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Review Paper

A review on perfect, pretty good, state transfers and their applications

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Abstract. In this study, we review two significant topics: perfect State Transfer (PST) and Pretty Good State Transfer (PGST). These concepts involve designing interactions within a chain of spins on graph structures of networks, enabling a quantum state initially placed at one end to be perfectly or pretty transferred to the opposite end within a specified timeframe. PST and PGST play crucial roles in applications such as quantum information processing, quantum communication networks, and quantum chemistry.

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Contents

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1 Introduction

Perfect state transfer (PST) is a quantum phenomenon in which a quantum state can be transferred from one place to another without losing information. In other words, start from the primary state, let the walk take place for a certain amount of time, and find the same state in another place. The study of PST on graphs has attracted much attention due to its applications in quantum information processing and computing. The history of the concept of PST can be traced back to the early 2000s When it was first introduced as a theoretical concept. Chris Godsil is a mathematician who has made significant contributions to the study of PST. According to him, the probability of PST occurrence in the network depends on several factors, including the network topology, the coupling strength between different network nodes, and the presence of any external noise or decoherence effects that may disrupt the transmission process. In general, PST happens more in networks with symmetrical topology. Since the exact set of conditions required for this phenomenon is relatively rare, we lower our expectations and aim for pretty good state transfer (PGST), i.e. the ability to find a state as close as possible to the initial state elsewhere in the network. In recent years, PST and PGST have been studied in the context of physical systems such as spin systems, molecular systems, etc. The term random walk was first introduced in 1905 by Pearson as a mathematical formalism to study the path of a particle that includes random walks. Quantum walks, as a generalization of classical walks in the quantum field, were first introduced by Aharonov [5]. The ability to use different aspects of quantum mechanics, such as interference, probability distributions, and their entanglement, lead to different behaviours in quantum walks. Proper utilization of the non-classical characteristics of this type of walk provides numerous advantages in fields such as encryption, algorithms, and quantum simulation. Several different models of quantum walks have been introduced and studied so far, with two main types being discrete quantum walks and continuous quantum walks. It is interesting to note that a similar idea can be found in Feynman's work as well [71]. Connections between these two types have been discovered, and it has recently been shown that both discrete and continuous quantum walks can serve as a basis for universal computations. Continuous quantum walks were introduced by Farhi and Gutman [68], while discrete quantum walks were introduced by Watrous [153]. A specific type of discrete quantum walk known as the Hadamard walk was introduced by Ambainis et al. [9]. A review article by Kempe elaborates on these two types, along with some algorithmic applications [99]. A continuous-time quantum walk on a graph Γ is defined using the Schr$'$odinger equation with the real symmetric adjacency matrix A_Γ as the Hamiltonian. If $|\psi(t)\rangle \in \mathbb{C}^{|V|}$ is a time-dependent amplitude vector on the vertices of Γ, then the evolution of the quantum walk is given by $|\psi(t)\rangle = e^{-itA_\Gamma}|\psi(0)\rangle$. Where $\psi(0)\rangle$ is the initial amplitude vector. We usually assume that $\psi(0)\rangle$ is a unit vector. The amplitude of the quantum walk on vertex a at time t is given by the amplitude of the quantum walk on vertex a at time t is given by $\psi_a(t) = \langle a \mid \psi(t)\rangle$, while the probability of vertex a at time t is $p_a(t) = |\langle a \mid \psi(t)\rangle|^2$. where $|a\rangle, |b\rangle$ denote the unit vectors corresponding to the vertices a and b, respectively. The graph Γ has perfect state transfer if there exist vertices a and b in Γ and a time $t \in \mathbb{R}^+$. We say that Γ has universal perfect state transfer if PST occurs between all distinct pairs of vertices a and b of Γ. We call a graph Γ periodic if for any state $|\psi\rangle$, there is a time t so that $\left|\left\langle \psi \left| e^{-itA_\Gamma} \right| \psi \right\rangle\right| = 1$. In correspondence with discrete-time classical random walks, a discrete-time quantum walk has a quantum coin that is tossed each time step to determine which direction to move in. The coin is a quantum system of size $d_{\max}$, the largest degree of any vertex in the graph. The full quantum system is thus a combination of the position and the coin, we write the basis states as $|x, c\rangle$ where the first label is the vertex and the second the coin. A general state of a discrete-time quantum walk at time t can thus be written $|\psi(t)\rangle = \sum_{x,c} \alpha_{x,c}(t)|x, c\rangle$, where $\alpha_{x,c}(t) \in \mathbb{C}$.

The coin is *"tossed"* by applying a unitary operator, usually designed to ensure that only available paths can be chosen, i.e., based on the degree $d \leq d_{\max}$ at each vertex. A common choice is the unitary based on the Grover diffusion operator, which has elements $C^{(\Gamma)}_{ij} = 2/d - \delta_{ij}$. Any implementation of quantum information processing that is not based on optical qubits will require a mechanism for transporting qubits between gates and processors. There have been several theoretical proposals for qubit transport which are based on a chain of spin-half particles that are coupled by Heisenberg or XY interactions. The first proposal [33] was a homogenous chain of particles coupled by homogeneous, nearest-neighbor interactions. Bose proposed using an unmodulated and unmeasured spin chain as a channel for short-distance quantum communications. The qubit to be transmitted is placed on one spin and received later on a distant spin with some fidelity. He showed that the highest fidelity is obtained for short spin chains (number of spins ~ 100).

Matthias Christendel et al. [49] found that chains of any length were able to transport qubits perfectly but only if the coupling between neighbouring particles was inhomogeneous and carefully engineered in such a way as to be strong at the middle of the chain and weaker towards the ends of the chain. they showed that not only does perfect state transfer exist under certain conditions, but instead of examining transition in a linear network, one can choose

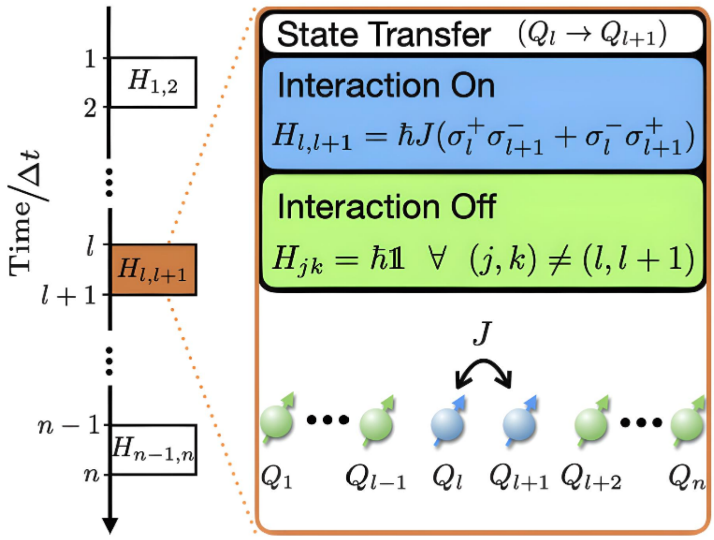


Figure 1. [84] Schematic illustration of the perfect n-qubit PST process Λ_t, which consists of several iterations of quantum operations. For the l-th iteration, the state is transferred from Q_l to Q_{l+1} by turning on the qubit-qubit interaction for a period of time $\Delta t = \pi/(2J)$, where J is $1/2$. Therefore, to transfer prepared states from Q_1 to Q_n, the iteration has to be performed $n-1$ times.

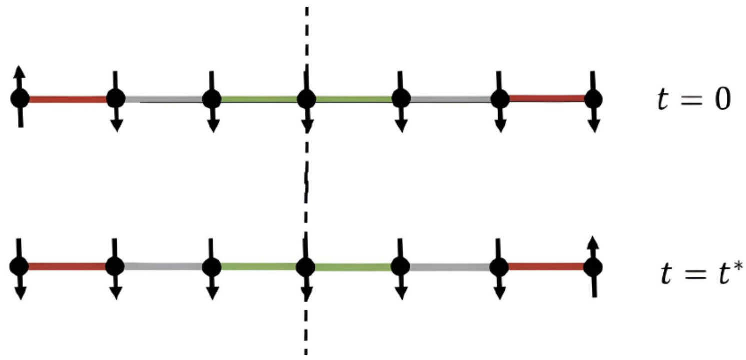


Figure 2. A schematic of an engineered chain with a mirror symmetric profile of the couplings (different colors correspond to different coupling strengths). Initially, the excitation is localized at the first site of the chain. At the retrieval time t^* we find that the excitation has been perfectly transferred to the other end of the chain.

a nonlinear network such as a hypercube network. A year later, they found in [50] that hypercubic networks of any dimension can transport qubits between pairs of antipodal nodes. They stated that if PST occurs in graphs with mirror symmetry, then a certain proportion of eigenvalue differences will be significant and used this fact to show that PST does not occur in the path P_n where $n \geq 4$ (where all coupling constants are identical).

To achieve PST on longer distances, they showed that if a graph admits PST then every Cartesian power of the graph also has PST. Therefore, they check PST for powers of P_2 and P_3 and use it to show that weighted paths of arbitrary weight may accept PST. In 2006, Alistair Kay et al. [96] studied the state transfer problem using other coupling models. It is obtained

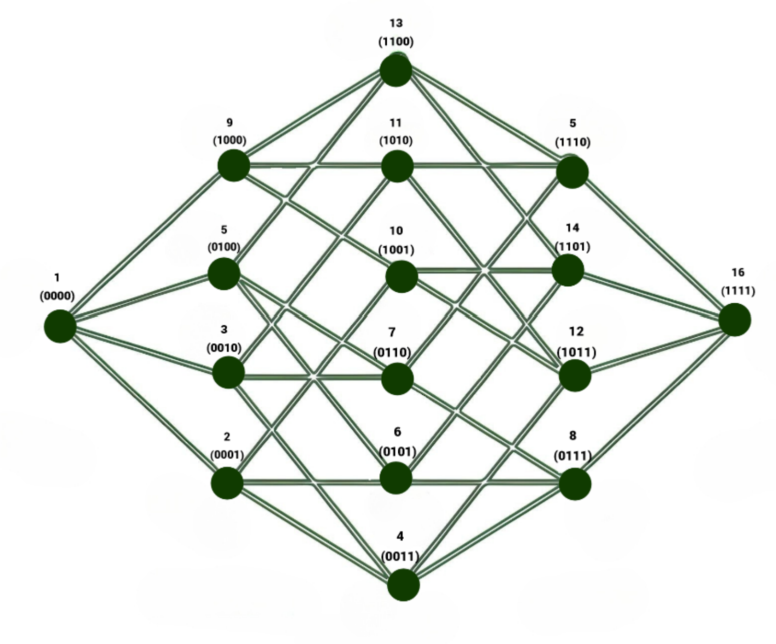


Figure 3. [67] Illustration of the binary labeling of the nodes of a d-dimensional hypercube (here $d = 4$). Exhibits perfect quantum transport between nodes $(0,0,\cdots,0)$ and $(1,1,\cdots,1)$.

through an iterative algorithm based on the concept of inverse eigenvalue problems (IEP).In 2007 Nitin Saxena et al. [130] took a step towards the classification of network topologies, which exhibit periodic quantum dynamics. They proposed circulant graphs as potential candidates for modelling quantum spin networks enabling the perfect state transfer between antipodal sites in a network and showed that PST does not occur if the circulant graph is of odd order. Furthermore, they proved that a quantum system, whose Hamiltonian is the same as the adjacency matrix of a circulant graph, is periodic if and only if all eigenvalues of the graph are integers (that is, the graph is integral). Therefore, circulant graphs having PST must be integral circulant graphs. A year later Christopher Fecer et al. [67] developed the results obtained [49]. They showed that by introducing additional links into a hypercubic network in a specific way, the destination node of a qubit can be changed. Thus if a user were able to choose which extra links in the network were "*switchedon*", they would be able to route a qubit to any desired destination within the network and in a duration that is independent of the network size.

Jafarizadeh and Soufiani [88] investigated the perfect transfer of quantum states in regular distance graphs. For these graphs, the vertices can be classified in such a way that the graph is layered into separate layers. That is, using the algebraic properties of these graphs, any regular distance network can be mapped onto a Linear chain to generate PST for long distances. In these graphs, the adjacency matrix is defined based on the shortest path between two vertices. Ann Bernasconi et al. [30] investigate the perfect state transfer between two particles in quantum networks modelled by a large class of cubic graphs, and generalize the results of the articles [49,67]. In 2009, Milan Bašić et al. [24] completed the results of [130].

They also proved that in the class of unitary Cayley graphs, PST can occur in only two of them (K_2 and C_4). Then they present [25] simple conditions to describe integral distance graphs that allow the occurrence of PST in terms of eigenvalues. They stated that integral distance graphs with minimum possible vertices that provide PST (except unitary Cayley graphs) $ICG_8(\{1,2\})$ and $ICG_8(\{1,4\})$ are Moreover, it is conjectured that the bi-class of integral distance graphs $ICG_n(\{1,\frac{n}{2}\})$ and $ICG_n(\{1,\frac{n}{4}\})$ when $n \in {}_8\mathbb{N}$, have PST. They proved this conjecture in the same article. In addition, it is shown that no circulant graph allows the occurrence of PST in a class of graphs whose number of vertices is not square-free. In the same year Andra Casaccina and other researchers showed in [42] by showing that it is possible to achieve perfect transfer by shifting (adding) energy to specific vertices. This technique appears to be a potentially powerful tool to change and in some cases improve, the transfer capabilities of quantum networks. Analytical results are presented for all-to-all networks and all-to-all networks with a missing link. Moreover, they evaluate the effect of random fluctuations on the transmission fidelity. Angeles-Canul et al. [10] investigate the complete transition of the quantum state on weighted graphs. They proved the join of a weighted two-vertex graph with any regular graph has PST. This generalizes a result of Casaccino et al. [42]. Where the regular graph is a complete graph or a complete graph with a missing link. In contrast, the half-join of a weighted two-vertex graph with any weighted regular graph has no PST. This implies that adding weights in a complete bipartite graph does not help in achieving PST. A Hamming graph has PST between each pair of its vertices. This is obtained using a closure property on weighted Cartesian products of perfect state transfer graphs. Moreover, on the hypercube, we show that PST occurs between uniform superpositions on pairs of arbitrary subcubes. This generalizes the results of Bernasconi et al. [30]. In 2010, Bašić and Petković [26] present more results on PST in integral circulant graphs. The non-existence of PST is proved for several classes of well-distance graphs that have a single divisor d_0 (a divisor that is prime to other divisors). The same result for classes of ICG with features NSF[a] (that is, for every $d \in D$, $\frac{n}{d}$ is not a perfect square) is obtained. A direct consequence of these results is the description of the ICG graph with two divisors that have PST. Finally, it is proved that ICG with the number of vertices $n = 2p^2$ do not have PST. In [11] Canul et al. proposed a new family of graphs in which PST occurs. Their structure is based on the adjoint operator on graphs, their circulant generalizations, and their Cartesian product. The results obtained in this article are based on the results of the articles [25, 26] and integral circulant graphs and regular graphs that have PST. Specifically, they showed that $ICG_n(\{2,\frac{n}{2^b}\} \cup Q)$ where $b \in \{1,2\}$ and $n \in {}_{16}\mathbb{N}$ and Q is a subset of odd divisors of n, has PST. In addition, they proved that there exists a family of non-periodic graphs formed by joining a bi-vertex graph without edges to a class of regular graphs, which have PST. Y. Ge et al. [74] described new constructions of graphs that show PST in time-continuous quantum excursions. Their structures are based on double cone types and the product of Cartesian graphs (such as n-cube Q_n). Some of their results are:

1. If Γ is a graph that has PST at time t_Γ and $t_\Gamma\mathrm{Spec}(\Gamma) \subseteq \mathbb{Z}\pi$, and H is also a distance

[a] Not Square-free

graph with are odd eigenvalues, then $\Gamma \times H$ has PST. Also, if H is a regular graph that has PST at time t_H and $t_H|V_H|\text{Spec}(\Gamma) \subseteq 2\mathbb{Z}\pi$, then the graph $\Gamma[H]$ has PST. For example $Q_{2n} \times H$, and $\Gamma[Q_n]$ have PST, whenever H is any circulant with odd eigenvalues and Γ is any integral graph, for integer $n \geq 2$. These complement constructions of perfect state transfer graphs based on Cartesian products.

2. Double cone $\overline{K}_2 + \Gamma$ (Γ connected graph), if the weight of the edges of the cone is proportional to the Peron eigenvector Γ, it has PST.

3. for an infinite family $\mathcal{G}$ from regular graphs, there is a distance connection. Therefore, the graph $K_1 + \mathcal{G} + \overline{K}_n + \mathcal{G} + K_1$ has PST.

In contrast, no perfect state transfer exists if a complete bipartite connection is used (even in the presence of weights). They also describe a generalization of the path-collapsing argument, which reduces questions about perfect state transfer to simpler (weighted) multigraphs, for graphs with equitable distance partitions. They show that cylindrical $K_1 + \mathcal{G} + \overline{K}_n + \mathcal{G} + K_1$ for every $\mathcal{G}$ family of regular graphs does not have PST. In [101], Kay reviewed what had been studied and studied from PST. He showed how one designs the (fixed) interactions of a chain of spins so that a quantum state, initially inserted on one end of the chain, is perfectly transferred to the opposite end in a fixed time. The perfect state transfer systems are then used as a constructive tool to design Hamiltonian implementations of other primitive protocols such as entanglement generation and signal amplification in measurements, before showing that universal quantum computation can be implemented in this way. In 2011, Bašić proved in [27] using the circulant graph that there exists between two distinct vertices a and b in the spin network PST, whenever $\tau \in \mathbb{R}^+$ exist that $|F(\tau)_{a,b}| = 1$ where $F(t) = \exp(iAt)$ and A is the circulant graph adjacency matrix. Saxena et al. [130] proved for the vertex a of the graph and the time $\tau \in \mathbb{R}^+$, $|F(\tau)_{a,a}| = 1$ if and only if all eigenvalues of the graph are integers. In the integral circulant graph n vertices $ICG_n(D)$ with set of vertices $\mathbb{Z}_n = \{0,1,\ldots,n-1\}$, vertices a and b are adjacent when $\gcd(a-b,n) \in D$ which $D = \{d \mid d \mid n, 1 \leq d \leq n\}$. These graphs are symmetric and have many applications in the theory of chemical graphs. In this paper, they show that $ICG_n(D)$ has PST if and only if $D = \tilde{D}_3 \cup D_2 \cup 2D_2 \cup 4D_2 \cup \{\frac{n}{2^\alpha}\}$, that $n \in {}_4\mathbb{N}$, $\alpha \in \{1,2\}$, $D_2 = \{d \in D \mid \frac{n}{d} \in {}_8\mathbb{N}+4\} \setminus \{\frac{n}{4}\}$, $\tilde{D}_3 = \{d \in D \mid \frac{n}{d} \in {}_8\mathbb{N}\}$. In this article, the distance between vertices that occurs between them PST is calculated and also the $ICG_n(D)$ graphs that have PST are determined. In the same year Cheong and Godsil in the article [48] examined PST in cubelike graph $X(C)$ They researched. Let C be an infinite vector set in the vector space $\mathbb{Z}_2^d$. The cubelike graph $X(C)$ has $\mathbb{Z}_2^d$ as its vertex set, and two elements of $\mathbb{Z}_2^d$ are adjacent if their difference is in C. they consider M is the $d \times |C|$ matrix where the elements of C are its columns and its row space of M the code of X. They used these codes to study PST in $X(C)$. Bernasconi et al. [30] proved that in the graph $X(C)$ in time $\frac{\pi}{2}$, PST occurs if and only if the elements of the sum be C is not zero. Cheung and Godsil, it is investigated what results if this sum is zero?. They proved that if PST occurs in a cubelike graph, it must hold in time $\tau = \frac{\pi}{2D}$, where D is the greatest common divisor of the weights of the code words. Furthermore, they show that PST occurs in time $\frac{\pi}{4}$

if and only if $D = 2$ and the code is self-orthogonal. Kay extended [97] in the article [98]. He showed that if PST occurs between two vertices in a simple graph, none of these two vertices can be involved in PST with a third vertex. Viven Kendon and Christino Tamon [103], investigated the results related to PST in adjoint structures, weighted paths and circulant graphs. They also discussed discrete-time quantum walks. In the article [130], Petković and Bašić extended and investigated the results of [26] about the necessary and sufficient conditions for the existence of PST in circulant graphs. Jiang Zhu et al. [159] investigated the necessary and sufficient conditions for the existence of period in a graph and the existence of PST between antipodal vertices in graphs with extreme diameter. In 2012 Bachmann et al. [21] investigated asymmetric graphs that have PST. In the same year, in the articles [76,77], Godsil investigated the necessary conditions for the occurrence of PST and its applications in cryptographic systems. In [76], he studied the results of PST according to algebraic graph theory with the help of the properties of the function $\exp(itA)$ where A is the adjacency matrix of the graph. It was also shown in [77] that if PST occurs in a graph, then the square of its spectral radius is either an integer or lies in a quadratic extension of the rationals. As a result, for any integer k, there are only finitely many graphs with maximum valency k on which PST occurs. He also showed that if there is a PST from vertex u to vertex v, then the graphs $\Gamma \setminus u$ and $\Gamma \setminus v$ are cospectral and any automorphism Γ that fixes u, must fix v (and conversely). Godsil et al. [79] by solving the problem posed by Bose, they determine the exact number of qubits in modulated XY-Hamiltonian chains, that permit the transfer with fidelity arbitrarily close to 1, a phenomenon called pretty good state transfer. They proved that in a spin chain, PGST occurs if and only if the number of vertices is $2p - 1$ or $p - 1$ (p is prime) or $2^m - 1$ be ($m \in \mathbb{N}$). In the article [89], Jafarizadeh et al. studied the engineering problems of Hamiltonians for the occurrence of PST. In the same year, Vinet and Zhedanov construct XX quantum systems with nearest-neighbor interactions that enable the occurrence of PST. Sets of orthogonal polynomials (OPS) are in correspondence with such systems. The key observation is that for any admissible excitation energy spectrum, the weight function of the associated OPS is uniquely prescribed. This entails the complete characterization of these PST models with the mirror symmetry property arising as a corollary. In 2013 fan and Godzil stated in [69] during research that PST is a rare phenomenon, therefore they investigated PGST and the conditions of its occurrence. In this article, the condition of existence of PGST in the double stars graph $S_{k,l}$ is investigated and it is proved that PST does not occur in this double stars graph. In 2014, Cameron et al. [36] investigated the concept of universal perfect state transfer, where there is PST between each pair of graph vertices. They proved the following results about graphs with these properties:

1. Graphs with universal state transfer have distinct eigenvalues and flat eigenbasis (each eigenvector has entries which are equal in magnitude).

