the proposition is complete. □

By employing Part 2 of Lemma 5.2 and Proposition 5.4, the following corollary will be achieved. (Note that by Lemmas 5.1 and 5.2 we have $|\mathcal{S}_{L,2\ell+1}(3)| = 1$.)

Corollary 5.5. *Suppose that $m \geq 2$ and $0 \leq \ell \leq m-2$ are given. Then for $k \geq 4$ we have*

$$|\mathcal{S}_{L,2\ell+1}(k)| = \begin{cases} 2\sum_{t=1}^{n-1} |\mathcal{S}_{L,L}(k-2t)| & \text{if } k = 2n \leq 2(m-\ell)-1, \\ 1 + 2\sum_{t=1}^{n-1} |\mathcal{S}_{L,L}(k-2t)| & \text{if } k = 2n+1 \leq 2(m-\ell)-1, \\ 2\sum_{t=1}^{m-\ell-1} |\mathcal{S}_{L,L}(k-2t)| & \text{if } k \geq 2(m-\ell). \end{cases}$$

Next, we need to provide a recursive formula for $|\mathcal{S}_{L,L}(k)|$ which is stated in the following corollary.

Corollary 5.6. *Suppose that $m \geq 2$ is given. Then for $k \geq 3$ we have*

$$|\mathcal{S}_{L,L}(k)| = \begin{cases} 2\sum_{t=0}^{n-1} |\mathcal{S}_{L,L}(k-2t-1)| & \text{if } k = 2n+1 \leq 2m, \\ 1 + 2\sum_{t=0}^{n-1} |\mathcal{S}_{L,L}(k-2t-1)| & \text{if } k = 2n+2 \leq 2m, \\ 2\sum_{t=0}^{m-1} |\mathcal{S}_{L,L}(k-2t-1)| & \text{if } k \geq 2m+1. \end{cases}$$

Proof. By setting $\ell = 0$ in Corollary 5.5 and employing Part 3 of Lemma 5.2, the corollary holds for $m \geq 2$ and $k \geq 5$. The cases $k = 3,4$, also, hold by applying Lemmas 5.1 and 5.2. □

Definition 5. Let $s \in \mathcal{A}_m$. We denote the set of periodic points of period k of $(\sigma, \widehat{\Sigma}_m)$ that begin with s by $\mathcal{P}_s(k)$ and its cardinality by $|\mathcal{P}_s(k)|$.

It is clear that $(\overline{L, a_1, a_2, \cdots, a_{k-1}}) \in \mathcal{P}_L(k)$ if and only if $(L, a_1, a_2, \cdots, a_{k-1}, L) \in \mathcal{S}_{L,L}(k+1)$. Hence, by Lemma 5.1, for each $k \geq 1$, we have

$$|\mathcal{P}_L(k)| = |\mathcal{P}_R(k)| = |\mathcal{S}_{L,L}(k+1)|. \tag{6}$$

It is clear that $|\mathcal{P}_L(1)| = |\mathcal{P}_R(1)| = 1$ and $|\mathcal{P}_\ell(1)| = 0$ for $\ell \in \{1, 2, \cdots, 2m-2\}$.

By employing (6) and Corollary 5.6 the following corollary is obtained.

Corollary 5.7. *Suppose that $m \geq 1$ is given. Then for $k \geq 2$ we have*

$$|\mathcal{P}_L(k)| = \begin{cases} 2\sum_{t=0}^{n-1} |\mathcal{P}_L(k-2t-1)| & \text{if } k = 2n \leq 2m-1, \\ 1 + 2\sum_{t=0}^{n-1} |\mathcal{P}_L(k-2t-1)| & \text{if } k = 2n+1 \leq 2m-1, \\ 2\sum_{t=0}^{m-1} |\mathcal{P}_L(k-2t-1)| & \text{if } k \geq 2m. \end{cases}$$

Proposition 5.8. *Suppose that $m \geq 2$ and $1 \leq \ell \leq m-1$ are given. Then for each $k \geq 1$ we have*

$$|\mathcal{P}_{2\ell-1}(k)| = |\mathcal{P}_{2\ell}(k)| = \begin{cases} 2|\mathcal{S}_{L,2\ell}(k-2\ell+1)| & \text{if } k \geq 2\ell+1, \\ 0 & \text{if } k \leq 2\ell. \end{cases}$$

Proof. Let $(\overline{a_0,a_1,a_2,\cdots,a_{k-2},a_{k-1}}) \in \mathcal{P}_{2\ell}(k)$. Then $a_0 = a_k = 2\ell$, $a_0,a_1,\cdots,a_{2\ell-1} = 2\ell,2\ell-1,\cdots,1$ and $a_{2\ell} \in \{L,R\}$. Now if $k \leq 2\ell$, then $1 \leq a_{k-1} \leq 2\ell$, a contradiction. Hence, for $k \leq 2\ell$, $|\mathcal{P}_{2\ell}(k)| = 0$. If $k \geq 2\ell+1$, then $(a_0,a_1,a_2,\cdots,a_{k-2},a_{k-1})$ corresponds to a sequence of $\mathcal{S}_{L,2\ell}(k-2\ell+1)$ if $a_{2\ell} = L$ and it corresponds to a sequence of $\mathcal{S}_{R,2\ell}(k-2\ell+1)$ if $a_{2\ell} = R$. Therefore, $|\mathcal{P}_{2\ell}(k)| = 2|\mathcal{S}_{L,2\ell}(k-2\ell+1)|$.