2. The switching automorphism group of a graph with universal state transfer is abelian and its order divides the size of the graph. Moreover, if the state transfer is perfect, the switching automorphism group is cyclic.

3. There is a family of complex oriented prime-length cycles which has universal PGST. This provides a concrete example of a family of graphs with this universal property.

4. matrices which has universal PGST.

On the other hand, Kay showed in [98] that no graph with a real-valued adjacency matrix can have universal perfect state transfer. Finally, they proved a spectral characterization of universal perfect state transfer for graphs switching equivalent to circulants. In [160], Zhou et al. investigated the existence of PST in the induced subgraph of a graph. They assumed that Γ is a graph with adjacency matrix A, for an eigenvalue μ of A with multiplicity k, a set of stars in Γ is considered a vertex set X in Γ with $|X| = k$ as so that the induced subgraph $\Gamma - X$ does not have μ as its eigenvalue. They proved that PST does not occur between any two vertices in X. In 2015, Coutinho et al. [54] sufficient conditions for the existence of PST on distance regular graphs. Using this condition, a new example of complete state transitions in simple graphs is presented. In the same year, Coutinho and Henry Liu in [55], investigated the complete transition of the Laplacian mode in trees. They stated that they are interested in investigating graphs that allow quantum state transfer with 1 accuracy. For this reason, enough to consider the action of the Laplacian matrix of the graph in a vector space of suitable dimension their main result is that if the underlying graph is a tree with more than two vertices, then perfect state transfer does not happen. They also explore related questions, such as what happens in bipartite graphs and graphs with an odd number of spanning trees. Finally, they consider the model based on the XY-Hamiltonian, whose action is equivalent to the action of the adjacency matrix of the graph. In this case, they conjecture that perfect state transfer does not happen in trees with more than three vertices. In the same year, Kirkland in [104] presented an interesting method to check the existence of PST. He stated that suppose Γ is a weighted graph with adjacency matrix A and state transition matrix $U(t) = e^{itA}$ which (s, r)-entry of $U(t)$ with symbol $|u(t)_{s,r}|^2$ is shown. Between these two vertices in the graph, PST exists at a specific time such as t_0, if $|U(t_0)_{s,r}| = 1$. A parallel set of results using the Laplacian matrix Γ is also developed, and examples illustrating the results are included. These techniques rely on the spectral decomposition of adjacency matrix (respectively, Laplacian) and perturbation theory for eigenvalues and eigenvectors of symmetric matrices. In 2016, Eckelsberg et al. [3], examined the Corona digraph product. They proved that the corona product of two graphs has no Laplacian perfect state transfer, whenever the first graph has at least two vertices. This complements a result of Coutinho and Liu in [55] who showed that no tree of size greater than two has Laplacian perfect state transfer. In contrast, they proved that the corona product of two graphs exhibits Laplacian pretty good state transfer, under some conditions. This provides the first known examples of families of graphs with Laplacian pretty good state transfer. Their result extends the work of Fan and Godsil in [69] on double stars to the Laplacian setting. Moreover, they also show that the corona product of any cocktail party graph with a single vertex graph has Laplacian pretty good state transfer, even though odd cocktail party graphs have no perfect state transfer. In the same year, Coutinho and Godsil [56] studied graphs whose adjacency matrix is a sum of tensor products of 01-matrices, focusing on the case where a graph is the tensor product of two other graphs. As a result, they constructed many new which have PST. In the same year in [116, 117] Pal et al., investigated PGST and PST differently. They stated in [116] that since there are very few graphs in which PST occurs, it is useful to find new graphs with the PST property. A good way to construct new graphs is by forming NEPS (Non-Complete Extended p-sum). A path on three vertices exhibits perfect state transfer and so they investigated some NEPS of the path on three vertices. A sufficient condition is found for an NEPS of the path on three vertices to have perfect state transfer. Using these NEPS, some other graphs are also constructed having perfect state transfer. In [117], they found that NEPS of the path on three vertices with basis containing tuples with hamming weights of both parties did not exhibit perfect state transfer. But these NEPS admit PGST with an additional condition. Further, we investigate PGST on the Cartesian product of graphs and they found that a graph can have PGST from a vertex u to two different vertices v and w. In 2017 Connelly et al. [52] proved properties of graphs with universal perfect state transfer that generalizes the results of Cameron et al. [36]. In this paper, they construct non-circulant families of graphs with universal perfect state transfer. All prior known constructions were circulants. Moreover, they proved that if a circulant, whose order is prime, prime squared, or a power of two, has universal perfect state transfer then its underlying graph must be complete. This is nearly tight since there are universal perfect state transfer circulants with non-prime-power order where some edges are missing. Coutinho et al. [57] investigated the study of a continuous-time quantum walk on a path graph. In this paper, they showed that for any odd prime p and every positive integer t, with $2^t p - 1$ number of vertices, PGST between vertices a and $n + 1 - a$ for each a that is a multiple of 2^{t-1} occurs. This gives the first examples of pretty good state transfer occurring between internal vertices on a path when it does not occur between the extremal vertices. Johnston et al. [91] focus on the Laplacian matrix and those graphs for which the Laplacian can be diagonalized by a Hadamard matrix. They gave a simple eigenvalue characterization for when such a graph has PST at time $\frac{\pi}{2}$; this characterization allows one to choose the integer eigenvalues to build graphs having perfect state transfer. They characterize the graphs that are diagonalizable by the standard Hadamard matrix, showing a direct relationship to cubelike graphs. Also, they give an optimality result, showing that among regular graphs of degree at most 4, the hypercube is the sparsest Hadamard diagonalizable connected unweighted graph with PST. In the same year, Pal et al. [115, 118] introduced a sufficient condition for the gcd-graph so that in time $\frac{\pi}{2}$, the graph has periodicity and PST and using it shows that deduce that there exists gcd-graph having PST over an abelian group of order divisible by 4. Also, they found a necessary and sufficient condition for a class of gcd-graphs to be periodic at π. Using this, they characterize a class of gcd-graphs not exhibiting PST at gcd-graphs that gives PST in time $\frac{\pi}{2^k}$ for every positive integer k. In addition, in the same year, they investigated the existence of PGST in C_n and $\bar{C}_n$ in [118]. They found that PGST occurs in a cycle on n vertices if and only if n is a power of two and it occurs between every pair of antipodal vertices. In addition, they look for PGST in more general circulant graphs. they proved that the union (edge-disjoint) of an integral circulant graph with a cycle, each has 2^k ($k \geq 3$) vertices, admits

PGST. The complement of such a union also admits PGST. This enables them to find some non-circulant graphs admitting PGST. Among the complement of cycles, they also found a class of graphs not exhibiting PGST. In 2018 Pal [120] presented a class of circulant graphs admitting pretty good state transfer. Also, he found some circulant graphs not exhibiting PGST. This generalizes several pre-existing results on circulant graphs admitting pretty good state transfer. In 2019 Tan et al. [140] present a characterization of connected simple Cayley graphs $\Gamma = \mathrm{Cay}(G,S)$, where G is an abelian group and S is a non-empty subset of G. They showed that many previous results on periodicity and existence of PST of circulant graphs (where the underlying group G is cyclic) and cubelike graphs $G = (F_{2n}, +)$ can be derived or generalized to arbitrary abelian case in unified and more simple ways from our characterization. Also, they gave a positive answer to the question: Are there cubelike graphs having PST at time t where t is arbitrarily small?. In the same year, Eisenberg et al. [63] constructed infinite families of graphs in which PGST can be induced by adding a potential to the nodes of the graph (i.e. adding a number to a diagonal entry of the adjacency matrix). Indeed, they showed that given any graph with a pair of cospectral nodes, a simple modification of the graph, along with a suitable potential, yields PGST between the nodes. This generalizes previous work, concerning graphs with an involution, to asymmetric graphs. In 2020 Bommel in [34], examined conditions for a pair of strongly cospectral vertices to have PGST in terms of minimal polynomials and provided cases PGST can be ruled out. He also provided new examples of simple, unweighted graphs exhibiting pretty good state transfer. Finally, they consider modifying paths by adding symmetric weighted edges and apply these results to this case. Cao et al. [37] investigated the existence of PST in the Cayley graph $\mathrm{Cay}(\mathrm{D_n},\mathrm{S})$ with non-normal S and they showed that $\mathrm{Cay}(\mathrm{D_n},\mathrm{S})$ cannot PST if n is odd. For even integers$n$, it is proved that if $\mathrm{Cay}(\mathrm{D_n},\mathrm{S})$ has PST, then S is normal. Gamol Moraby et al. [109] studied the spectral features, on fractal-like graphs, of Hamiltonians which exhibit the special property of PST: the transmission of quantum states without dissipation. The essential goal is to develop the theoretical framework for understanding the interplay between perfect quantum state transfer, spectral properties and the geometry of the underlying graph, to design novel protocols for applications in quantum information science. They presented a new lifting and glueing construction and used this to prove results concerning an inductive spectral structure, applicable to a wide variety of fractal-like graphs. They illustrated this construction with explicit examples for several classes of diamond graphs. In the same year, Godsil et al. [80], studied PST, using techniques in algebraic graph theory. They are motivated by the study of state transfer in continuous-time quantum walks, which is understood to be a rare and interesting phenomenon they consider a perturbation on an edge uv of a graph where they add a weight γ to the edge and a loop of weight γ to each of u and v. They characterize when this perturbation results in strongly cospectral vertices u and v. Applying this to strongly regular graphs, they gave infinite families of strongly regular graphs where some perturbation results in perfect state transfer. Further, they showed that for every strongly regular graph, there is some perturbation which results in PGST. In the same year, Godsil et al. [81] investigated the existence of PST on directed graphs. They studied the phenomena, unique to oriented graphs, of multiple state transfer, where there is a set of vertices such that perfect state transfer occurs between every pair in that set. They gave a characterization of multiple state transfer and a new example of a graph where it occurs. Cao et al. [38,39], investigated the existence of PGST and PST on Cayley graphs over dihedral groups. Ada Chan et al. [46] initiated the study of pretty good quantum fractional revival in graphs, a generalization of pretty good quantum state transfer in graphs. They gave a complete characterization of pretty good fractional revival in a graph in terms of the eigenvalues and eigenvectors of the adjacency matrix of a graph. This characterization follows from a lemma due to Kronecker on Diophantine approximation and is similar to the spectral characterization of PST in graphs. Using this, they gave complete characterizations of when pretty good fractional revival can occur in paths and cycles. Li et al. [105], gave a few sufficient conditions for NEPS of complete graphs to be periodic or exhibit PST. Luo et al. [106] consider the existence of PST on Cayley graphs over semi-dihedral groups which are non-abelian. Using the representations of semi-dihedral groups, they proved some necessary and sufficient conditions for Cayley graphs over semi-dihedral groups admitting PST. By those conditions, they presented examples of PST on Cayley graphs over semi-dihedral groups. In addition, they proposed results about whether some new Cayley graphs over non-abelian groups admit PST. In 2022, Arezoomand et al. [16] established a characterization of Cayley graphs over dicyclic groups T_{4n}, having PST. In the same year, he [17], gave a characterization of Cayley graphs over groups with an abelian subgroup of index 2 having PST, which improves the earlier results on Cayley graphs over abelian groups, dihedral groups and dicyclic group and determines Cayley graphs over generalized dihedral groups and generalized dicyclic groups having PST. Cao et al. [41], showed that graphs $\mathrm{Cay}(\mathrm{SD}_{8\mathrm{n}},\mathrm{S})$ have PGST for some suitable subsets S if n is a power of 2. Moreover, they presented a sufficient and necessary condition for a non-integral graph $\mathrm{Cay}(\mathrm{SD}_{8\mathrm{n}},\mathrm{S})$ to admit PGST. Some concrete constructions of Cayley graphs over semi-dihedral groups having PGST are also presented. Kubota et al. [104] study PST in Grover walks, which are typical discrete-time quantum walk models. In particular, they focused on states associated with the vertices of a graph. They call such states vertex-type states. PST between vertex-type states can be studied via Chebyshev polynomials. They derived a necessary condition on the eigenvalues of a graph for PST between vertex-type states to occur. In addition, they perfectly determined the complete multipartite graphs whose partite sets are the same size on which PST occurs between vertex-type states, together with the time. Zhang et al. [158] investigated the unsigned LPST and LPGST in Q-graph of the graphs. They showed that if all the signless Laplacian eigenvalues of a regular graph Γ are integers, then the Q-graph of Γ has no signless Laplacian perfect state transfer. They also gave a sufficient condition that the Q-graph of a regular graph has signless LPST when Γ has signless LPST between two specific vertices. In the same year, Pal in [125] investigated the existence of quantum state transfer between a pair of twin vertices in a graph when the edge between the vertices is perturbed. He found that the removal of any set of pairwise non-adjacent edges from a complete graph with several vertices divisible by 4 results LPST at $\frac{\pi}{2}$ time between the end vertices of every edge removed. Further, He showed that all Laplacian integral graphs with a pair of twin vertices exhibit LPST when the edge between the vertices is perturbed. Wang et al. [152] first gave a necessary and sufficient condition for a graph to have LPGST. As an application of such results, they gave a complete characterization of LPGST in paths. Miki et al. in [109] studied the existence of PST on a new solvable two-dimensional spin lattice model defined on a regular triangular lattice. In 2023 Aquaviva et al. in [4] proved the following results:

(1) that oriented graphs, the oriented 3-cycle and the oriented edge are the only graphs where PST occurs between every pair of vertices.

(2) This settles a conjecture of Cameron et al. [36]. On the other hand, they constructed an infinite family of oriented graphs with PST between any pair of vertices on a subset of size four. There are infinite families of Hermitian graphs with one-way PST, where PST occurs without periodicity. In contrast, PST implies periodicity whenever the adjacency matrix has algebraic entries (as Godsil shows [78]).

(3) There are infinite families with non-monogamous PGST in rooted graph products. In particular, they generalized known results on double stars (due to Fan and Godsil [69]) and on paths with loops (due to Kempton, Lippner and Yau [100]). The latter extends the experimental observation of quantum transport (made by Zimboras et al. [161]) and showed non-monogamous PGST can occur amongst distant vertices.

Anuradha et al. in [14] demonstrated that PST can be achieved in an optical waveguide lattice governed by a Hamiltonian with modulated nearest-neighbor couplings. Eda Chan et al. [47] established the theory for PGST in discrete-time quantum walks. For a class of walks, they showed that PGST is characterized by the spectrum of certain Hermitian adjacency matrix of the graph. More specifically, the vertices involved in PGST must be strongly cospectral relative to this matrix, and the *Arccos* of its eigenvalues must satisfy some number theoretic conditions. Using normalized adjacency matrices, cyclic covers and the theory on linear relations between geodetic angles, they construct several infinite families of walks that exhibit this phenomenon. Coutinho et al. [58] proved that the only trees that admit perfect state transfer according to the adjacency matrix model are P_2 and P_3. This answers a question first asked by Godsil in [47] and proved conjecture by Coutinho and Liu in [55]. Wang et al. [152], established the necessary and sufficient condition for a bi-Cayley graph having perfect state transfer over any given finite abelian group. As corollaries, many known and new results are obtained on Cayley graphs having perfect state transfer over abelian groups, (generalized) dihedral groups, semi-dihedral groups and generalized quaternion groups. In particular, they gave an example of a connected non-normal Cayley graph over a dihedral group having perfect state transfer between two distinct vertices, which was thought impossible.

This paper is organized as follows. In Sections 2 and 3 we review the fundamentals of PST and PGST. In Section 4, we review the proven results in context of PST and PGST. In Section 5, we peruse the application of quantum computing in different fields. Finally, in Section 6, the article concludes.

2 Definitions and Preliminaries

In this section, we review standard facts and notation used. Our notation for representations of finite groups is based on the notations introduced in [87].

2.1 Representation and Character of finite groups

Let G be a finite group, $\mathbb{C}$ the field of complex numbers, and V a $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} V = n < \infty$. A $\mathbb{C}$-representation of G on V is a group homomorphism $T : G \to \mathrm{GL}(V)$ of G into the group of all Linear mapping of V onto itself. Then we call V a G-module over $\mathbb{C}$ for T and $n = \dim_{\mathbb{C}} V$ is the degree of T. Now we can consider V as a $\mathbb{C}G$-module. Thus by choosing a $\mathbb{C}$-basis for V, clearly the $\mathbb{C}$-representation given a matrix $\mathbb{C}$-representation $D : G \to \mathrm{GL}_n(\mathbb{C})$ of G into the multiplicative group of non-singular $n \times n$ matrices over $\mathbb{C}$ $(D(g) = (D_{ij}(g))_{1\le i,j\le n})$. We call two matrix representations D_1 and D_2 are equivalent, if the corresponding $\mathbb{C}G$-module $V_i (i = 1,2)$ are isomorphic. This yields the existence of a non-singular matrix P such that

$$P^{-1}D_1(g)P = D_2(g) \quad \text{for all} \;\; g \in G.$$

Every representation $D : G \to \mathrm{GL}_n(\mathbb{C})$ given the character $\chi : G \to \mathbb{C}$, which this function gives to each $g \in G$, $\chi(g) = \mathrm{tr}(D(g))$, where $\mathrm{tr}(D(g))$ is the trace of matrix $D(g)$. Let $M_{mn}(\mathbb{C})$ be the set of all $m \times n$ matrices over the field $\mathbb{C}$ and $M_{nn}(\mathbb{C}) = M_n(\mathbb{C})$. The representation D is reducible if it is equivalent to a block upper triangular representation, namely to a representation of the form

$$g \mapsto \begin{pmatrix} A(g) & B(g) \\ 0 & C(g) \end{pmatrix},$$

where $A(g) \in M_{n_1}(\mathbb{C}), B(g) \in M_{n_1\times n_2}(\mathbb{C}), C(g) \in M_{n_2}(\mathbb{C})$, and where $n_1, n_2 \in \mathbb{N}$ are independent of $g \in G$. Otherwise, it is called irreducible.

Let V be a finite-dimensional $\mathbb{C}$-vector space. A map

$$\langle .,. \rangle : V \times V \to \mathbb{C}$$

is called an inner product on V if the following holds for all $v, w, v_1, v_2 \in V$ and $c_1, c_2 \in \mathbb{C}$:

(1) $\langle c_1v_1 + c_2v_2, w\rangle = c_1\langle v_1, w\rangle + c_2\langle v_2, w\rangle$;

(2) $\langle w, v\rangle = \overline{\langle v, w\rangle}$;

(3) $\langle v, v\rangle \ge 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$

Notice that the norm $||v||$ of a vector v in an inner product space is defined by $|v| = \sqrt{\langle v, v\rangle}$. Furthermore, the standard inner product on $\mathbb{C}^n$ is given by

$$\langle (a_1, ..., a_n), (b_1, ..., b_n)\rangle = \sum_{i=1}^{n} a_i\overline{b_i}.$$

Recall that if $A = (a_{ij}) \in M_{mn}(\mathbb{C})$ is a matrix, then its transpose is the matrix $A^T = (a_{ji}) \in M_{nm}(\mathbb{C})$. The conjugate of A is $\overline{A} = (\overline{a_{ij}})$. The conjugate-transpose or adjoint of A is the matrix $A^* = \overline{A^T}$.

A Linear operator $U \in \mathrm{GL}(V)$ is said to be unitary if $\langle Uv, Uw \rangle = \langle v, w \rangle$ for every $v, w \in V$. Moreover, concerning the standard inner product on $\mathbb{C}^n$, the Linear transformation associated with a matrix $A \in \mathrm{GL}_n(\mathbb{C})$ is unitary if and only if $A^{-1} = A^*$, such a matrix is thus called unitary. The unitary $n \times n$ matrices form a subgroup $U_n(\mathbb{C})$ of $\mathrm{GL}_n(\mathbb{C})$. A representation $D : G \to \mathrm{GL}_n(\mathbb{C})$ is said to be unitary if $D(g) \in \mathrm{U}(n)$ for all $g \in G$. Every complex representation $D : G \to \mathrm{GL}_n(\mathbb{C})$ is equivalent to a unitary representation.

2.2 Cayley graph

The Cayley graph was introduced by Arthur Cayley in 1878 to explain the concept of pure groups described by a set of generators. Let G is group. A symmetric subset of group G is a subset $S \subseteq G$, where $1 \notin S$ and $S = S^{-1}$. The Cayley graph $\Gamma = \mathrm{Cay(G,S)}$ with respect to S is a graph whose vertex set is $G = V(\Gamma)$ and edge set is $E(\Gamma) = \{(x,y) | x, y \in G, yx^{-1} \in S\}$. If S is a union of conjugacy classes in G, i.e. S is a normal Cayley subset, then we call Γ a quasi-abelian Cayley graph of G with respect to S. Note that, since S is symmetric, $\Gamma = \mathrm{Cay(G,S)}$ is a simple graph. The adjacency matrix of Γ is defined by $A = A(\Gamma) = (a_{x,y})_{x,y \in G}$

$$a_{x,y} = \begin{cases} 1 & \text{if } yx^{-1} \in S \\ 0 & \text{else.} \end{cases}$$

The circulant matrix is a square matrix in which every row of the matrix is a right cyclic shift of the row above it. If the first row is $(c_0, c_1, \ldots, c_n)$, then we denote it by $C(c_0, c_1, \ldots, c_n)$. Furthermore, the anti-circulant matrix is a square matrix in which every row of the matrix is a left cyclic shift of the row above it.

2.3 Spectra of Cayley graph

Proposition 2.1. *[16] Let $\Gamma = \mathrm{Cay(G,S)}$ be an undirected Cayley graph over a finite group G with irreducible unitary matrix representations $\rho^{(1)}, \ldots, \rho^{(m)}$. Let d_l be the degree of $\rho^{(l)}$. For each $l \in \{1, \ldots, m\}$, define a $d_l \times d_l$ block matrix $A_l := \rho^{(l)}(S)$. Let $\chi_{A_l}(\lambda)$ and $\chi_A(\lambda)$ be the characteristic polynomial of A_l and A, the adjacency matrix of Γ, respectively. Then*

(1) there exists a basis $\mathcal{B}$ such that $[A]_{\mathcal{B}} = \mathrm{Diag}(A_1 \otimes I_{d_1}, \ldots, A_m \otimes I_{d_m})$.

(2) $\chi_A(\lambda) = \Pi_{l=1}^m \chi_{A_l}(\lambda)^{d_l}$.

(3) Let $v_{(k)}$ be an eigenvector of A_k, $1 \le k \le m$, associated with λ. Then the following vectors are distinct linearly independent d_k eigenvectors of Γ associated with λ:

$$v^j_{(k)} := \sum_{g \in G} \left[v_{(k)} \cdot \rho_j^{(k)}(g) \right] e_g, \quad 1 \le j \le d_k$$

where $\cdot$ is the usual inner product and $\rho_j^{(k)}(g)$ is a vector whose coordinates are the coordinates of jth column of $\rho^{(k)}(g)$.

Corollary 2.2. *[16] Keeping the notations of Proposition 2.1 and considering fixed ordering $g_1 = 1, g_2, \cdots, g_n$ of all elements of G, we have*

(1) Let $U = (v^j_{(k)})^T$ and $U.U^* = [u_{rs}]$. Then

$$u_{rs} = [v_{(k)}.\rho_j^{(k)}(g_r)][\bar{v}_{(k)}.\bar{\rho}_j^{(k)}(g_s)];$$

(2) If $\rho^{(k)}$ is 1-dimensional representation of G, then $\lambda = \rho^{(k)}(S)$ is an eigenvalue of A_k, $v_{(k)} = 1$ and $v^1_{(k)} = \sum_{g\in G} \rho^{(k)}(g)e_g$ is an eigenvector of Γ associated to the eigenvalue $\rho^{(k)}(S)$. Furthermore, by the above notation $u_{rs} = \rho^{(k)}(g_r)\bar{\rho}^{(k)}(g_s) = \rho^{(k)}(g_r {g_s}^{-1})$.

(3) If for every $g \in G$, we have that $\sum_{s\in S} \rho^{(k)}(gsg^{-1}) = \sum_{s\in S} \rho^{(k)}(s)$, then $\lambda_k = \frac{\chi_k(S)}{d_k} = \frac{\sum_{s\in S}\chi(s)}{d_k}$ is an eigenvalues of Γ with multiplicity ${d_k}^2$ and standard basis $e_1, e_2, \cdots e_{d_k}$ are eigenvectors of A_k associated to $\lambda_k = \frac{\chi_k(S)}{d_k}$. Furthermore, the eigenvectors $v_{ij}^{(k)} = \sqrt{\frac{d_k}{|G|}} \sum_{g\in G} \rho_{ij}^{(k)}(g)e_g = \sqrt{\frac{d_k}{|G|}}(\rho_{ij}^{(k)}(g_1), \cdots, \rho_{ij}^{(k)}(g_n))$, $1 \le i, j \le d_k$, which are associated to λ_k form an orthonormal basis for the eigenspa V_{λ_k}, where $\rho_{ij}^{(k)}(g)$ is the ij-entry of the matrix $\rho^{(k)}(g)$. Also, by the notation of (1), we have $u_{rs} = {\rho_{ij}}^{(k)}(g_r)\bar{\rho_{ij}}^{(k)}(g_s) = \rho_{ij}^{(k)}(g_r g_s^{-1})$.

The following result enables us to compute explicitly the eigenvalues of any normal Cayley graph using character values of the underlying group. This result was proved by Diaconis and Shahshahani [61] and Zieschang [157].

Lemma 2.3. *[157] Let $G = \{g_1, \cdots, g_n\}$ be a finite group of order n and $\rho^{(1)}, \cdots, \rho^{(t)}$ be a complete set of unitary representatives of the equivalence classes of irreducible representations of G. Let χ_i be the character of $\rho^{(i)}$ and d_i be the degree of $\rho^{(i)}$. Let $S \subseteq G$ be a conjugation-closed. Then the eigenvalues of the Cayley graph of* Cay(G,S) *with respect to S are $\lambda_1, \cdots, \lambda_t$, where*

$$\lambda_k = \frac{1}{d_k} \sum_{s\in S} \chi_k(s), \;\; 1 \le k \le t,$$

and that λ_k has multiplicity d_k^2.

2.4 PST on Cayley graphs

For a simple graph Γ with n vertices, Spec(Γ) denotes the set of all eigenvalues of Γ. For any symmetric matrix A, assume that its eigenvalues are λ_i's for $1 \le i \le n$. There is a unitary matrix $P = (v_1, \cdots, v_n)$, where each v_i is an eigenvector of $\lambda_i, (1 \le i \le n)$. Thus we have the following spectral decomposition of A

$$A = \lambda_1 E_1 + \cdots + \lambda_n E_n,$$

where $E_i = v_i v_i^*, (1 \le i \le n)$ satisfies

$$E_i E_j = \begin{cases} E_i & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

Definition 2.4. *Let Γ be a graph. For two distinct vertices $u, v \in V(\Gamma)$, we say that Γ has a perfect state transfer (PST) from u to v at the time $t(>0)$ if the (u,v)-entry of $H(t)$, denoted by $H(t)_{u,v}$, has absolute value 1. We say that Γ is periodic at u with period t if $H(t)_{u,v}$ has absolute value 1. If Γ is periodic with period t at every point, then Γ is said to be periodic.*

Let Γ be an undirected simple graph whose vertex set is denoted by $V(\Gamma)$ and $A = A(\Gamma)$ be the adjacency matrix of Γ. For a real number t, the *transfer matrix* of Γ is defined as the following $n \times n$ matrix:

$$H(t) = H_\Gamma(t) = \exp(-itA) = \sum_{s=0}^{+\infty} \frac{(-itA)^s}{s!} = (H(t))_{u,v \in V(\Gamma)},$$

where $i = \sqrt{-1}$ and $n = |V(\Gamma)|$ is the number of vertices in Γ. Therefore, we have the decomposition of the transfer matrix

$$H(t) = \exp(-i\lambda_1 t)E_1 + \cdots + \exp(-i\lambda_n t)E_n.$$

We also need notation of the 2-adic exponential valuation of rational numbers which is a mapping defined by $\eta_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$,
which $\eta_2(0) = \infty$, *and* $\eta_2(2^l \frac{a}{b}) = l$ *($a, b, l \in \mathbb{Z}$ and $2 \nmid ab$).*

We assume that $\infty + \infty = \infty + l = \infty$ and $\infty > l$ for any $l \in \mathbb{Z}$. Then for $\beta, \beta' \in \mathbb{Q}$, the following properties yield for η_2:

(1) $\eta_2(\beta\beta') = \eta_2(\beta) + \eta_2(\beta')$, (2.4)

(2) $\eta_2(\beta + \beta') \ge \min(\eta_2(\beta), \eta_2(\beta'))$ and the equality holds if $\eta_2(\beta) \ne \eta_2(\beta')$.

2.5 PGST on Cayley graphs

If a graph Γ has n vertices, we also denote its vertices as u, $1 \le u \le n$. For integer u, $1 \le u \le n$, we use e_u to denote the unit norm vector whose u-th entry is 1. PGST was first introduced by Godsil in the article [76]. Suppose that Γ has PGST from u to v, then there is a sequence $\{t_k\}$ of real numbers such that $\lim_{k\to\infty} H(t_k)e_u = \gamma e_v$, where $||\gamma|| = 1$. In other words, for every $\varepsilon > 0$ there exists $t \in \mathbb{R}$ and $\gamma \in \mathbb{C}$ with $||\gamma|| = 1$ such that

$$\left| e_u^T H(t) e_v - \gamma \right| < \varepsilon.$$

Now we introduce the Kronecker approximation theorem on simultaneous approximation of numbers. This will be used later to find graphs allowing PGST.

Lemma 2.5 (Kronecker Approximation Theorem). *[8] If $\alpha_1, \ldots, \alpha_s$ are arbitrary real numbers and if $1, \beta_1, \ldots, \beta_s$ are real, algebraic numbers Linearly independent over $\mathbb{Q}$, then for any $\varepsilon > 0$, there exist $q, p_1, \ldots, p_s \in \mathbb{Z}$ such that*

$$\left|q\beta_j - p_j - \alpha_j\right| < \varepsilon, \quad 1 \leq j \leq s.$$

Note that there is a strong version of the Kronecker Approximation Theorem as follows:

Lemma 2.6 (Strong version of Kronecker Approximation Theorem). *[23] Let $x_0, \ldots, x_N$ and $a_0, \ldots, a_N$ be fixed real numbers. For every $\delta > 0$ there exists a real t such that*

$$|x_s t - a_s| < \delta \qquad (mod\ 2\pi), \quad s = 0, 1, 2, \ldots, N$$

hold if and only if, for integers $l_0, \ldots, l_N$, if

$$l_0 x_0 + l_1 x_1 + \cdots + l_N x_N = 0$$

, then

$$l_0 a_0 + l_1 a_1 + \cdots + l_N a_N \equiv 0 \quad (mod\ 2\pi).$$

Lemma 2.7. *[23] Let Γ be a simple connected graph. Then Γ admits PGST between two points u and v if and only if u and v are cospectral. In this case, let $\lambda_1, \ldots, \lambda_d$ be the eigenvalues in their support, and for $j = 1, \ldots, d$, σ_j is defined*

$$\sigma_j = \begin{cases} 1 & \text{if } E_j e_u = E_j e_v \\ 0 & \text{else} \end{cases}.$$

If there is a sequence of integers $\sum_{j=1}^{d} l_j \sigma_j$ such that $\sum_{j=1}^{d} l_j \lambda_j = 0$ is odd then $\sum_{j=1}^{d} l_j \neq 0$.

Theorem 2.8. *[38] Let the notations be defined as above. Suppose that for j, $\lambda_j^{(2)} = \lambda_j^{(1)} + t_j$ where $t_j \in \mathbb{Z}$. Then we have*

(1) Γ_1 has PGST (PST) from u to v at the time in $2\pi z$ if and only if so does Γ_2.

(2) Assume that Γ_1 has PGST from u to v, then Γ_2 has PGST from u to v if, for every sequence of integers $l_1, \ldots, l_n$, if $\sum_{j=1}^{n} l_j \lambda_j^{(2)} = 0$ and $\sum_{j=1}^{n} l_j \sigma_j a$ odd, then $\sum_{j=1}^{n} l_j t_j = 0$.

3 Results

3.1 The results of PST

3.1.1 PST in Cayley graphs on abelian groups

These graphs are highly symmetric and have important applications in chemical graph theory. We restate some results proved. These results establish necessary and sufficient conditions for PST. In paper [24], the present authors proved that an integral circulant graph with a square-free number of vertices does not have PST. Two classes of integral circulant graphs having PST were also found. They showed that there exists an integral circulant graph with n vertices having a perfect state transfer if and only if $4|n$. Several integral circulant graphs have been found to have a perfect state transfer for the values of n divisible by 4. Moreover, they proved the non-existence of PST for several other classes of integral circulant graphs whose order is divisible by 4. These classes cover the class of graphs where the divisor set contains exactly two elements. The obtained results partially answer the main question of which integral circulant graphs have a perfect state transfer. A Cayley graph over a finite abelian group $(G,+)$ with the connection set S, where $0 \notin S \subseteq G$ and $\{-s : s \in S\} = S$, is denoted by Cay(G,S). The elements of G are the vertices of the graph, where two vertices $a, b \in G$ are adjacent if and only if $a - b \in S$. If $G = \mathbb{Z}_n$ then the Cayley graph is called a circulant graph. In particular, a Cayley graph over $\mathbb{Z}_n$ with $S = \{1, n-1\}$ is called a cycle which is denoted by C_n. The eigenvalues of C_n are given by

$$\lambda_l = 2\cos\left(\frac{2l\pi}{n}\right), \quad 0 \le l \le n-1 \tag{3.1.1}$$

and the associated eigenvectors are $\mathbf{v}_l = \left[1, \omega_n^l, \ldots, \omega_n^{l(n-1)}\right]^T$, where $\omega_n = \exp\left(\frac{2\pi i}{n}\right)$ is the primitive n-th root of unity.

Theorem 3.1. *[24] There exists a PST in graph $ICG_n(D)$ between vertices 0 and a, if and only if there are integers p and q such that $\gcd(p,q) = 1$ and $j = 0, 1, \ldots, n-2$,*

$$\frac{p}{q}\left(\lambda_{j+1} - \lambda_j\right) + \frac{a}{n} \in \mathbb{Z}$$

.

Theorem 3.2. *[24] There is no PST in $ICG_n(D)$ if $\frac{n}{d}$ is odd for every $d \in D$. For n even, if there exists a PST in $ICG_n(D)$ between vertices 0 and a then $a = \frac{n}{2}$.*

According to Theorem 3.2, PST may exists in $ICG_n(D)$ only between vertices $\frac{n}{d}$ and 0 (i.e., between b and $\frac{n}{d} + b$). Hence we will avoid referring to the input and output vertex and will just say that there exists a PST in $ICG_n(D)$.
For a given prime number p and integer $n \in \mathbb{N}_0$, denote by $S_p(n)$ the maximal number α such that $p^\alpha \mid n$ if $n \in \mathbb{N}$, and $S_p(0) = +\infty$ for an arbitrary prime number p. The following result is proven in [24] and is further used as the criterion for the existence of PST.

Lemma 3.3. *[24] There exists PST in $ICG_n(D)$ if and only if there exists a number $m \in \mathbb{N}_0$ such that the following holds for all $j = 0, 1, \ldots, n-2$,*

$$S_2\left(\lambda_{j+1} - \lambda_j\right) = m.$$

Corollary 3.4. *[125] Let $ICG_n(D)$ have PST. One of the following two statements must hold:*
(1). $\lambda_j \equiv \lambda_{j+1} \pmod 2$ for every $0 \le j \le n-1$ (i.e., all eigenvalues λ_j have the same parity).
(2). $\lambda_j \equiv \lambda_{j+1} + 1 \pmod 2$ for every $0 \le j \le n-1$ (i.e., λ_j are alternatively odd and even).

We end this section with the following result concerning unitary Cayley graphs.
The unitary Cayley graphs are the special case of integral circulant graphs with $D = \{1\}$.

Proposition 3.5. *[24] If both n and $\frac{n}{2}$ are not square-free integers, there is no PST in $ICG_n(\{1\})$.*

Theorem 3.6. *[25] There is no PST in graph $ICG_n(D)$ if n is an even square-free integer.*

Theorem 3.7. *[125] Let $S_2(n) = 2$ and D contains exactly one even divisor. Then $ICG_n(D)$ has no PST.*

Proposition 15. [25] The minimal number of vertices of a non-unitary integral circulant graph allowing PST is $n = 8$.

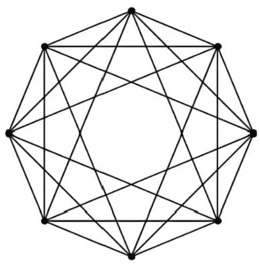
Theorem 3.8. *[25] Integral circulant graph $ICG_n\left(\left\{1, \frac{n}{2}\right\}\right)$ where $S_2(n) \geq 3$ has PST.*

Theorem 3.9. *[25] The integral circulant graph $ICG\left(\left\{1, \frac{n}{4}\right\}\right)$ where $S_2(n) \geq 3$ has PST.*

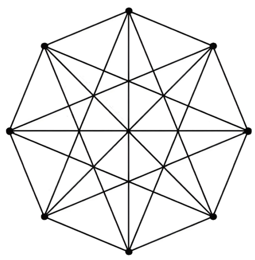
The return of theorems 3.8 and 3.9 proved in the article [125] by Petković and Bašić;

Theorem 3.10. *[125] The integral circulant graph $ICG_n\left(\left\{1, \frac{n}{2}\right\}\right)$ has a PST if and only if $S_2(n) \geq 3$.*

Theorem 3.11. *[125] The integral circulant graph $ICG_n\left(\left\{1, \frac{n}{4}\right\}\right)$ has a PST if and only if $S_2(n) \geq 3$.*



$ICG_8\left(\left\{1, \frac{1}{2}\right\}\right)$



$ICG_8\left(\left\{1, \frac{1}{4}\right\}\right)$

Corollary 3.12. *[125] The integral circulant graph $ICG_n(D)$ where D contains an even divisor which is relatively prime to all other divisors in D, has a PST if and only if $S_2(n) \geq 3$ and $D = \{1, \frac{n}{2}\}, D = \{1, \frac{n}{4}\}$.*

Theorem 3.13. *[125] The integral circulant graph $ICG_n\left(\left\{1, 2, \frac{n}{2}\right\}\right)$ has a PST if and only if $S_2(n) \geq 4$.*

Theorem 3.14. *[125] Let n be a positive integer such that $S_2(n) = 2$. Then graph $ICG_n\left(\left\{1, 2, 4, \frac{n}{4}\right\}\right)$ has a PST.*

Theorem 3.15. *[125] Let n be a positive integer such that $S_2(n) = 2$. Then graph $ICG_n\left(\left\{1, 2, 4, \frac{n}{2}\right\}\right)$ has a PST.*

For a given integral circulant graph $ICG_n(D)$, we define that $d_0 \in D$ is an isolated divisor if $\gcd(d_0, d) = 1$ for every $d \in D \setminus \{d_0\}$. We will investigate the existence of PST in the integral circulant graphs $ICG_n(D)$ having an isolated divisor d_0.