By a similar argument we find that

$$|\mathcal{P}_{2\ell-1}(k)| = \begin{cases} 2|\mathcal{S}_{L,2\ell-1}(k-2\ell+2)| & \text{if } k \geq 2\ell+1, \\ 0 & \text{if } k \leq 2\ell. \end{cases}$$

By part 2 of Lemma 5.2 we have $|\mathcal{S}_{L,2\ell-1}(k-2\ell+2)| = |\mathcal{S}_{L,2\ell}(k-2\ell+1)|$, therefore, the assertion holds. □

By (6) and Propositions 5.4 and 5.8 we obtain the following corollary.

Corollary 5.9. *Suppose that $m \geq 2$ and $1 \leq \ell \leq m-1$ are given. Then for $k \geq 2\ell+2$ we have*

$$|\mathcal{P}_{2\ell-1}(k)| = |\mathcal{P}_{2\ell}(k)| = \begin{cases} 4\sum_{t=0}^{n-2} |\mathcal{P}_L(k-2\ell-2t-1)| & \text{if } k = 2\ell+2n-2 \leq 2m-1, \\ \\ 2+4\sum_{t=0}^{n-2} |\mathcal{P}_L(k-2\ell-2t-1)| & \text{if } k = 2\ell+2n-1 \leq 2m-1, \\ \\ 4\sum_{t=0}^{m-\ell-1} |\mathcal{P}_L(k-2\ell-2t-1)| & \text{if } k \geq 2m. \end{cases}$$

Moreover, $|\mathcal{P}_{2\ell-1}(2\ell+1)| = |\mathcal{P}_{2\ell}(2\ell+1)| = 2|\mathcal{S}_{L,2\ell}(2)| = 2$.

Let $\widehat{T}_m(k)$ be the set of periodic points of period k of $(\sigma,\widehat{\Sigma}_m)$ and $|\widehat{T}_m(k)|$ be its cardinality. By (6) and Proposition 5.8 that show $|\mathcal{P}_L(k)| = |\mathcal{P}_R(k)|$ and $|\mathcal{P}_{2\ell-1}(k)| = |\mathcal{P}_{2\ell}(k)|$, we have

$$|\widehat{T}_m(k)| = \begin{cases} 2\Big(|\mathcal{P}_L(k)| + \sum_{t=1}^{m-1} |\mathcal{P}_{2t-1}(k)|\Big) & \text{if } m \geq 2, \\ 2|\mathcal{P}_L(k)| & \text{if } m = 1. \end{cases}$$

Note that $\mathbf{s} \in \Sigma_m \setminus \{(\overline{E,W}),(\overline{W,E})\}$ is a periodic point of $\sigma|_{\Sigma_m}$ if and only if $\mathbf{s}$ is a periodic point of $\sigma|_{\widehat{\Sigma}_m}$. Therefore, if we denote the number of the periodic points of period k of (σ,Σ_m) by $|T_m(k)|$, then

$$|T_m(k)| = \begin{cases} |\widehat{T}_m(k)| + 2 & \text{if } k \text{ is even}, \\ |\widehat{T}_m(k)| & \text{if } k \text{ is odd}. \end{cases}$$

Now by employing the above results we can have an algorithm to compute the number of the periodic points of $(\sigma,\widehat{\Sigma}_m)$. Here we present this algorithm for $m = 2,3$.

Case $m = 2$. In this case $\mathcal{A}_2 = \{R,L,1,2\}$,

$$|\mathcal{P}_L(k)| = \begin{cases} 1 & \text{if } k = 1, \\ 2|\mathcal{P}_L(1)| & \text{if } k = 2, \\ 2|\mathcal{P}_L(2)| + 1 & \text{if } k = 3, \\ 2\big(|\mathcal{P}_L(k-1)| + |\mathcal{P}_L(k-3)|\big) & \text{if } k \geq 4, \end{cases}$$

$$|\mathcal{P}_1(k)| = \begin{cases} 0 & \text{if } k = 1,2, \\ 2 & \text{if } k = 3, \\ 4|\mathcal{P}_L(k-3)| & \text{if } k \geq 4, \end{cases}$$

and

$|\widehat{T}_2(k)| = 2|\mathcal{P}_L(k)| + 2|\mathcal{P}_1(k)|$.