Corollary 3.16. *[26] All connected graphs $ICG_n(D)$, where $\frac{n}{2}$ is an even square-free integer and every two divisors from D are relatively prime, have no PST.*

Corollary 3.17. *[26] Let $ICG_n(D)$ be an integral circulant graph such that $D \neq \{1, \frac{n}{2}\}$ and $D \neq \{1, \frac{n}{4}\}$ and $\gcd(d_1, d_2) = 1$ for every $d_1, d_2 \in D$, $d_1 \neq d_2$. Then $ICG_n(D)$ has no PST.*

Recall that in [25] it was proved that ICG-graphs whose order n is a square-free integer do not have PST. In the general case, when n is not square-free, the situation is much more complicated and complete consideration requires a lot of cases. They will illustrate this fact in a simple case when $n = 2p^2$, where p is prime. In the following theorem, they show that these graphs do not have PST either, but the complete proof requires a total of 10 different cases.

Theorem 3.18. *[26] Let p be an arbitrary prime number and $n = 2p^2$. There is no integral circulant graph $ICG_n(D)$ allowing PST for any set of divisors D.*

Bašić in [27] provided a complete characterization for ICG that admit PST.

Lemma 3.19. *[27] $ICG_n(D)$ has PST if and only if one of the following conditions holds:*

(1) $\lambda_{2j} \in {}_4\mathbb{N} + 2$ and $\lambda_{2j+1} = 0$, if $n/2 \notin D$,

(2) $\lambda_{2j} \in {}_4\mathbb{N} + 1$ and $\lambda_{2j+1} = -1$, if $n/2 \in D$,
for $0 \leq j \leq \frac{n}{2}$.

Theorem 3.20. *[27] Let D be a set of divisors of n such that $\frac{n}{2}, \frac{n}{4} \notin D$. Then$ICG_n\left(D \cup \left\{\frac{n}{4}\right\}\right)$ has PST if and only if $ICG_n\left(D \cup \left\{\frac{n}{2}\right\}\right)$ has PST.*

Let $ICG_n(D)$ denote an arbitrary integral circulant graph from order n. We define sets $D_i \subseteq D$ for $0 \leq i \leq l$, where $l = S_2\left(\frac{n}{d}\right)$, as follows: $D_i = \left\{d \in D \middle| S_2\left(\frac{n}{d}\right) = i\right\}$.

Theorem 3.21. *[27] $ICG_n(D)$ has PST if and only if $n \in 4\mathbb{N}$ and $D_1^* = 2D_2^*$ and $D_0 = 4D_2^*$ and either $\frac{n}{4} \in D$ or $\frac{n}{2} \in D$, where $D_1^* = D_1 \setminus \left\{\frac{n}{2}\right\}$, $D_2^* = D_2 \setminus \left\{\frac{n}{4}\right\}$.*

By below theorem, the minimum time to find PST in a cubelike graph $\Gamma = \mathrm{Cay(G,S)}$ is πM where $M = \gcd(d - \alpha_z : z \in G)$, $d = |S|$, and $\alpha_z = \chi_z(S)$.

Theorem 3.22. *[140] Let $\Gamma = \mathrm{Cay(G,S)}$ be a connected simple abelian Cayley graph with $n = |G| \geq 3$. Then for $g, h \in G$ and $a = g - h \neq 0$, Γ has PST between g and h if and only if the following three conditions hold:*

(1) Γ is an integral graph. Namely, the eigenvalues $\alpha_x = \sum_{g \in G} \chi_x(g) \in \mathbb{Z}$, for all $x \in G$;

(2) the order of α is two;

(3) $\eta_2(d - \alpha_x)$ for all $x \in G_1$ are the same number, say, ρ, and $\eta_2(M_0) \geq \rho + 1$, where $M_0 = \gcd(d - \alpha_x \colon x \in G_0)$ and $G_1 = \{x \in G \colon \chi_a(x) = -1\}$.

Corollary 3.23. *[140] Let $G = \mathbb{Z}_n$, $n = 2m$, $S \subseteq G$, let $\Gamma = \mathrm{Cay(G,S)}$ be an integral connected (circulant) Cayley graph. Then Γ has PST between g and $g+m$ (for each $g \in G$) if and only if there exists $\rho \in \mathbb{Z}$ such that*

$$\eta_2(\alpha_j - \alpha_{j+1}) = \rho, \qquad (0 \le j \le n-2)$$

where $\alpha_j = \chi_j(S) = \sum_{g \in S} w_n^{jg}$.

Tan et al. [140] present a non-existence result on PST in abelian Cayley graphs $\Gamma = \mathrm{Cay(G,S)}$, they have shown that if Γ has PST between two distinct vertices g and h in G, then the order of $a = g - h$ should be two so that $n = |G|$ is even. For the circulant case, the stronger necessary condition $4|n$ has been proved in [125] by heavy computations.

Theorem 3.24. *[140] Let $\Gamma = \mathrm{Cay(G,S)}$ be an integral abelian Cayley graph, $n = |G| \equiv 2$ (mod4) and $n \ge 6$, Then Γ has no PST between any distinct vertices.*

Let G be the additive group of $\mathbb{F}_2^n$. For any subset S of $\mathbb{F}_2^n, 0 \notin S, \Gamma = \mathrm{Cay(G,S)}$ is an integral simple graph and the order of any non-zero element in G is two. The character group of G is
$\hat{G} = \hat{\mathbb{F}}_2^n = \{\chi_z : z \in \mathbb{F}_2^n\}$, where for $g = (g_1, \cdots, g_n), z = (z_1, \cdots, z_n) \in \mathbb{F}_2^n$,
$\chi_z(g) = (-1)^{z \cdot g}, \quad z \cdot g = \sum_{j=1}^n z_j g_j \in \mathbb{F}_2$
If we view $\mathbb{F}_2^n$ as the additive group of the finite field $\mathbb{F}_q$ with $q = 2^n$, then
$\hat{G} = \left(\widehat{\mathbb{F}_q, +}\right) = \{\lambda_z : z \in \mathbb{F}_q\}$
where for $g, z \in \mathbb{F}_q, \lambda_z(g) = (-1)^{T(zg)}$, and $T : \mathbb{F}_q \to \mathbb{F}_2$ is the trace mapping.

Lemma 3.25. *[140] Let $\Gamma = \mathrm{Cay}(\mathbb{F}_q, S)$ be a connected graph, where $S \subseteq \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, $q \equiv 2^n \ge 2$, and let $a \in \mathbb{F}_q^*$. Then for each $c \in \mathbb{F}_q^*$, Γ, has PST between vertices g and $g+a$ at time t if and only if $\Gamma' = \mathrm{Cay}(\mathbb{F}_q, S')$ has PST between g' and $g' + a'$ at time t where*

$$S' = cS = \{cz \mid z \in S\} \quad g' = c^{-1}g, \; a' = c^{-1}a.$$

Tan et al. When answering the question: Are there cubelike graphs having PST at time t where t is arbitrarily small?, firstly, they showed that M should be a power of 2 (Lemma 3.26), then they provided a lower bound on t (Theorem 3.27) and showed that this lower bound is tight (Theorems 3.28 and 3.29).

Lemma 3.26. *[140] Let $G = \mathbb{F}_n^2$, $0 \notin S \subseteq \mathbb{F}_n^2$ and $d = |S| \ge 1$. Then, $M = \gcd(d - \alpha_z : z \in G)$ is a power of 2 where $\alpha_z = \chi_z(S)$.*

Then in the following theorem they provided a lower bound on t.

Theorem 3.27. *[140] Let $G = \mathbb{F}_2^n$, where $n \ge 2$, $0 \notin S \subset G$ and let $\Gamma = \mathrm{Cay}(G, S')$ be a connected graph. If Γ has PST between two distinct elements g and $g+a$ at time t, then the minimum time t is $\frac{\pi}{M}$, where $M = 2^l$, $1 \le l \le \left\lfloor \frac{n}{2} \right\rfloor$ where $s = \left\lfloor \frac{n}{2} \right\rfloor$ if $n = 2s+1$ or $n = 2s$.*

And in the end, they proved that this lower bound is tight. For $n = 2m$ $(m \ge 2), f \in B_n$ is called a bent function if $|W_f(y)| = 2^m$ for all $y \in \mathbb{F}_2^n$.

Bent functions exist for all $m \ge 1$ and many series of bent functions have been constructed in the past forty years.

Theorem 3.28. *[140] Let $n = 2m + 1 (m \geq 2)$, and f be a bent function in $B_{n-1} = B_{2m}$. Denote $S' = \mathrm{Supp}(f) = \left\{z' \in \mathbb{F}_2^{n-1} : f(z') = 1\right\}$, and assume that $0 \notin S'$. Denote $S_\varepsilon = (\varepsilon, S')$ $(\varepsilon = 0,1)$ and $S = S_0 \cup S_1$. Then*

(1) The cubelike graph $\Gamma = \mathrm{Cay}(\mathbb{F}_2^n, S)$ is connected.

(2) For $a = (1,0), 0 \in \mathbb{F}_2^{n-1}$, Γ has PST between g and $g + a$ for any $g \in \mathbb{F}_2^n$ at time $\frac{\pi}{2^m}$.

(3) The minimum period of any vertex in Γ is $\frac{\pi}{2^m}$.

Theorem 3.29. *[140] Let $n = 2m$ with $m \geq 2$, $f(x)$ be a bent function with n variables, $f(0) = 0$, and let $S = \mathrm{Supp}(f) = \{x \in \mathbb{F}_2^n : f(x) = 1\}$. Then the cubelike graph $\Gamma = \mathrm{Cay}(\mathbb{F}_2^n, S)$ is connected and the minimum period of each vertex $g \in \mathbb{F}_2^n$ in Γ is $\frac{\pi}{2^m}$.*

As a generalization of Cayley graphs, see [15], semi-Cayley graphs are introduced. Let G be a finite group, R, L and S be subsets of G such that R and L are inverse-closed subsets not containing the identity element of G. The semi-Cayley graph over G with respect to R, L and S, denoted by $\mathrm{SC}(G,R,L,S)$ is an undirected graph with vertex set $\{(g,0),(g,1) \mid g \in G\}$ and edge set consists of three sets:

$$\begin{array}{ll} \left\{\{(x,0),(y,0)\} \mid yx^{-1} \in R\right\} & \text{(right edges)},\\ \left\{\{(x,1),(y,1)\} \mid yx^{-1} \in L\right\} & \text{(left edges)},\\ \left\{\{(x,0),(y,1)\} \mid yx^{-1} \in S\right\} & \text{(spoke edges)}. \end{array}$$

Clearly $\mathrm{SC}(G,R,L,S)$ is a regular graph if and only if $|R| = |L|$. By the following result, every Cayley graph over a group having a subgroup of index 2 is a semi-Cayley graph. Note that the converse is not true, since as an example the Petersen graph is a semi Cayley graph over the cyclic group $\mathbb{Z}_5$ and it is not a Cayley graph. We will use this fact frequently without referring to it. As a result below, give some relation between PST on Cayley graphs and semi-Cayley graphs.

Corollary 3.30. *[17] Let $\Gamma = \mathrm{SC}(G,R,L,S)$ be a semi-Cayley graph over an abelian group G and Γ has a PST between two vertices (g,r) and (h,s), with $r \neq s$. Then $R = L$ and Γ is a Cayley graph over a group isomorphic to $G \rtimes \mathbb{Z}_2$.*

The following results, characterize Cayley graphs over finite groups with abelian groups of index 2 having PST.

Corollary 3.31. *[17] Let $\Gamma = \mathrm{Cay}(G,T)$ be an undirected Cayley graph, where G has an abelian subgroup H of index 2, $G = H \cup xH, T = T_1 \cup xT_2$, where $T_1, T_2 \subset H$ (if $T_2 = \emptyset$ then we put $xT_2 = \emptyset$), $\mathrm{Irr}(H) = \{\chi_1, \ldots, \chi_n\}$ and $X = \{i \mid \chi_i(T_2) = 0\}$. Then eigenvalues of Γ are*

$$\lambda_i^+ = \frac{\chi_i(T_1) + \chi_i(xT_1x^{-1}) + \sqrt{(\chi_i(T_1) - \chi_i(xT_1x^{-1}))^2 + 4|\chi_i(x^2T_2)|}}{2}, \quad i = 1, \ldots, n,$$

and

$$\lambda_i^- = \frac{\chi_i(T_1) + \chi_i(xT_1x^{-1}) - \sqrt{(\chi_i(T_1) - \chi_i(xT_1x^{-1}))^2 + 4|\chi_i(x^2T_2)|}}{2}, \quad i = 1, \ldots, n,$$

furthermore, Γ has a PST between two vertices g_1 and g_2 at time t if and only if one of the following holds:

- *(1) $g_1, g_2 \in H$, and* $\begin{cases} \chi_i(a) = \exp\left(-\mathbf{i}\left(\lambda_1^+ - \lambda_i^+\right)t\right), & \forall i = 1, \ldots, n, \\ \chi_j(a) = \exp\left(-\mathbf{i}\left(\lambda_1^+ - \lambda_j^-\right)t\right), & \forall j \notin X, \end{cases}$

  where $a = g_1^{-1}g_2$.

- *(2) $x^{-1}g_1, x^{-1}g_2 \in H$, and* $\begin{cases} \chi_i(a) = \exp\left(-\mathbf{i}\left(\lambda_1^- - \lambda_i^-\right)t\right) & \forall i = 1, \ldots, n \\ \chi_j(a) = \exp\left(-\mathbf{i}\left(\lambda_1^- - \lambda_j^+\right)t\right) & \forall j \notin X \end{cases}$

  where $a = g_1^{-1}g_2$.

- *(3) $g_1 \in H, x^{-1}g_2 \in H$, and for all $j = 1, \ldots, n, \chi_j\left(x^2T_2\right) \neq 0$ and*

$$\chi_j(a) = \frac{|\chi_j(x^2T_2)|}{\chi_j(x^2T_2)} \exp\left(-\mathbf{i}\left(\lambda_1^+ - \lambda_j^+\right)t\right), \exp\left(-\mathbf{i}\left(\lambda_j^+\right)t\right) = -\exp\left(-\mathbf{i}\lambda_j^- t\right),$$

  and $T_1x = xT_1$, where $a = (xg_1)^{-1}g_2$.

  (4) $x^{-1}g_1 \in H, g_2 \in H$ and for all $j = 1, \ldots, n, \chi_j\left(x^2T_2\right) \neq 0$ and

$$\chi_j(a) = \frac{|\chi_j(x^2T_2)|}{\overline{\chi_j(x^2T_2)}} \exp\left(-\mathbf{i}\left(\lambda_1^+ - \lambda_j^+\right)t\right) \exp\left(-\mathbf{i}\left(\lambda_j^+\right)t\right) = -\exp\left(-\mathbf{i}\lambda_j^- t\right)$$

  and $T_1x = xT_1$, where $a = (x^{-1}g_1)^{-1}g_2$.

Arezoomand in the Theorem 3.32 gave a characterization of SC(G,R,L,S) having PST, where G is abelian and $R = L$.

Theorem 3.32. *[17] Let G be a finite abelian group of order n, $Irr(G) = \{\chi_1, \ldots, \chi_n\}$, where χ_1 is the trivial character of G, and $\Gamma = \mathrm{SC}(G,R,R,S)$. Let $u = (g,r)$ and $v = (h,r)$ are vertices of Γ and $a = g^{-1}h$. Then Γ has a PST between two distinct vertices u and v if and only if the following conditions hold:*

(1) a has order 2,

(2) Γ is integral, and for each i, $\chi_i(R)$ and $|\chi_i(S)|$ are integers.

(3) $\eta_2(|R|+|S|-\lambda_i^+) = \eta_2(|R|+|S|-\lambda_i^-) = k$ *is the same integer, say* k*, for all* i *that* $\chi_i(g_x^{-1}g_y) = 1$ *and for all* i *with* $\chi_i(g_x^{-1}\cdot g_y) = -1$, $\eta_2(|R|+|S|-\lambda_i^-) > k$, $\eta_2(|R|+|S|-\lambda_i^+) > k$. *Also,* Γ *is periodic if and only if* Γ *is integral. Furthermore, the minimum period of the vertices is* $\frac{2\pi}{M}$*, where* $M = \gcd(\lambda - \lambda_i^+ : \lambda \in Spec(\Gamma) \setminus \{\lambda_i^+\})$.

Theorem 3.33. *[17] Let* $\Gamma = \mathrm{SC(G,R,L,S)}$*, and* $u = (g,r)$*,* $v = (h,s)$*, where* $r \neq s$*, be two distinct vertices of* Γ*. Then* Γ *has a PST between* u *and* v *at time* t *if and only if the following conditions hold:*

1. $R = L$,
2. $\chi_j(S) \neq 0$ *for each* j,
3. *if* $r = 0, s = 1$ *then* $\chi_j(g^{-1}h) = \frac{\chi_j(S)}{\chi_j(S)} \exp\big(-i(\lambda_1^+ - \lambda_j^+)t\big)$*, and if* $r = 1, s = 0$ *then* $\chi_j(g^{-1}h) = \frac{|\chi_j(S)|}{\chi_j(S)} \exp\big(-i(\lambda_1^+ - \lambda_j^+)t\big)$*, where in both cases* $\exp\big(-i(\lambda_1^+ - \lambda_j^+)t\big) = \pm 1$.
4. $|\chi_j(S)|, \chi_j(R) \in \mathbb{Z}$ *for each* j*, in particular* Γ *is integral,*
5. $\eta_2(|S|) = \eta_2(|\chi_j(S)|)$ *for all* j.

Since SC(G,R,L,G) is the join graph of Cay(G,R) and Cay(G,L), the following result gives a characterization of the join of two Cayley graphs over the same abelian group.

Corollary 3.34. *[17] Let* $\Gamma = \mathrm{SC(G,R,L,G)}$*, where* G *is an abelian group of order* n*, and* $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_n\}$*. Then* Γ *has a PST between vertices* $u = (g,r)$ *and* $v = (h,s)$ *at time* t *if and only if one of the following holds:*

1. $g = h, r = s = 0, \big(\lambda_1^+ - \chi_i(R)\big)t \in 2\pi\mathbb{Z}$ *for all* i *and* $t\sqrt{(|R|-|L|)^2 + 4n^2} \in \pi\mathbb{Z}$.
2. $g = h, r = s = 1, \big(\lambda_1^- - \chi_i(L)\big)t \in 2\pi\mathbb{Z}$ *for all* i *and* $t\sqrt{(|R|-|L|)^2 + 4n^2} \in \pi\mathbb{Z}$.

3. $g \neq h, r = s = 0, \sum_{i=1}^{n} \exp\big(\mathbf{i}\lambda_i^+ t\big) = 0$ *and* $t\sqrt{(|R|-|L|)^2 + 4n^2} \in \pi\mathbb{Z}$.

4. $g \neq h, r = s = 1, \sum_{i=1}^{n} \exp\big(\mathbf{i}\lambda_i^- t\big) = 0$ *and* $t\sqrt{(|R|-|L|)^2 + 4n^2} \in \pi\mathbb{Z}$,

where $\lambda_1^+ = \frac{|R|+|L|+|\sqrt{(|R|-|L|)^2+4n^2}}{2}, \lambda_1^- = \frac{|R|+|L|-|\sqrt{(|R|-|L|)^2+4n^2}}{2}$, $\lambda_i^+ = \chi_i(R)$ *and* $\lambda_i^- = \chi_i(L), i = 2, \ldots, n$ *are eigenavlues of* Γ.

A graph is said to be a bi-Cayley graph (or semi-Cayley graph) over a group G if it admits G as a semiregular automorphism group with two orbits of equal size. Let R, L and T be subsets of a group G such that $R = R^{-1}, L = L^{-1}$ and $R \cup L$ does not contain the identity element of G. Define the graph BiCay(G;R,L,T) to have vertex set the union of the right part $G_0 = \{g_0 \mid g \in G\}$ and the left part $G_1 = \{g_1 \mid g \in G\}$, and edge set the union of the right edges $\{\{h_0, g_0\} \mid gh^{-1} \in R\}$, the left edges $\{\{h_1, g_1\} \mid gh^{-1} \in L\}$ and the spokes $\{\{h_0, g_1\} \mid gh^{-1} \in T\}$. For convenience, for $g \in G$, when we say $g \in G_i$ with $i = 0, 1$, it means $g_i \in G_i$. Let Γ be a bi-Cayley graph over an abelian group of order n with the adjacency matrix D. Then D has eigenvalues

$$\lambda_{2k-j} = \frac{\chi_k(R) + \chi_k(L) + (-1)^j \sqrt{(\chi_k(R) - \chi_k(L))^2 + 4|\chi_k(T)|^2}}{2},$$

where $k = 1, 2, ..., n$ and $j = 0, 1$.

Theorem 3.35. *[149] Let $\Gamma = \mathrm{BiCay(G;R,L,T)}$ be a bi-Cayley graph over an abelian group G of order n. Then Γ has PST between vertices g_p and g_q with $(g_p, g_q) \in (G_0 \times G_1) \cup (G_1 \times G_0)$ at the time t if and only if the following conditions hold, $M = \gcd(\lambda_2 - \lambda_{2k} \mid 1 \leqslant k \leqslant n)$ and $M_T = \gcd(|\chi_k(T)| \mid 1 \leqslant k \leqslant n)$*

1. *Γ is an integral graph and $R = L$.*
2. *For each $1 \leqslant k \leqslant n$,*

2.1) $\chi_k\left(g_p g_q^{-1}\right) = \begin{cases} \frac{|\chi_k(T)|}{\chi_k(T)}, & \text{if } \exp\left(\mathbf{i}t\left(\lambda_2 - \lambda_{2k}\right)\right) = 1 \\ -\frac{|\chi_k(T)|}{\chi_k(T)}, & \text{if } \exp\left(\mathbf{i}t\left(\lambda_2 - \lambda_{2k}\right)\right) = -1 \end{cases}$
2.2) $\chi_k(T) \neq 0$;
2.3) $\eta_2(|\chi_k(T)|) = \eta_2(|T|)$.

3. If $M > 0$, then $v_2(M) > v_2(|T|)$.
4. $t \in \left\{ \frac{(1+2z)\pi}{\gcd(2M_T, M)} \,\middle|\, z \in \mathbb{Z} \right\}$.

Theorem 3.36. *[149] Let $\Gamma = \mathrm{BiCay(G;R,L,T)}$ be an integral bi-Cayley graph over an abelian group G of order n. Write*

$$H = \{k \mid \chi_k(T) = 0, 1 \leqslant k \leqslant n\}.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_{2n}$ be as in Lemma 3.1. Let

$$M_0 = \gcd\left(\lambda_{2k} - \lambda_{2k-1} \mid 1 \leqslant k \leqslant n, k \notin H\right),$$

and

$$M_1 = \gcd\left(\lambda_2 - \lambda_{2k-1} \mid 1 \leqslant k \leqslant n, k \notin H\right).$$

Then Γ has PST between vertices g_p and g_q with $(g_p, g_q) \in (G_0 \times G_0) \cup (G_1 \times G_1)$ at the time t if and only if the following conditions hold, where for $j \in \{-1, 1\}$,

$$\Omega_j = \left\{k \mid \chi_k\left(g_p g_q^{-1}\right) = j, 1 \leqslant k \leqslant n\right\}$$

$X = R$, if $(g_p, g_q) \in G_0 \times G_0$ and $X = L$ if $(g_p, g_q) \in G_1 \times G_1$;

$$M_{\emptyset,X} = \gcd\left(|X| - \chi_k(X) \mid 1 \leqslant k \leqslant n\right)$$

and

$$M_X = \begin{cases} \gcd\left(\lambda_2 - \chi_k(X) \mid k \in H\right), & \text{if } H \neq \emptyset \\ 0, & \text{if } H = \emptyset \end{cases}$$

1. *For each $1 \leqslant k \leqslant n$, $\chi_k\left(g_p {g_q}^{-1}\right) = \pm 1$.*

2. *There exists an integer μ such that*

$$2.1)\begin{cases} \eta_2\left(\lambda_2 - \lambda_{2k-1}\right) = \mu, & \text{for all } k \in \Omega_{-1} \backslash H; \\ \eta_2\left(\lambda_2 - \chi_k(X)\right) = \mu, & \text{for } T \neq \emptyset \text{ and all } k \in \Omega_{-1} \cap H; \\ \eta_2\left(|X| - \chi_k(X)\right) = \mu, & \text{for } T = \emptyset \text{ and all } k \in \Omega_{-1}; \end{cases}$$

$$2.2)\begin{cases} \eta_2\left(\lambda_{2k} - \lambda_{2k-1}\right) \geqslant \mu + 1, & \text{for all } k \notin H; \\ \eta_2\left(\lambda_2 - \lambda_{2k-1}\right) \geqslant \mu + 1, & \text{for all } k \in \Omega_1 \backslash H; \\ \eta_2\left(\lambda_2 - \chi_k(X)\right) \geqslant \mu + 1, & \text{for } T \neq \emptyset \text{ and all } k \in \Omega_1 \cap H; \\ \eta_2\left(|X| - \chi_k(X)\right) \geqslant \mu + 1, & \text{for } T = \emptyset \text{ and all } k \in \Omega_1. \end{cases}$$

3. *If $T \neq \emptyset$, then* $t \in \begin{cases} \left\{ \frac{(1+2z)\pi}{\gcd(M_0, M_1, M_X)} \,\middle|\, z \in \mathbb{Z} \right\}, & \text{when } \Omega_{-1} \neq \emptyset \\ \left\{ \frac{2\pi z}{\gcd(M_0, M_1, M_X)} \,\middle|\, z \in \mathbb{Z} \right\}, & \text{when } \Omega_{-1} = \emptyset \end{cases}$

4. *If $T = \emptyset$, then*
$$t \in \begin{cases} \left\{ \frac{2\pi z}{M_{\emptyset,X}} \,\middle|\, z \in \mathbb{Z} \right\}, & \text{when } \Omega_{-1} = \emptyset \text{ or for each } k \in \Omega_{-1}, \chi_k(X) = |X| \\ \left\{ \frac{(1+2z)\pi}{M_{\emptyset,X}} \,\middle|\, z \in \mathbb{Z} \right\}, & \text{otherwise.} \end{cases}$$

An extension of a group N by a group F is a group $\tilde{G}$ that has a normal subgroup $G \cong N$ such that $\tilde{G}/G \cong F$. Let N be a finite abelian group and let $\tilde{G}$ be an extension of N by the cyclic group $\mathbb{Z}_2$. Then $\tilde{G}$ has a normal subgroup $G \cong N$ such that $\tilde{G}/G \cong \mathbb{Z}_2$. We can assume that $\tilde{G} = \langle bG \rangle = \{g, bg \mid g \in G\}$ with $b^2 \in G$. When $b^2 = 1$, by establishing a relation between a Cayley graph over $\tilde{G}$ and a bi-Cayley graph over G, the following theorem gives a non-existence result on PST over $\tilde{G}$. Roughly speaking, a Cayley graph over $\tilde{G}$ can be viewed as a bi-Cayley graph over G whose right part of the vertex set is G and the left part is bG.

Theorem 3.37. *[149] Let N be an abelian group of order n. Let $\tilde{G}$ be an extension of N by $\mathbb{Z}_2$ such that $\tilde{G}$ has a subgroup $G \cong N$ and $\tilde{G} = G : \langle b \rangle$ with $b^2 = 1$. Let $\Gamma = \mathrm{Cay}(\tilde{G}, \tilde{S})$ be a Cayley graph satisfying that*

1. $1 \notin \tilde{S} = \tilde{S}^{-1}$;
2. *for* $g \in G, bg \in \tilde{S}$ *if and only if* $gb \in \tilde{S}$.

If $n \equiv 1 (\mathrm{mod}\, 2)$, then Γ has no PST between any pair of distinct vertices.

3.1.2 PST in Cayley graphs on non-abelian groups

In the literature, there are relatively few results on perfect state transfer in Cayley graphs on non-abelian groups. In this section, we restate some results proved .
In [37, 39], necessary and sufficient conditions for a Cayley graph on a dihedral group (that is, a dihedrant) to admit perfect state transfer was obtained and explicit constructions were given. As one may expect, these results rely on the irreducible representations of dihedral groups. As before, in this section, let $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$ be the dihedral group of order $2n \geq 4$ and S be a subset of $D_{2n} \setminus \{1_{D_{2n}}\}$ with $S^{-1} = S$. The following result examines the case when n is odd.

Theorem 3.38. *[37] Let $n = 2m + 1$ and let S be a non-empty subset of D_n. Let $\Gamma = \mathrm{Cay}(\mathrm{D_n}, \mathrm{S})$ be a connected Cayley graph with connection set S. Then Γ has no PST between two distinct vertices, and Γ is periodic if and only if it is integral and $S_2 = \emptyset$ or $S_2 = \langle a \rangle$. The minimum period of the vertices is $\frac{2\pi}{M}$, where*

$$M = \gcd(\lambda - \lambda_1 : \lambda \in Spec(\Gamma) \setminus \{\lambda_1\})$$

In the case when $n = 2m$ is even, following [37] let ψ_1 denote the trivial representation of D_{2n} and ψ_4 the one-dimensional irreducible representation of D_{2n} defined by

$$\psi_4\left(a^i\right) = (-1)^i, \psi_4\left(ba^i\right) = (-1)^{i+1}, 0 \leq i \leq n-1.$$

If S is closed under conjugation, then $\mathrm{Cay}\left(\mathrm{D_{2n}}, \mathrm{S}\right)$ has four (not necessarily distinct) eigenvalues $\lambda_1 = |S|, \lambda_2, \lambda_3, \lambda_4$ which correspond to ψ_1 and ψ_4, respectively, and some eigenvalues μ_j corresponding to the two-dimensional representations $\rho_j, 1 \leq j \leq m-1$, where ρ_j is defined by

$$\rho_j\left(a^i\right) = \begin{pmatrix} \omega_n^{ij} & 0 \\ 0 & \omega_n^{-ij} \end{pmatrix}, \rho_j\left(ba^i\right) = \begin{pmatrix} 0 & \omega_n^{-ij} \\ \omega_n^{ij} & 0 \end{pmatrix}, 0 \leq i \leq n-1.$$

Identify the elements of D_{2n} with integers $0, 1, \ldots, 2n-1$ in the follow way: for $0 \leq u \leq n-1$, a^u corresponds to u, and for $n \leq u \leq 2n-1$, ba^u corresponds to u. Recall from (2.4) the 2-adic valuation $\eta_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ of rational numbers. With these notations we now present the result from [37] for even n which covers [39], in the special case when S is closed under conjugation.

Theorem 3.39. *[37] Let $n = 2m$ and let S be a non-empty subset of D_n. Let $\Gamma = \mathrm{Cay}(\mathrm{D_n}, \mathrm{S})$ be a connected Cayley graph with the connection set S. Then Γ cannot have PST between two distinct vertices if S is not conjugation-closed. Conversely, if S is conjugation-closed, then Γ has four eigenvalues (not necessarily distinct) which correspond to the one-dimensional representations ψ_1 to ψ_4, respectively. One eigenvalue is $\lambda_1 = |S|$ and the other three eigenvalues are denoted by $\lambda_2, \lambda_3, \lambda_4$, and some multiple eigenvalues corresponding to the two-dimensional representations ρ_h, which are denoted by $(\leq h \leq m-1)$. Γ is periodic if and only if it is integral. The minimum period of the vertices is $\frac{2\pi}{M}$, where $M = \gcd(\lambda - \lambda_1, \lambda \in \mathrm{Spec}(\Gamma) \setminus \{\lambda_1\})$ Meanwhile,*

1. *when m is even, Γ has PST between two distinct vertices u and v if and only if*
   - *(a) all eigenvalues of Γ are integers, namely, Γ is integral,*
   - *(b) $v = u + m$,*
   - *(c) there is a constant α such that $\eta_2(\mu_{2h'} - \lambda_1) = \alpha$, for every $1 \leq h' \leq \frac{m}{2}$ and for each eigenvalue$\lambda \neq \mu_{2h'-1}$, we have that $\eta_2(\lambda - \lambda_1) > \alpha$.*
2. *when m is odd, Γ has PST between two distinct vertices u and v if and only if the following conditions hold:*
   - *(a) all the eigenvalues of Γ are integers,*
   - *(b) $v = u + m$,*
   - *(c) $\eta_2(\lambda_3 - \lambda_1)$, $\eta_2(\lambda_4 - \lambda_1)$ and $\eta_2(\mu_{2h-1} - \lambda_1)$ are the same for all $1 \leq h' \leq \frac{m-1}{2}$, say, β, and $\eta_2(\mu_{2h'} - \lambda_1)$ and $\eta_2(\lambda_2 - \lambda_1)$ are bigger than β for all $1 \leq h' \leq \frac{m-1}{2}$.*

   Furthermore, when the conditions hold, the minimum time at which Γ has PST between u and v is $\frac{\pi}{M}$, where $M = \gcd(\lambda - \lambda_1, \lambda \in \mathrm{Spec}(\Gamma) \setminus \{\lambda_1\})$.

Wang et al. [149] gave an example of a connected non-normal Cayley graph over a dihedral group having PST between two distinct vertices by applying the relationship between Cayley graphs and bi-Cayley graphs, which produces a counterexample of [37].

Example 3.40. *Let m be a positive integer and $\tilde{G} = D_{8m} = \langle a, b \mid a^{8m} = b^2 = 1, ab = ba^{-1} \rangle$ be a dihedral group of order $16m$. Let $\tilde{S} = \{a^{2j-1} \mid 1 \leqslant j \leqslant 4m\} \cup \{ba^{2m}, ba^{6m}\}$ and $\Gamma = \mathrm{Cay}(\mathrm{D_{8m}}, \tilde{\mathrm{S}})$. Then for any $1 \leqslant i \leqslant 8m$ and $j \in \{0, 1\}$, Γ has PST between vertices $b^j a^i$ and $b^j a^{i+4m}$ at any time $t \in \left\{ \frac{(1+2z)\pi}{2} \middle| z \in \mathbb{Z} \right\}$. Moreover, Γ is periodic at any time $t \in \{z\pi \mid z \in \mathbb{Z}\}$.*

Several families of dihedrants which admit perfect state transfer or are periodic were constructed in [39].

Theorem 3.41. *[39] Let $n = 2m$ and S be a non-empty subset of D_n satisfying $gSg^{-1} = S$ for all $g \in D_n$. Let $\Gamma = \mathrm{Cay}(\mathrm{D_n}, \mathrm{S})$ be a connected Cayley graph with connection set S. Then Γ has four (not necessarily distinct) eigenvalues which correspond to the one-dimensional representations ψ_1 to*

ψ_4, respectively, with one is $\lambda_1 = |S|$ and the three other eigenvalues which are denoted by $\lambda_2, \lambda_3, \lambda_4$, and some multiple eigenvalues corresponding to the two-dimensional representations ρ_h, which are denoted by μ_h $(1 \le h \le m-1)$. Moreover, Γ is periodic if and only if it is integral. The minimum period of the vertices is $\frac{2\pi}{M}$, where $M = \gcd(\lambda - \lambda_1, \lambda \in \mathrm{Spec}(\Gamma) \setminus \{\lambda_1\})$. Meanwhile,

1. *(i) when m is even, Γ has PST between two distinct vertices u and v if and only if*
   - *(a) all eigenvalues of Γ are integers, namely, Γ is integral,*
   - *(b) $v = u + m$,*
   - *(c) there is a constant α such that $\eta_2(\mu_{2h'-1} - \lambda_1) = \alpha$, for every $1 \le h' \le \frac{m}{2}$ and for each eigenvalue$\lambda \ne \mu_{2h'-1}$ $(1 \le h' \le \frac{m}{2})$, it holds that $\eta_2(\lambda - \lambda_1) > \alpha$;*
2. *(ii) when m is odd, Γ has PST between two distinct vertices u and v if and only if the following conditions hold:*
   - *(a) all the eigenvalues of Γ are integers,*
   - *(b) $v = u + m$,*
   - *(c) $\eta_2(\lambda_3 - \lambda_1)$, $\eta_2(\lambda_4 - \lambda_1)$ and $\eta_2(\mu_{2h'-1} - \lambda_1)$ are the same for all $1 \le h' \le \frac{m-1}{2}$, say, β, and $\eta_2(\lambda_2 - \lambda_1)$, $\eta_2(\mu_{2h'} - \lambda_1)$ are bigger than β for all $1 \le h' \le \frac{m-1}{2}$.*

Furthermore, when the conditions hold, the minimum time at which Γ has PST between u and v is $\frac{\pi}{M}$, here $M = \gcd(\lambda - \lambda_1, \lambda \in \mathrm{Spec}(\Gamma) \setminus \{\lambda_1\})$.

Theorem 3.42. *[39] Let $n = 2m+1$ and S be a non-empty subset of D_n satisfying $gSg^{-1} = S$ for all $g \in D_n$. Let $\Gamma = \mathrm{Cay(D_n,S)}$ be a connected Cayley graph with connection set S. Then Γ cannot have PST between two distinct vertices, and Γ is periodic if and only if it is integral. The minimum period of the vertices is $\frac{2\pi}{M}$, where $M = \gcd(\lambda - \lambda_1, \lambda \in \mathrm{Spec}(T) \setminus \{\lambda_1\})$.*

In the following theorem Luo et al. gave a necessary and sufficient condition for a connected normal Cayley graph on SD_{8n} to admit perfect state transfer.