Case $m = 3$. In this case $\mathcal{A}_3 = \{R, L, 1, 2, 3, 4\}$,

$$|\mathcal{P}_L(k)| = \begin{cases} 1 & \text{if } k = 1, \\ 2|\mathcal{P}_L(1)| & \text{if } k = 2, \\ 2|\mathcal{P}_L(2)| + 1 & \text{if } k = 3, \\ 2\big(|\mathcal{P}_L(3)| + |\mathcal{P}_L(1)|\big) & \text{if } k = 4, \\ 2\big(|\mathcal{P}_L(4)| + |\mathcal{P}_L(2)|\big) + 1 & \text{if } k = 5 \\ 2\big(|\mathcal{P}_L(k-1)| + |\mathcal{P}_L(k-3)| + |\mathcal{P}_L(k-5)|\big) & \text{if } k \geq 6, \end{cases}$$

$$|\mathcal{P}_1(k)| = \begin{cases} 0 & \text{if } k = 1,2, \\ 2 & \text{if } k = 3, \\ 4|\mathcal{P}_L(1)| & \text{if } k = 4, \\ 2 + 4|\mathcal{P}_L(2)| & \text{if } k = 5, \\ 4\big(|\mathcal{P}_L(k-3)| + |\mathcal{P}_L(k-5)|\big) & \text{if } k \geq 6, \end{cases}$$

$$|\mathcal{P}_3(k)| = \begin{cases} 0 & \text{if } k \leq 4, \\ 2 & \text{if } k = 5, \\ 4|\mathcal{P}_L(k-5)| & \text{if } k \geq 6, \end{cases}$$

and

$|\widehat{T}_3(k)| = 2|\mathcal{P}_L(k)| + 2|\mathcal{P}_1(k)| + 2|\mathcal{P}_3(k)|$.

The results of this algorithm for $2 \leq m \leq 4$ are shown in Table 1.

In the following theorem we determine the relation between the number of the periodic points of period k in (σ, Σ) and in (σ, Σ_m) for suitable m.

Theorem 5.10. *Let $|T(k)|$ be the number of the periodic points of period k of (σ, Σ). Then $|T(k)| = |T_m(k)|$ if $k = 2m - 1$ or $k = 2m$.*

Proof. We have $|T(1)| = |T_1(1)| = 2$ and $|T(2)| = |T_1(2)| = 6$. Therefore, let $k \geq 3$ and $\mathbf{s} = (\overline{s_0, s_1, \cdots, s_{k-1}})$ be a periodic point of (σ, Σ), where $k = 2m$ or $k = 2m - 1$ for some $m \geq 2$. If there is i such that $s_i \geq 2m - 1$, then for suitable $n \geq 0$, the periodic point $\sigma^n(\mathbf{s})$ of period k will begin with $2m, 2m - 1, \cdots, 2, 1, L$ or $2m, 2m - 1, \cdots, 2, 1, R$ which is impossible. Thus, $s_i \in \mathcal{A}_m \cup \{E, W\}$ and the theorem holds (see Table 1). □

		$\lvert\mathcal{P}_L(k)\rvert$	$\lvert\mathcal{P}_1(k)\rvert$	$\lvert\mathcal{P}_3(k)\rvert$	$\lvert\mathcal{P}_5(k)\rvert$	$\lvert\widehat{T}_m(k)\rvert$	$\lvert T_m(k)\rvert$	$\lvert T(k)\rvert$
m=2	k=1	1	0	-	-	2	2	-
	k=2	2	0	-	-	4	6	-
	k=3	5	2	-	-	14	14	14
	k=4	12	4	-	-	32	34	34
	k=5	28	8	-	-	72	72	-
	k=6	66	20	-	-	172	174	-
	k=7	156	48	-	-	408	408	-
	k=8	368	112	-	-	960	962	-
m=3	k=1	1	0	0	-	2	2	-
	k=2	2	0	0	-	4	6	-
	k=3	5	2	0	-	14	14	-
	k=4	12	4	0	-	32	34	-
	k=5	29	10	2	-	82	82	82
	k=6	70	24	4	-	196	198	198
	k=7	168	56	8	-	464	464	-
	k=8	404	136	20	-	1120	1122	-
m=4	k=1	1	0	0	0	2	2	-
	k=2	2	0	0	0	4	6	-
	k=3	5	2	0	0	14	14	-
	k=4	12	4	0	0	32	34	-
	k=5	29	10	2	0	82	82	-
	k=6	70	24	4	0	196	198	-
	k=7	169	58	10	2	478	478	478
	k=8	408	140	24	4	1152	1154	1154

Table 1. The results of the algorithm for $\lvert\widehat{T}_m(k)\rvert$, $\lvert T_m(k)\rvert$ and $\lvert T(k)\rvert$ when $2 \le m \le 4$ and $1 \le k \le 8$.

6 Conclusion

In this article we investigate the dynamics of the family $f_a(x) = ax^d(x-1) + x$ when $a < 0$ is a real number and $d \ge 2$ is an even integer. The dynamical behavior of the orbit of the positive critical point of f_a changes by varying a. We consider the case that the orbit of this critical point converges to 0 and also the case that this critical point is mapped to a repelling periodic point of period 2. In each case we show that there is a closed and totally disconnected invariant set on which the set of the periodic points of f_a is dense and f_a has a dense orbit. Moreover, we give a recursive formula for counting the number of the periodic points of f_a.

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