Theorem 3.43. *[108] Suppose that $n > 1$ is an odd number and S is a subset of SD_{8n} such that the cardinality of S is $d > 0$ and $gSg^{-1} = S$ for all $g \in D_{8n}$. Let $\mathrm{Cay(SD_{8n},S)}$ be a simple connected Cayley graph with the connection set S. Let $Q_1 = \{2,4,\ldots,2n-2\}$ and*

$$Q_3 = \{1,3,\ldots,n-2\} \cup \{2n+1, 2n+3, \ldots, 3n-2\}$$

be the sets. Then $\mathrm{Cay(SD_{8n},S)}$ has eight (not necessarily distinct) eigenvalues $\lambda_1 = d, \lambda_2, \ldots, \lambda_8$ which correspond to the representations $\sigma_1, \ldots, \sigma_8$ of degree one, respectively, and $2n-2$ eigenvalues δ_j $(j \in Q_1 \cup Q_3)$ with multiplicity 4 corresponding to the representations ρ_j of degree two, respectively. Furthermore, if $k = \gcd\big(\lambda - d, \lambda \in \mathrm{Spec}(\mathrm{Cay}(SD_{8n}, S) \setminus \{\lambda_1\})\big)$, then

1. *the graph $\mathrm{Cay(SD_{8n},S)}$ is periodic with minimum period $\frac{2\pi}{k}$ if and only if it is an integral graph.*

2. *the graph* $\mathrm{Cay}(\mathrm{SD_{8n}},\mathrm{S})$ *has PST from a to b at time t if and only if*

   (a) *the graph* $\mathrm{Cay}(\mathrm{SD_{8n}},\mathrm{S})$ *is integral;*

   (b) $a-b=2n$ *or* $a-b=-2n$ *with* $0\le a,b\le 4n-1$ *or* $4n\le a,b\le 8n-1$*;*

   (c) *For each* $j\in Q_3$ *and* $z=5,6,7,8$, $\eta_2(\delta_j-d)=\eta_2(\lambda_z-d)=r$ *and* $\eta_2(\lambda-d)>r$ *for any other eigenvalues* $\lambda=\delta_j$ *with* $j\in Q_3$ *and* $\lambda\neq\lambda_z$ *with* $z=5,6,7,8$*. In addition, the minimum time* $t=\frac{\pi}{k}$*.*

Theorem 3.44. *[108] Assume that* $n>0$ *is an even number and* S *is a subset of* SD_{8n} *such that the cardinality of* S *is* $d>0$ *and* $gSg^{-1}=S$ *for all* $g\in D_{8n}$*. Let* $\mathrm{Cay}(\mathrm{SD_{8n}},\mathrm{S})$ *be a simple connected Cayley graph with the connection set S. Let* $Q_1=\{2,4,\dots,2n-2\}$ *and*

$$Q_2=\{1,3,\dots,n-1\}\cup\{2n+1,2n+3,\dots,3n-1\}$$

be the sets. Then $\mathrm{Cay}(\mathrm{SD_{8n}},\mathrm{S})$ *has four (not necessarily distinct) eigenvalues* $\lambda_1=d,\lambda_2,\lambda_3,\lambda_4$ *corresponding to the representations* $\sigma_1,\dots,\sigma_4$ *of degree one, respectively, and* $2n-1$ *eigenvalues* δ_j *(*$j\in Q_1\cup Q_2$*) with multiplicity* 4 *which correspond the representations* ρ_j *of degree two, respectively. Furthermore, if* $k=\gcd\left(\lambda-d,\lambda\in\mathrm{Spec}\left(\mathrm{Cay}(SD_{8n},S)\right)\setminus\{\lambda_1\}\right)$*, then*

1. *the graph*$\mathrm{Cay}(\mathrm{SD_{8n}},\mathrm{S})$ *is periodic with minimum period* $\frac{2\pi}{k}$ *if and only if it is an integral graph.*

2. *the graph* $\mathrm{Cay}(\mathrm{SD_{8n}},\mathrm{S})$ *has PST from a to b at time t if and only if*

   (a) *the graph* $\mathrm{Cay}(\mathrm{SD_{8n}},\mathrm{S})$ *is integral;*

   (b) $a-b=2n$ *or* $a-b=-2n$ *with* $0\le a,b\le 4n-1$ *or* $4n\le a,b\le 8n-1$*;*

   (c) *For each* $j\in Q_2$, $\eta_2(\delta_j-d)=r$ *and* $j\in Q_2$, $\eta_2(\lambda-d)>r$ *for any other eigenvalues* $\lambda\neq\delta_j$ *with* $j\in Q_2$*.*
   Additionally, the minimum time $t=\frac{\pi}{k}$*.*

Arezoomand et al. [16] gave a complete characterization of existence of *PST* for quasiabelian Cayley graphs over T_{4n} groups. As one may expect, these results rely on the irreducible representations of T_{4n}. $T_{4n}=\langle a,b\mid a^{2n}=1,a^n=b^2,b^{-1}ab=a^{-1}\rangle$, where $n\ge 2$ and $\omega=e^{2\pi i/2n}=\cos(\pi/n)+i\sin(\pi/n)$ be a $2n$-th root of unity which is neither 1 nor -1. The irreducible representations and characters of the dicyclic group T_{4n} are listed in the Tables 1, 2.

Theorem 3.45. *[16] Let* $\Gamma=\mathrm{Cay}(T_{4n},S)$ *be a quasiabelian Cayley graph with respect to S. Then* Γ *is periodic if and only if it is integral. The minimum period of the vertices is* $\frac{2\pi}{M}$*, where*

$$M=\gcd\left(\lambda-\lambda_1,\lambda\in\mathrm{Spec}(\Gamma)\setminus\{\lambda_1\}\right).$$

Furthermore,

1. *when n is even,* Γ *has PST between two distinct vertices u and v if and only if*

Table 1. Irreducible representation of T_{4n} for n even.

	a	b
ψ_1	(1)	(1)
ψ_2	(1)	(-1)
ψ_3	(-1)	(1)
ψ_4	(-1)	(-1)
$\gamma_r,\ _{(1\leq r\leq n-1)}$	$\begin{pmatrix}\omega^r & 0\\ 0 & \omega^{-r}\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ \omega^{rn} & 0\end{pmatrix}$

Table 2. Irreducible representation of T_{4n} for n odd.

	a	b
ψ_1	(1)	(1)
ψ_2	(1)	(-1)
ψ_3	(-1)	(i)
ψ_4	(-1)	$(-i)$
$\gamma_r,\ _{(1\leq r\leq n-1)}$	$\begin{pmatrix}\omega^r & 0\\ 0 & \omega^{-r}\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ \omega^{rn} & 0\end{pmatrix}$

(a) *all eigenvalues of* Γ *are integers, namely,* Γ *is integral,*

(b) $u = v + n$ *when* $0 \leq u, v \leq 2n-1$ *or* $-2n \leq u, v \leq 4n-1$,

(c) *there is a constant* λ *such that* $1 \leq r' \leq \frac{n}{2}$ *for every* $\eta_2(\mu_{2r'-1} - \lambda_1) = \lambda$ *and for each eigenvalue* $\lambda \neq \mu_{2r'-1}$, $(1 \leq r' \leq \frac{n}{2})$, *it holds that* $\eta_2(\lambda - \lambda_1) > \lambda$;

2. *when* n *is odd,* Γ *has PST between two distinct vertices* u *and* v *if and only if*

(a) *all eigenvalues of* Γ *are integers, namely,* Γ *is integral,*

(b) $u = v + n$ *when* $0 \leq u, v \leq 2n-1$ *or* $-2n \leq u, v \leq 4n-1$,

(c) $\eta_2(\lambda_3 - \lambda_1)$, $\eta_2(\lambda_4 - \lambda_1)$ *and* $\eta_2(\mu_{2r'-1} - \lambda_1)$ *are the same for the* $1 \leq r' \leq \frac{(n-1)}{2}$, *say* β, *and* $\eta_2(\lambda_2 - \lambda_1)$, $\eta_2(\mu_{2r'} - \lambda_1)$ *are bigger than* β *for all* $1 \leq r' \leq \frac{(n-1)}{2}$.

Recently Wang et al. [17, 149] investigated perfect state transfer on semi-Cayley graphs over abelian groups. Semi-Cayley graphs, also known as bi-Cayley graphs, are a generalization of Cayley graphs. A graph is said to be a semi-Cayley graph over a group G if it admits G as a semiregular subgroup of the full automorphism group with two orbits of equal size. In the following theorems, they proved some necessary and sufficient conditions for a quasi-abelian semi-Cayley graph having perfect state transfer. They also gave an example for the application of theorem 3.47.

Theorem 3.46. *[150] Let* $\Gamma = SC(G, R, L, S)$ *be a quasi-abelian semi-Cayley graph over a group* G *of order* n. *Suppose that* G *has* $m (m \leq n)$ *non-equivalent irreducible representations. For* $Y \in \{R, L, S\}$, *let $\lambda_{k,Y}$ be an eigenvalue of* $\mathrm{Cay}(G,Y)$ *with multiplicity d_k^2 where $1 \le k \le m$ satisfying $\sum_{k=1}^{m} d_k^2 = n$. Write*

$$\Delta = \{k \mid \lambda_{k,S} = 0, 1 \le k \le m\}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_{2m}$ are

$$\lambda_{2k-j} = \frac{\lambda_{k,R} + \lambda_{k,L} + (-1)^j \sqrt{(\lambda_{k,R} - \lambda_{k,L})^2 + 4|\lambda_{k,S}|^2}}{2}$$

whose multiplicity is d_k^2, where $k = 1,2,\ldots,m$ and $j = 0,1$. Then Γ has PST between vertices g_p and g_q at time t if and only if the following hold for every $1 \le k \le m$, where $X = R$, if $(g_p, g_q) \in G_0 \times G_0$ and $X = L$ if $(g_p, g_q) \in G_1 \times G_1$.

1.
$$\chi_k\left(g_p g_q^{-1}\right) = \begin{cases} \pm d_k, & \text{if } (g_p,g_q) \in (G_0 \times G_0) \cup (G_1 \times G_1) \\ d_k \frac{|\lambda_{k,S}|}{\lambda_{k,S}}, & \text{if } (g_p,g_q) \in G_0 \times G_1, \exp(\mathrm{it}(\lambda_2 - \lambda_{2k})) = 1 \\ -d_k \frac{|\lambda_{k,S}|}{\lambda_{k,S}}, & \text{if } (g_p,g_q) \in G_0 \times G_1, \exp(\mathrm{it}(\lambda_2 - \lambda_{2k})) = -1 \\ d_k \frac{|\lambda_{k,S}|}{\lambda_{k,S}}, & \text{if } (g_p,g_q) \in G_1 \times G_0, \exp(\mathrm{it}(\lambda_2 - \lambda_{2k})) = 1 \\ -d_k \frac{|\lambda_{k,S}|}{\lambda_{k,S}}, & \text{if } (g_p,g_q) \in G_1 \times G_0, \exp(\mathrm{it}(\lambda_2 - \lambda_{2k})) = -1 \end{cases}$$

2. *If $(g_p, g_q) \in (G_0 \times G_0) \cup (G_1 \times G_1)$, then*
$$\begin{cases} t(\lambda_{2k} - \lambda_{2k-1}) \in \{2z\pi \mid z \in \mathbb{Z}\} \text{ and} \\ t(\lambda_2 - \lambda_{2k-1}) \in \left\{ \left(2z - \frac{\chi_k(g_p g_q^{-1}) - d_k}{2d_k}\right)\pi \,\middle|\, z \in \mathbb{Z} \right\}, & \text{if } S \ne \emptyset \text{ and } k \notin \Delta; \\ t(|X| - \lambda_{k,X}) \in \left\{ \left(2z - \frac{\chi_k(g_p g_q^{-1}) - d_k}{2d_k}\right)\pi \,\middle|\, z \in \mathbb{Z} \right\}, & \text{if } S = \emptyset; \\ t(\lambda_2 - \lambda_{k,X}) \in \left\{ \left(2z - \frac{\chi_k(g_p g_q^{-1}) - d_k}{2d_k}\right)\pi \,\middle|\, z \in \mathbb{Z} \right\}, & \text{if } S \ne \emptyset \text{ and } k \in \Delta. \end{cases}$$

3. *If $(g_p, g_q) \in (G_0 \times G_1) \cup (G_1 \times G_0)$, then $R = L, \lambda_{k,S} \ne 0, t(\lambda_{2k} - \lambda_{2k-1}) \in \{(2z+1)\pi \mid z \in \mathbb{Z}\}$ and $t(\lambda_2 - \lambda_{2k}) \in \{z\pi \mid z \in \mathbb{Z}\}$.*

Theorem 3.47. *[150] Let $\Gamma = SC(G,R,L,S)$ be a quasi-abelian semi-Cayley graph over a group G of order n. Suppose that G has m non-equivalent irreducible representations. For $Y \in \{R,L,S\}$, let $\lambda_{k,Y}$ be an eigenvalue of* $\mathrm{Cay}(G,Y)$ *with multiplicity d_k^2 where $1 \le k \le m$ satisfying $\sum_{k=1}^{m} d_k^2 = n$. Let*

$$M_S = \gcd(|\lambda_{k,S}| \mid 1 \le k \le m)$$

and

$$M = \gcd(\lambda_2 - \lambda_{2k} \mid 1 \le k \le m)$$

where $\lambda_1, \lambda_2, \ldots, \lambda_{2m}$ are

$$\lambda_{2k-j} = \frac{\lambda_{k,R} + \lambda_{k,L} + (-1)^j \sqrt{(\lambda_{k,R} - \lambda_{k,L})^2 + 4|\lambda_{k,S}|^2}}{2}$$

whose multiplicity is d_k^2, where $k = 1,2,\ldots,m$ and $j = 0,1$.. Then Γ has PST between vertices g_p and g_q with $(g_p, g_q) \in (G_0 \times G_1) \cup (G_1 \times G_0)$ at time t if and only if the following hold.

1. *Γ is an integral graph and $R = L$.*

2. *For each $1 \leq k \leq m$,*

   (a) $$\chi_k\left(g_p g_q^{-1}\right) = \begin{cases} d_k \frac{\left|\lambda_{k,S}\right|}{\lambda_{k,S}}, & \text{if } \left(g_p, g_q\right) \in G_0 \times G_1, \exp\left(\mathrm{it}\left(\lambda_2 - \lambda_{2\mathrm{k}}\right)\right) = 1; \\ -d_k \frac{\left|\lambda_{k,S}\right|}{\lambda_{k,S}}, & \text{if } \left(g_p, g_q\right) \in G_0 \times G_1, \exp\left(\mathrm{tt}\left(\lambda_2 - \lambda_{2\mathrm{k}}\right)\right) = -1; \\ d_k \frac{\left|\lambda_{k,S}\right|}{\lambda_{k,S}}, & \text{if } \left(g_p, g_q\right) \in G_1 \times G_0, \exp\left(\mathrm{it}\left(\lambda_2 - \lambda_{2\mathrm{k}}\right)\right) = 1; \\ -d_k \frac{\left|\lambda_{k,S}\right|}{\lambda_{k,S}}, & \text{if } \left(g_p, g_q\right) \in G_1 \times G_0, \exp\left(\mathrm{tt}\left(\lambda_2 - \lambda_{2\mathrm{k}}\right)\right) = -1. \end{cases}$$

   (b) $\lambda_{k,S} \neq 0$

   (c) $v_2\left(\left|\lambda_{k,S}\right|\right) = v_2(|S|)$

   (d) *If $M > 0$, then $v_2(M) > v_2(|S|)$.*

3. $t \in \left\{ \frac{(1+2z)\pi}{\gcd(2M_S, M)} \,\middle|\, z \in \mathbb{Z} \right\}$.

Theorem 3.48. *[150] Let $\Gamma = SC(G, R, L, S)$ be an integral quasi-abelian semi-Cayley graph over a group G of order n. Suppose that G has m non-equivalent irreducible representations. Let*

$$\begin{aligned} M_0 &= \gcd\left(\lambda_{2k} - \lambda_{2k-1} \mid 1 \leq k \leq m, k \notin \Delta\right), \\ M_1 &= \gcd\left(\lambda_2 - \lambda_{2k-1} \mid 1 \leq k \leq m, k \notin \Delta\right), \\ M_{\emptyset,X} &= \gcd\left(|X| - \lambda_{k,X} \mid 1 \leq k \leq m\right), \end{aligned}$$

and

$$M_X = \begin{cases} \gcd\left(\lambda_2 - \lambda_{k,X} \mid k \in \Delta\right), & \text{if } \Delta \neq \emptyset; \\ 0, & \text{if } \Delta = \emptyset. \end{cases}$$

Then Γ has PST between vertices g_p and g_q with $\left(g_p, g_q\right) \in (G_0 \times G_0) \cup (G_1 \times G_1)$ at time t if and only if the following hold.

1. *For each $1 \leq k \leq m, \chi_k\left(g_p g_q^{-1}\right) = \pm d_k$.*

2. *There exists an integer μ such that*

   (a) $$\begin{cases} v_2\left(\lambda_2 - \lambda_{2k-1}\right) = \mu \text{ for } S \neq \emptyset \text{ and every } k \in \Omega^- \setminus \Delta; \\ v_2\left(\lambda_2 - \lambda_{k,X}\right) = \mu \quad \text{for } S \neq \emptyset \text{ and every } k \in \Omega^- \cap \Delta; \\ v_2\left(|X| - \lambda_{k,X}\right) = \mu \text{ for } S = \emptyset \text{ and every } k \in \Omega^-; \end{cases}$$

   (b) $$\begin{cases} v_2\left(\lambda_{2k} - \lambda_{2k-1}\right) \geq \mu + 1 & \text{for } S \neq \emptyset \text{ and every } k \notin \Delta; \\ v_2\left(\lambda_2 - \lambda_{2k-1}\right) \geq \mu + 1 & \text{for } S \neq \emptyset \text{ and every } k \in \Omega^+ \setminus \Delta; \\ v_2\left(\lambda_2 - \lambda_{k,X}\right) \geq \mu + 1 & \text{for } S \neq \emptyset \text{ and every } k \in \Omega^+ \cap \Delta; \\ v_2\left(|X| - \lambda_{k,X}\right) \geq \mu + 1 & \text{for } S = \emptyset \text{ and every } k \in \Omega^+. \end{cases}$$

3. *If $S \neq \emptyset$, then*

$$t \in \begin{cases} \left\{ \frac{(1+2z)\pi}{\gcd(M_0, M_1, M_X)} \,\middle|\, z \in \mathbb{Z} \right\}, \text{ when } \Omega^- \neq \emptyset; \\ \left\{ \frac{2\pi z}{\gcd(M_0, M_1, M_X)} \,\middle|\, z \in \mathbb{Z} \right\}, \text{ when } \Omega^- = \emptyset. \end{cases}$$

4. *If $S = \emptyset$, then*

$$t \in \begin{cases} \left\{ \frac{2\pi z}{M_{\emptyset,X}} \,\middle|\, z \in \mathbb{Z} \right\}, \quad \text{when } \Omega^- = \emptyset \text{ or for each } k \in \Omega^-, \lambda_{k,X} = |X|; \\ \left\{ \frac{(1+2z)\pi}{M_{\emptyset,X}} \,\middle|\, z \in \mathbb{Z} \right\}, \text{ otherwise.} \end{cases}$$

Example 3.49. *[150] Let $G = D_{12} = \langle a, b \mid a^6 = b^2 = 1, bab = a^{-1} \rangle$, $R = \langle a \rangle b$, $L = \{b, a^2b, a^4b\}$ and $S = \{a, a^5\}$. Let $\Gamma = SC(G, R, L, S)$. Then Γ is an integral quasiabelian semi-Cayley graph and has no PST between two distinct vertices. However, Γ is periodic at time $t \in \{2z\pi \mid z \in \mathbb{Z} \backslash \{0\}\}$.*

Since $R \neq L$, by Theorem 3.47, Γ cannot have PST between any pair of vertices in $(G_0 \times G_1) \cup (G_1 \times G_0)$.

In the following theorem they focused on the existence of perfect state transfer on quasi-abelian Cayley graphs. Examples are provided in to illustrate our results. For instance, Example 3.51 gives a necessary and sufficient condition for a quasi-abelian Cayley graph over a dihedral group having perfect state transfer.

Theorem 3.50. *[150] Let $\mathrm{Cay}(G, R)$ be a quasi-abelian Cayley graph over a group G of order n. Suppose that G has m non-equivalent irreducible representations. For $1 \leq k \leq m$, let λ_k be an eigenvalue of $\mathrm{Cay}(G, R)$ with multiplicity d_k^2 satisfying $\sum_{k=1}^{m} d_k^2 = n$. For $g, h \in G$, $\mathrm{Cay}(G, R)$ has PST between g and h at time t if and only if the following hold, where $M = \gcd(|R| - \lambda_k \mid 1 \leq k \leq m)$ and $\Omega^- = \{k \mid \chi_k(gh^{-1}) = -d_k, 1 \leq k \leq m\}$*

1. *$\mathrm{Cay}(G, R)$ is an integral graph.*

2. *For every $1 \leq k \leq m$, $\chi_k(gh^{-1}) = \pm d_k$.*

3. *There exists an integer μ such that $v_2(|R| - \lambda_k) = \mu$ for $k \in \{1 \leq j \leq m \mid \chi_j(gh^{-1}) = -d_j\}$ and $v_2(|R| - \lambda_k) \geq \mu + 1$ for $k \in \{1 \leq j \leq m \mid \chi_j(gh^{-1}) = d_j\}$*

4. $t \in \begin{cases} \left\{ \frac{2z\pi}{M} \,\middle|\, z \in \mathbb{Z} \right\}, \quad g = h \text{ or for each } k \in \Omega^-, \lambda_k = |R| \\ \left\{ \frac{(1+2z)\pi}{M} \,\middle|\, z \in \mathbb{Z} \right\}, \quad \text{otherwise.} \end{cases}$

   Moreover, the order of gh^{-1} is two and $\mathrm{Cay}(G, R)$ is periodic if and only if it is an integral graph and the period is $t \in \{ \frac{2z\pi}{M} \mid z \in \mathbb{Z} \backslash \{0\} \}$.

Example 3.51. *[150] Let $n \geq 3$ and $G = D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Let S be a subset of G satisfying $gSg^{-1} = S$ for all $g \in G$. Suppose that G has m non-equivalent irreducible representations. Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph. Let $\{\lambda_k \mid 1 \leq k \leq m\}$ be the spectrum of Γ where $\lambda_k = \frac{\chi_k(S)}{d_k}$ is $\lambda_k = \frac{1}{d_k} \sum_{g \in S} \chi_k(g) := \frac{\chi_k(S)}{d_k}$, and has multiplicity d_k^2. Then for distinct vertices g and h of G, $\mathrm{Cay}(G, S)$ has PST between g and h at time t if and only if the following hold;*

1. $\mathrm{Cay}(G,S)$ *is an integral graph.*
2. n *is even and* $gh^{-1} = a^{\frac{n}{2}}$.
3. *There exists an integer* μ *such that*
   (a) *if* $n \equiv 2(\bmod 4)$, *then for* $1 \le i \le \frac{1}{2}\left(\frac{n}{2}-1\right)$, $v_2(|S|-\lambda_3) = v_2(|S|-\lambda_4) = v_2(|S|-\lambda_{2i+3}) = \mu$ *and* $v_2(|S|-\lambda_2)$ *and* $v_2(|S|-\lambda_{2i+4})$ *are greater than* μ;
   (b) *if* $n \equiv 0(\bmod 4)$, *then* $v_2(|S|-\lambda_{2i+3}) = \mu$ *and* $v_2(|S|-\lambda_{2i+4}) \ge \mu+1$ *for* $1 \le i \le \frac{n}{4}$, *and* $v_2(|S|-\lambda_j) \ge \mu+1$ *for* $2 \le j \le 4$.
4. $t \in \left\{ \frac{(1+2z)\pi}{M} \,\middle|\, z \in \mathbb{Z} \right\}$, *where* $M = \gcd(|S|-\lambda_k \mid 1 \le k \le m)$.

Furthermore, if $\mathrm{Cay}(G,S)$ *is an integral graph, then it is periodic at time* $t \in \left\{ \frac{2z\pi}{M} \,\middle|\, z \in \mathbb{Z}\backslash\{0\} \right\}$

Khalilipour et al. [102] gave a complete characterization of existence of *PST* for quasiabelian Cayley graphs over U_{6n} groups. Let $U_{6n} = \langle a,b \mid a^{2n} = b^3 = 1, | \, a^{2n} = b^3 = 1, a^{-1}ba = b^{-1}\rangle$. be the group of order $6n$ and S be a subset of $U_{6n} \setminus \{1_{U_{6n}}\}$ with $S^{-1} = S$.

Table 3. Irreducible representation of U_{6n}, $\omega = e^{2\pi i/3}, \varepsilon = e^{2\pi i/2n}$

	a^{2j}	a^{2j+1}	$a^{2j}b$
$\varphi_k(0 \le k \le 2n-1)$	ε^{2kj}	$\varepsilon^{k(2j+1)}$	ε^{2kj}
$\gamma_l(0 \le l \le n-1)$	$\begin{pmatrix} \varepsilon^{2lj} & 0 \\ 0 & \varepsilon^{2lj} \end{pmatrix}$	$\begin{pmatrix} 0 & \varepsilon^{l(2j+1)} \\ \varepsilon^{l(2j+1)} & 0 \end{pmatrix}$	$\begin{pmatrix} \varepsilon^{2lj}\omega & 0 \\ 0 & \varepsilon^{2lj}\omega^2 \end{pmatrix}$

Theorem 3.52. *[102] Let* $\Gamma = \mathrm{Cay}(U_{6n}, S)$ *be a quasiabelian Cayley graph with respect to* S. *Then* Γ *has* $2n$ *(not necessarily distinct) eigenvalues which correspond to the one-dimensional representations* φ_k $(0 \le k \le 2n-1)$, *respectively, with one is* $\lambda_0 = |S|$ *and the* $2n-1$ *other eigenvalues which are denoted by* λ_k, *and some multiple eigenvalues corresponding to the two-dimensional representations* γ_l $(0 \le l \le n-1)$, *which are denoted by* μ_l. *Moreover,* Γ *is periodic if and only if it is integral. The minimum period of the vertices is* $2\pi/M$, *where* $M = \gcd(\lambda - \lambda_0, \lambda \in \mathrm{Spec}(\Gamma) \setminus \{\lambda_0\})$. *Furthermore, For each* n, Γ *has PST between two vertices* u *and* v *if and only if*

1. *all eigenvalues of* Γ *are integers, namely,* Γ *is integral,*
2. $v = u + n$ *when* $0 \le u, v \le 3n-1$ *or* $3n \le u, v \le 6n-1$,
3. $\eta_2(\lambda_1 - \lambda_0)$, $\eta_2(\lambda_{2k'+1} - \lambda_0)$ *and* $\eta_2(\mu_{2l'-1} - \lambda_0)$ *are the same for all* $1 \le k' \le n-1$ *and* $1 \le l' \le n/2$, *say,* α, *and* $\eta_2(\mu_{2l'} - \lambda_0)$ *and* $\eta_2(\lambda_{2k'} - \lambda_0)$ *are bigger than* α *for all* $1 \le l' \le n/2$ *and* $1 \le k' \le n-1$.

In [6], necessary and sufficient conditions for a Cayley graph on V_{8n} to admit perfect state transfer was obtained and explicit constructions were given. Let $V_{8n} = \langle a,b \mid a^{2n} = b^4 = 1, ba = a^{-1}b^{-1}, b^{-1}a = a^{-1}b\rangle$ be the group of order $8n$ and S be a subset of $V_{8n} \setminus \{1_{V_{8n}}\}$ with $S^{-1} = S$. The following results handle two cases when n is odd or even.

Table 4. Irreducible representations of V_{8n}, for n odd, $\omega = e^{2\pi i/2n}$

	a	b
χ_1	1	1
χ_2	1	-1
χ_3	-1	1
χ_4	-1	-1
$\psi_j (0 \leq j \leq n-1)$	$\begin{pmatrix} \omega^{2j} & 0 \\ 0 & -\omega^{-2j} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$\phi_k (1 \leq k \leq n-1)$	$\begin{pmatrix} \omega^{k} & 0 \\ 0 & \omega^{-k} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Theorem 3.53. *[6] Let S be a non-empty subset of V_{8n} such that $1 \notin S$ and $Sg = gS$ for all $g \in V_{8n}$. Let $\Gamma = \mathrm{Cay}(V_{8n}, S)$ be a connected Cayley graph with connection set S, where n is odd. Then Γ has four distinct eigenvalues which corresponds to the one-dimensional representations χ_1, χ_2, χ_3 and χ_4, respectively, with one is $\alpha_1 = |S|$ and the other three eigenvalues are denoted by α_2, α_3 and α_4, and some multiple eigenvalues corresponding to the two-dimensional representations ψ_j and ϕ_k, denoted by β_j and γ_k, respectively, for $0 \leq j \leq n-1$ and $1 \leq k \leq n-1$.*

1. *If $u \in V_1, v \in V_2$ or $u \in V_1, v \in V_4$ or $u \in V_2, v \in V_1$ or $u \in V_2, v \in V_3$ or $u \in V_3, v \in V_2$ or $u \in V_3, v \in V_4$ or $u \in V_4, v \in V_1$ or $u \in V_4, v \in V_3$, then Γ cannot have PST between two distinct vertices u and v.*

2. *If $u, v \in V_1$ or $u, v \in V_2$ or $u, v \in V_3$ or $u, v \in V_4$, then Γ cannot have PST between two distinct vertices u and v.*

3. *If $u \in V_1, v \in V_3$ or $u \in V_2, v \in V_4$ or $u \in V_3, v \in V_1$ or $u \in V_4, v \in V_2$, then Γ has PST between the vertices u and v if and only if the following three conditions hold.*

   (a) All the eigenvalues of Γ are integers, namely, Γ is integral,

   (b) $u = v + 4n$ and

   (c) $\eta_2(\alpha_1 - \beta_j)$ is a constant, say μ, and $\eta_2(\alpha_1 - \alpha_2), \eta_2(\alpha_1 - \alpha_3)$, $\eta_2(\alpha_1 - \alpha_4)$ and $\eta_2(\alpha_1 - \gamma_k)$ are all bigger than μ, for $0 \leq j \leq n-1$ and $1 \leq k \leq n-1$.

   Furthermore, when the conditions (a), (b) and (c) hold, the minimum time at which Γ has PST between u and v is $\frac{\pi}{M}$, where $M = \gcd(\alpha - \alpha_1, \alpha \in \mathrm{Spec}(\Gamma) \setminus \{\alpha_1\})$.

Theorem 3.54. *[6] Let S be a non-empty subset of V_{8n} such that $1 \notin S$ and $Sg = gS$ for all $g \in V_{8n}$. Let $\Gamma = \mathrm{Cay}(V_{8n}, S)$ be a connected Cayley graph with connection set S, where n is even. Then Γ has eight distinct eigenvalues which corresponds to the one-dimensional representations $\chi_1, \ldots, \chi_8$, respectively, with one is $\alpha_1 = |S|$ and the other three eigenvalues are denoted by $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ and α_8, and some multiple eigenvalues corresponding to the two-dimensional representations ψ_j and ϕ_k, denoted by β_j and γ_k, respectively, for $1 \leq j \leq n-1$ and $1 \leq k \leq n-1$.*

Table 5. Irreducible representations of V_{8n}, for n even, $\omega = e^{2\pi i/2n}$

	a	b
χ_1	1	1
χ_2	$\mathbf{i}$	$-\mathbf{i}$
χ_3	-1	-1
χ_4	$-\mathbf{i}$	$\mathbf{i}$
χ_5	1	-1
χ_6	$\mathbf{i}$	$\mathbf{i}$
χ_7	-1	1
χ_8	$-\mathbf{i}$	$-\mathbf{i}$
$\psi_j (1 \leq j \leq n-1)$	$\begin{pmatrix} \omega^j & 0 \\ 0 & \omega^{-j} \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{pmatrix}$
$\phi_k (1 \leq k \leq n-1)$	$\begin{pmatrix} \mathbf{i}\omega^k & 0 \\ 0 & \mathbf{i}\omega^{-k} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

1. *If $u \in V_1, v \in V_2$ or $u \in V_1, v \in V_4$ or $u \in V_2, v \in V_1$ or $u \in V_2, v \in V_3$ or $u \in V_3, v \in V_2$ or $u \in V_3, v \in V_4$ or $u \in V_4, v \in V_1$ or $u \in V_4, v \in V_3$, then Γ cannot have PST between distinct vertices u and v.*

2. *If $u, v \in V_1$ or $u, v \in V_2$ or $u, v \in V_3$ or $u, v \in V_4$, then Γ has PST between distinct vertices u and v if and only if the following three conditions hold.*

   (a) *All the eigenvalues of Γ are integers.*

   (b) *$u = v + n$.*

   (c) *If $n \equiv 0 (\mathrm{mod} 4)$, then $\eta_2\left(\alpha_1 - \beta_{2j'-1}\right)$ and $\eta_2(\alpha_1 - \gamma_{2k'-1})$ are the same, say μ_1, and $\eta_2(\alpha_1 - \alpha_2), \eta_2(\alpha_1 - \alpha_3), \eta_2(\alpha_1 - \alpha_4)$, $\eta_2(\alpha_1 - \alpha_5), \eta_2(\alpha_1 - \alpha_6), \eta_2(\alpha_1 - \alpha_7), \eta_2(\alpha_1 - \alpha_8), \eta_2\left(\alpha_1 - \beta_{2j'}\right)$ and $\eta_2(\alpha_1 - \gamma_{2k'})$ are all strictly greater than μ_1, for $1 \leq j' \leq \frac{n-1}{2}$ and $1 \leq k' \leq \frac{n-1}{2}$.*

   (d) *If $n \equiv 2 (\mathrm{mod} 4)$, then $\eta_2(\alpha_1 - \alpha_2), \eta_2(\alpha_1 - \alpha_4), \eta_2(\alpha_1 - \alpha_6)$, $\eta_2(\alpha_1 - \alpha_8), \eta_2\left(\alpha_1 - \beta_{2j'-1}\right)$ and $\eta_2(\alpha_1 - \gamma_{2k'})$ are the same, say μ_2, and $\eta_2(\alpha_1 - \alpha_3)$, $\eta_2(\alpha_1 - \alpha_5), \eta_2(\alpha_1 - \alpha_7), \eta_2\left(\alpha_1 - \beta_{2j'}\right)$ and $\eta_2(\alpha_1 - \gamma_{2k'-1})$ are all strictly greater than μ_2, for $1 \leq j' \leq \frac{n-1}{2}$ and $1 \leq k' \leq \frac{n-1}{2}$.*

3. *If $u \in V_1, v \in V_3$ or $u \in V_2, v \in V_4$ or $u \in V_3, v \in V_1$ or $u \in V_4, v \in V_2$, then Γ has PST between distinct vertices u and v if and only if the following three conditions hold.*

   (a) *All the eigenvalues of Γ are integers.*

   (b) *$u = v + 4n$.*

(c) $\eta_2(\alpha_1-\alpha_2), \eta_2(\alpha_1-\alpha_4), \eta_2(\alpha_1-\alpha_6), \eta_2(\alpha_1-\alpha_8)$ and $\eta_2(\alpha_1-\gamma_k)$ are the same, say μ_3, and $\eta_2(\alpha_1-\alpha_3), \eta_2(\alpha_1-\alpha_5), \eta_2(\alpha_1-\alpha_7)$ and $\eta_2(\alpha_1-\beta_j)$ are all strictly greater than μ_3, for $1 \le j \le n-1$ and $1 \le k \le n-1$

Furthermore, the minimum time at which Γ has PST between u and v is $\frac{\pi}{M}$, where

$$M = \gcd(\alpha-\alpha_1, \alpha \in \mathrm{Spec}(\Gamma) \setminus \{\alpha_1\})$$

3.2 The results of PGST

The notion of PGST was introduced by Godsil [76] and Vinet and Zhedanov [146] independently in 2012 as a relaxation of perfect state transfer, where in the latter paper the term almost state transfer was used for PGST. In this section, we restrict our attention to PGST in Cayley graphs. Recall that a regular graph is periodic if and only if it is integral. As observed in [118], a periodic graph exhibits PGST if and only if it admits perfect state transfer. Since all integral circulant graphs admitting perfect state transfer have been characterized, it follows that integral circulant graphs exhibiting PGST are all known and are given in Theorem 3.21. So the existence of PGST in circulant graphs is reduced to that in non-integral circulant graphs. Unfortunately, as far as we are aware, there is no known necessary and sufficient condition for a non-integral circulant graph to admit PGST.

3.2.1 PGST in Cayley graphs on abelian groups

Pal et al. [118] that in a circulant graph $\mathrm{Cay}(\mathbb{Z}_n, S)$, PGST occurs exclusively from u to $u+\frac{n}{2}$ for $u \in \mathbb{Z}_n$. This condition necessitates $\frac{n}{2} \in S$ and n being even. Additionally, they demonstrated that the cycle C_n of length n possesses PGST if and only if its complement does, which happens if and only if $n = 2k$ for some $k \ge 2$. Among the complement of cycles, we also find a class of graphs not exhibiting pretty good state transfer. A circulant graph G being vertex-transitive, for any pair of vertices u, v in G there exists an automorphism mapping u to v. Let A be the adjacency matrix of G. Note that the transition matrix $H(t)$ of G can be realized as a polynomial in A. If P is the matrix of an automorphism of G then P commutes with A as well as $H(t)$. If G exhibits PGST between two vertices u and v then

$$\lim_{k\to\infty} H(t_k)(Pe_u) = \gamma(Pe_v)$$

by Lemma 3.55, it is enough to find PGST in a circulant graph between the pair of vertices 0 and $\frac{n}{2}$. Let the spectral decomposition of the adjacency matrix of C_n be $A = \sum_{l=0}^{n-1} \lambda_l E_l$, where $E_l = \frac{1}{n} v_l v_l^*$ and λ_l, v_l is as mentioned in 3.1.1. Therefore, the transition matrix of C_n is evaluated as

$$H(t) = \exp(-itA) = \sum_{l=0}^{n-1} \exp(-i\lambda_l t) E_l$$

Note that $\left(0, \frac{n}{2}\right)$-th entry of E_l is $\frac{1}{n}\omega_n^{-\frac{nl}{2}}$. Hence $\left(0, \frac{n}{2}\right)$-th entry of $H(t)$ is given by

$$H(t)_{0,\frac{n}{2}} = \frac{1}{n}\sum_{l=0}^{n-1} \exp(-i\lambda_l t) \cdot \omega_n^{-\frac{nl}{2}} = \frac{1}{n}\sum_{l=0}^{n-1} \exp[-i(\lambda_l t + l\pi)]$$

Lemma 3.55. *[118] If PGST in a C_n or the complement of C_n then n is even and it occurs only between the pair of vertices u and $u+\frac{n}{2}$ where $u, u+\frac{n}{2} \in \mathbb{Z}_n$.*

If d is a proper divisor of n, we define $S_n(d) = \{x \in \mathbb{Z}_n : \gcd(x,n) = d\}$. For any set D containing proper divisors of n, we define $S_n(D) = \bigcup_{d \in D} S_n(d)$. The set $S_n(D)$ is called a greatest common divisor (gcd) set of $\mathbb{Z}_n$. A gcd graph over $\mathbb{Z}_n$ is a circulant graph whose connection set is a gcd set. We denote a gcd graph with the connection set $S_n(D)$ by $G(n,D)$.

Theorem 3.56. *[118] Let $n = 2k$ with $k \geq 3$. If D is a set of proper divisors of n not containing 1, then the circulant graph $C_n \cup G(n,D)$ along with its complement admits a pretty good state transfer concerning the same sequence in $2\pi\mathbb{Z}$.*

Corollary 3.57. *[118] Let a graph Γ_1 be periodic at a vertex at time 2π. If $\Gamma_2 \in \Gamma$, then the Cartesian product $\Gamma_1 \square \Gamma_2$ admits pretty good state transfer. If Γ_1 is regular, then the complement of $\Gamma_1 \square \Gamma_2$, denoted as $\overline{\Gamma_1 \square \Gamma_2}$, also exhibits pretty good state transfer.*

Theorem 3.58. *[118] A cycle C_n admits PGST if and only if $n = 2^k$ where $k \geq 2$.*

A graph Γ is said to be almost periodic if there exists a sequence t_k of real numbers and a complex number γ of unit modulus such that $\lim_{k\to\infty} H(t_k) = \gamma I$, where I is the identity matrix of appropriate order. Since circulant graphs are vertex-transitive, we observe that a circulant graph is almost periodic if and only if $\lim_{k\to\infty} H(t_k)e_0 = \gamma e_0$. This implies that if a cycle admits Perron-Frobenius theory, then it is necessarily almost periodic. Hence, by Theorem , all cycles of size $n = 2k$, $k \geq 2$, are almost periodic.

In the next result, we observe that the complement of all cycles does not possess PGST. This provides another class of circulant graphs that do not allow PGST.

Corollary 3.59. *[118] Let $m \in \mathbb{N}$ with $m \neq 2$ such that $n = mp$ for some odd prime p. Then, the complement of the cycle C_n does not exhibit Pretty Good State Transfer (PGST).*

Pal et al. in the following theorem extend Theorem 3.2.1 to obtain more circulant graphs (apart from the cycles) on $2k$ vertices exhibiting PGST.

Theorem 3.60. *[119] Let $k \in \mathbb{N}$ and $n = 2^k$. Also let $\mathrm{Cay}(\mathbb{Z}_n, S)$ be a non-integral circulant graph. Let d be the least among all the divisors of n so that $S \cap S_n(d)$ is a non-empty proper subset of $S_n(d)$. If $|S \cap S_n(d)| \equiv 2 \pmod 4$ then $\mathrm{Cay}(\mathbb{Z}_n, S)$ admits PGST with respect to a sequence in $2\pi\mathbb{Z}$. Moreover, if $|S \cap S_n(d)| \equiv 0 \pmod 4$ then $\mathrm{Cay}(\mathbb{Z}_n, S)$ is almost periodic with respect to a sequence in $2\pi\mathbb{Z}$.*

Pal et al. illustrate Theorem 3.60 by the following example:

Example 3.61. *[119] Let $n = 16$ and consider $\mathrm{Cay}(\mathbb{Z}_{16}, S)$ where $S = \{1,2,3,5,11,13,14,15\}$. Here $S \cap S_{16}(1) = \{1,3,5,11,13,15\} \subsetneq S_{16}(1)$ and $S \cap S_{16}(2) = \{2,14\} \subseteq S_{16}(2)$. In this case, we have $d = 1$. Notice that $|S \cap S_{16}(1)| \equiv 2 \pmod 4$. Hence, by the above Theorem, we conclude that $\mathrm{Cay}(\mathbb{Z}_{16}, S)$ admits PGST concerning a sequence in $2\pi\mathbb{Z}$. Observe that if we consider the $\mathrm{Cay}(\mathbb{Z}_{16}, S')$ with $S' = \{1,2,3,13,14,15\}$, then proceeding as above, we obtain $|S' \cap S_{16}(1)| \equiv 0 \pmod 4$. Hence, by the above Theorem, we find that $\mathrm{Cay}(\mathbb{Z}_{16}, S')$ is almost periodic concerning a sequence in $2\pi\mathbb{Z}$.*

In Corollary 3.59, Pal et al. observe that if $m \in \mathbb{N}$ and $n = mp$, where p is an odd prime, then the Cayley graph $\mathrm{Cay}(\mathbb{Z}_n, \{1, n-1\})$ does not exhibit PGST. Now, Pal et al. [119] show that $\mathrm{Cay}(\mathbb{Z}_n, S)$ does not admit PGST if $p \nmid s$ for all $s \in S$. Hence, Corollary 3.59 becomes a special case of the following Theorem.

Theorem 3.62. *[119] Let $m \in \mathbb{N}$ and $n = mp$, where p is an odd prime. Then the circulant graph$\mathrm{Cay}(\mathbb{Z}_n, S)$ does not exhibit PGST if $p \nmid s$ for all $s \in S$.*

3.2.2 PGST in Cayley graphs on non-abelian graphs

Cao et al. [38] investigated the possibility of $\mathrm{Cay}(D_n, S)$ having PGST, where D_n is a dihedral group and S is a conjugation-closed subset in D_n. They showed that when n is odd, there is no PGST in $\mathrm{Cay}(D_n, S)$, while it has PGST for some connection set S if n is a power of 2. Some concrete constructions are provided. They proved that this is the only case (n is a power of 2) for $\mathrm{Cay}(D_n, S)$ to have PGST except for the so-called *Power two case*.

Theorem 3.63. *[38] Let n be an odd number and S be a conjugation-closed subset of D_n. Let $\Gamma = \mathrm{Cay}(D_n, S)$ be a connected Cayley graph with connection set S. Then Γ cannot have PGST between two distinct vertices.*

Theorem 3.64. *[38] Suppose that $n = 2m = 2^k$, $k \geq 3$, D_n is a dihedral group of order $2n$ and S is a conjugation-closed subset in D_n. Let $S = S_1 \cup S_2$, $S_1 = S \cap \langle a \rangle$ and $S_2 = S \cap b\langle a \rangle$. Then $\mathrm{Cay}(D_n, S)$ admits PGST with respect to a time sequence in $2\pi\mathbb{Z}$ under the following two cases:*

1. *$S_1 = \{a^{\pm k_1}\}$ with the k_1 is an odd integer and $1 \leq k_1 \leq m-1$ and $S_2 = b\langle a \rangle$;*
2. *$S_2 = b\langle a \rangle$ and $S_1 = \{a^{\pm k_1}\} \cup (\cup_{j=1}^{r} \{a^{2^{m_j} l} : \gcd(l, 2) = 1, 1 \leq l \leq 2^{k-m_j} - 1\})$, k_1 ($1 \leq k_1 \leq m-1$) is an odd integer and $1 \leq m_1 < \cdots < m_2 \leq k-1$.*

Suppose that $D_n = \langle a, b \mid a^n = b^2 = 1, bab = a^{-1} \rangle$ is a dihedral group of order $2n$ and S is a conjugation-closed subset in D_n. Let $S_1 = \{a^{\pm k_1}, \ldots, a^{\pm k_r}\} \subset \langle a \rangle$. If for every k_i, $1 \leq i \leq r$, it holds that $n = 2^{v_2(n) - v_2(k_i)} k_i$, then we say that S_1 is of the Power-two case.

Wang and Cao in [41] studied PGST on Cayley graphs $\Gamma = \mathrm{Cay}(SD_{8n}, S)$, where SD_{8n} is a semi-dihedral group of order $8n$ and S is a conjugation-closed and symmetric subset of SD_{8n}. When n is a power of 2, a non-integral Γ may have PGST for some connection sets S. When n is odd, they found that the non-integral Cayley graph Γ cannot exhibit PGST. Moreover, a sufficient and necessary condition of the graph Γ admitting PGST is proposed. Some concrete constructions of Cayley graphs over semi-dihedral groups having PGST are provided as well.

Theorem 3.65. *[41] Let $SD_{8n} = \langle a, b | a^{4n} = b^2 = 1, bab = a^{2n-1} \rangle$ be a semi-dihedral group and $S = S_1 \cup S_2$ a conjugation-closed subset of SD_{8n}. Assume that $n = 2^k$, $k \geq 2$. Then graph $\mathrm{Cay}(SD_{8n}, S)$ exhibits PGST with respect to a sequence in $2\pi z$ under the following two cases:*

1. *$S_2 = b\langle a \rangle$, $S_1 = \{a^{\pm k_1}, a^{\pm k_1(2n-1)}\}$ with k_1 an odd integer and $1 \leq j_1 \leq 2n-1$;*

2. *$S_2 = b\langle a\rangle$ and $S_1 = \{a^{\pm k_1}, a^{\pm k_1(2n-1)}\} \cup \{a^{2^{m_j}l} a^{2^{m_j}l(2n-1)} : 1 \le l \le 2^{k+2-m_j} - 1, \gcd(l,2) = 1\}$, with k_1 $(1 \le k_1 \le 2n-1)$ an odd integer and $1 \le m_1 < \cdots < m_2 \le k+1$.*

Theorem 3.66. *[41] Assume that $n = mp$, where m is even and p is an odd prime number. Let $SD_{8n} = \langle a, b | a^{4n} = b^2 = 1, bab = a^{2n-1}\rangle$ and $S = S_1 \cup S_2$ a conjugation-closed subset of SD_{8n}. Suppose that $S_1 = \left\{a^{\pm k_1}, a^{\pm a_1(2n-1)}, \ldots, a^{\pm k_r}, a^{\pm k_r(2n-1)}\right\}$, $S_2 = b\langle a\rangle$. Then the non-integral graph $\mathrm{Cay}(\mathrm{SD_{8n}}, \mathrm{S})$ cannot exhibit PGST if S_1 is not of the Power-two case.*

Theorem 3.67. *[41] Assume that $n = mp$, where m is odd and p is an odd prime number. Let $SD_{8n} = \langle a, b | a^{4n} = b^2 = 1, bab = a^{2n-1}\rangle$ and $S = S_1 \cup S_2$ a conjugation-closed subset of SD_{8n}. Suppose that $S_1 = \{a^{\pm k_1}, a^{\pm a_1(2n-1)}, \ldots, a^{\pm k_r}, a^{\pm k_r(2n-1)}\}$, $S_2 = b\langle a\rangle$. Then the non-integral graph $\mathrm{Cay}(\mathrm{SD_{8n}}, \mathrm{S})$ cannot have PGST between two distinct vertices.*

Example 3.68. *Let $n = 12$ and $SD_{8n} = \langle a, b \mid a^{4n} = b^2 = 1, bab = a^{2n-1}\rangle$. Choose $S_1 = a^{\pm 3}, a^{\pm 3(2n-1)}$. The spectrum of $\Gamma = \mathrm{Cay}(\mathrm{SD_{8n}}, \mathrm{S})$ is given as follows: $\{52^{(1)}, -44^{(1)}, -4^{(6)}, 4^{(4)}, 2\sqrt{2}^{(12)}, -2\sqrt{2}^{(12)}, 0^{(60)}\}$. Then Γ has PGST at time $t \in 2\pi\mathbb{Z}$.*

4 Application

Potential, Challenges, and the Path Forward in recent years, there has been tremendous progress in the development of quantum computing hardware, algorithms and services leading to the expectation that in the near future, quantum computers will be capable of performing simulations for natural science applications, operations research, and machine learning at scales mostly inaccessible to classical computers. Whereas the impact of quantum computing has already started to be recognized in fields such as cryptanalysis, natural science simulations, and optimization among others, very little is known about the full potential of quantum computing simulations and machine learning in the realm of healthcare and life science (HCLS).

Applications of quantum field theory to finance, pure mathematics and so on show that quantum field theory is a discipline that spans many domains of knowledge that go beyond quantum physics. The utility and facility offered by quantum field theory in the study of diverse disciplines from the natural to social sciences points to the fact that quantum field theory should rightly be considered to be a subject of quantum mathematics. Quantum mathematics is based on state space, operators, Lagrangians, Hamiltonians and Feynman path integrals and encompasses systems with both finite and infinitely many coupled degrees of freedom. The term *quantum* is retained in referring to quantum mathematics, and in particular, the term *quantum field* is retained in applications outside quantum physics so that there is continuity in the terminology being used which in turn facilitates the transfer of results and structures from quantum physics to other disciplines, and vice-versa.

Quantum State Transfer (QST) is a crucial protocol in quantum communication and computation, enabling the transmission of quantum information between computational components. This process has contributed to various engineering fields, including physics, mathematics, chemistry, computer science, materials science, and more, while also advancing classical computing. Quantum computing can enhance algorithms and discover structures and patterns efficiently, benefiting computer science, cryptography, and other domains.

Technology Research in quantum algorithms has addressed numerous questions in computer science, and it plays a significant role in information technology security. Encryption and mathematical algorithms, such as factoring large numbers using the RSA technique, rely on quantum computing. These algorithms are currently challenging to break with conventional computers, taking a considerable amount of time. Moreover, quantum computing can create new cryptographic protocols resistant to quantum attacks, like secure key distribution over quantum channels, offering advantages over classical channels in detecting eavesdropping.

Another potential application of the engineered state transfer protocol is the generation of entanglement. The generation of bipartite and multipartite entanglement is of crucial importance for many quantum information tasks, ranging from quantum teleportation [29] to quantum error correction codes. Entanglement is generated by realizing the phenomenon of fractional revival in the spin chain, where the wavefunction, initially localized on one site of the chain, is found after some time perfectly splitted between the initial and the target site. Fractional revivals have been intensively studied in molecular [147] and atomic [19,122,156] systems and can be related to the Talbot effect [31] as well as to pattern formations of the spatial wavefunction. Quantum computing offers promising solutions for cryptanalysis and enhancing the security and efficiency of cryptographic systems. It can solve optimization problems exponentially faster than classical computers, benefiting areas such as quantum semi-definite programming, quantum data fitting, and quantum combinatorial optimization. In aerospace engineering, quantum computing can tackle computational challenges and improve performance over classical algorithms.

The quantum internet is a vision for a global-scale quantum network that would allow for quantum communication and quantum computing on a large scale. It would be based on the development of quantum communication technologies, such as quantum key distribution, as well as the development of quantum computers and quantum sensors. The quantum internet would have applications in areas such as secure communication, distributed quantum computing, and quantum sensing. With the advent of quantum computers, there has been an enormous interest in merging quantum computing with ML, leading to the thriving field of Quantum Machine Learning (QML) [32,44,45,95,131]. Rapid progress has been made in this field, largely fueled by the hope that QML may provide a quantum advantage in the near-term for some practically-relevant problems. While the prospects for such a practical quantum advantage remain unclear [132], a num ber of promising analytical results have already been put forward [1,13,85,93]. Still, much remains to be known about QML models. Quantum computing has shown a potential to improve machine learning model accuracy, with quantum neural networks demonstrating reduced training time and improved accuracy in various datasets. However, further development of quantum hardware is necessary for quantum machine learning to reach its full potential. Cryptography is defined as art of writing and solving codes. Conventional cryptography is based on algorithms and mathematical problems. If someone finds an efficient way to crack the algorithm, the information is no longer secure. Here is where the beauty of entanglement lies. Utilizing the method of quantum key distribution, a random secret key is established for cryptography [2]. Any attempt of manipulating will alter the state and can be detected making the communication secure without complex procedures and algorithms [70]. Although, it is very difficult to have realization of this process but cleverly designed systems are used to generate entangled photos such that one can be stored in the memory and other can travel through the fiber [126] ensuring the way for miracle process. Teleportation involves the transferring of quantum state of a particle to another particle over a distance i.e transmission of quantum information [121]. It is possible only due to the entanglement property of quantum particles. Quantum teleportation can serve as an elementary operation in quantum computers and basic ingredient in distributed quantum networks. First it was demonstrated as a transfer of quantum state of light onto another light beam [137] later developments used optical rays [128] and material particles [127].

Biological Challenges in healthcare and life sciences present opportunities to leverage the unique features of quantum computing to derive novel biological insights to improve patient care. New technologies have made it possible to create detailed maps of human cells, tissues, and organs, leading to advances in understanding diseases such as cancer [123,141,148], cardiovascular disease [113], and diabetes [134]. For example, artificial intelligence (AI) (herein defined as intelligent software automating routine labor, understanding and/or recognizing images, text patterns, etc.) and machine learning machine learning (ML) (herein defined as the set of algorithms and the mathematical and statistical methods allowing the computer to learn from data) have accelerated discovery in healthcare and life sciences by providing data-driven solutions. One prominent example of the data-driven solutions provided by AI is in the field of structural biology, where the longstanding problem of predicting the three-dimensional (3D) structure of a protein given a sequence [62] has seen significant improvement via transformer-based architectures [92].This work has had a profound effect on the field of structural biology by showcasing the potential for using data-driven approaches based on ML methods to solve scientific problems. AI/ML has improved protein structure prediction, generated large protein complexes, and designed de novo proteins and enzymes [12,35,60,86,107,136,151,154,155]. ML has been used to predict the effects of noncoding variants, diagnose diseases, and predict disease outcomes. However, AI has limitations that hinder its application to the clinic, including the complexity of biological systems, shortcomings of AI algorithms, and limitations of data availability. Despite these challenges, researchers continue to explore the potential of AI and ML in healthcare and life sciences. Quantum computing enables researchers to overcome existing limitations by tackling complex problems in areas such as biomarker discovery, clinical trial optimization, imaging analysis, and drug protein design and discovery [7,59,139]. Quantum computing through protein structure prediction, molecular docking and quantum simulation and quantum-classical combined techniques has significant potential in pharmaceutics and drug discovery, it also speeds up the drug development process and reduces costs [22, 43, 64, 66, 73, 94]. Scientists a vision for *Quantum Enabled Cell − Centric Therapeutics*, which aims to leverage advancements in single cell and spatial single-cell technologies to create a holistic view of cellular and metabolic activities in disease tissue. Cell therapies are a new approach to medicine, where human cells are reprogrammed to perform specific functions, such as killing cancer cells. This technology has the potential to treat various diseases, including autoimmune disorders, inflammation, and neurodegeneration. The scientis suggest that Quantum Neural Networks (QNNs) could provide advancements in this problem domain. Adding to the power of classical neural network models, QNNs utilize quantum mechanical effects such as superposition, entanglement, and interference to represent complex relations among data. As such, certain QNN architectures have been shown to have greater expressivity than some of their classical counterparts, allowing them to capture more complex probability distributions than classical models.

In particular, Quantum Convolutional Neural Networks (QCNNs) can be used to improve the accuracy of predicting CAR T cell phenotype [51]. QCNNs have several useful properties, including the number of variational parameters that scale logarithmically with the number of qubits and the absence of barren plateaus during training that can affect other types of QNNs [124]. scientis also emphasize the importance of collaboration between quantum and HCLS (healthcare and life science) researchers to develop new biology-inspired quantum algorithms and proof-of-concepts.

In general quantum computing enables medical practitioners to model atomic-level molecular interactions, which is necessary for medical research [110]. This will be particularly essential for diagnosis, treatment, drug discovery, and analytics. Due to the advancements in quantum computing, it is now possible to encode tens of thousands of proteins and simulate their interactions with drugs, which has not been possible before [83]. Quantum computing helps process this information more effectively by orders of magnitude as compared with conventional computing capabilities [111]. Quantum computing allows doctors to simultaneously compare large collections of data and their permutations to identify the best patterns. Detection of biomarkers specific to a disease in the blood is now possible through gold nanoparticles by using known methods, such as bio-barcode assay. In this situation, the goal could be to exploit the comparisons used to help the identification of a diagnosis [72].

Chemistry A central task of quantum theory, in the fields of quantum condensed matter physics and quantum chemistry, consists of describing the properties of interacting atomic, molecular, amorphous, and crystalline systems. Apart from advancing our fundamental understanding of quantum mechanics, a generic solution to the quantum problem would allow progress in application fields ranging from materials discovery (better magnets, solar cells, catalysts, or qubit hardware) to drug design. The required operation count and the number of quantum degrees of freedom required for the accurate ab initio description of such systems puts them far outside of the accessible regime on current or near-term quantum hardware, except for very small systems. These requirements were elucidated by many authors includ-

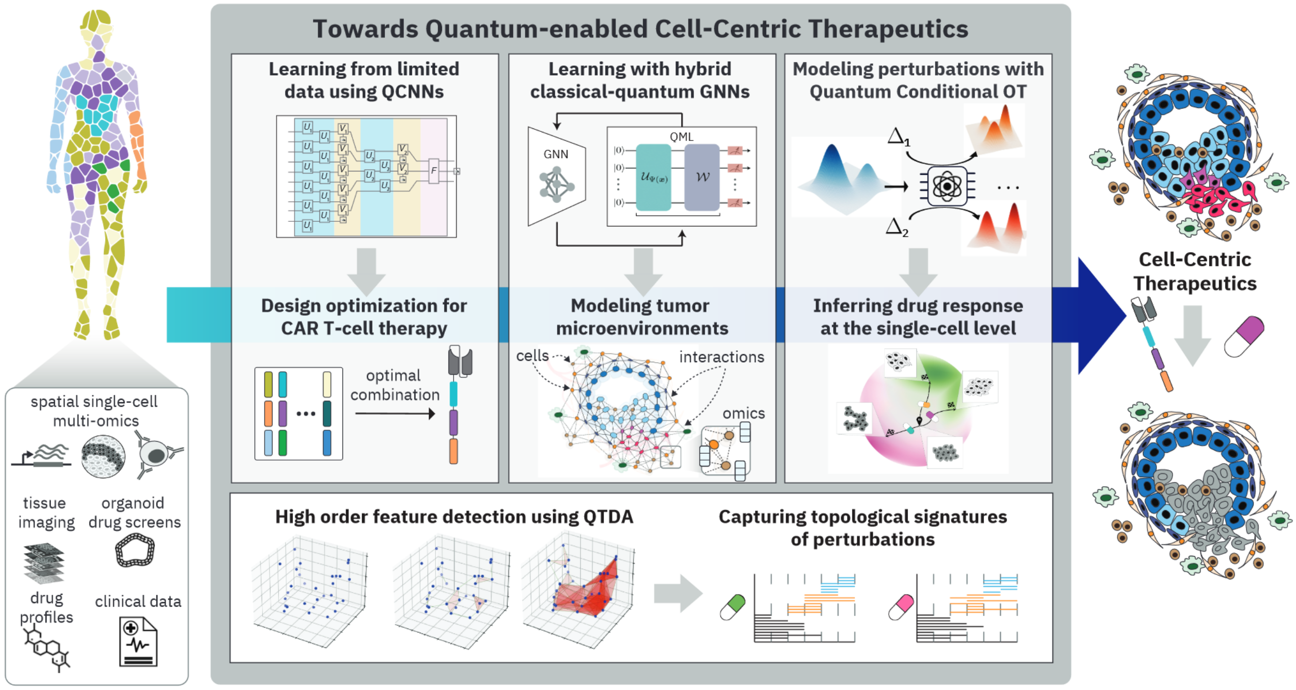


Figure 4. [28] Spatiotemporal single-cell, cell-line, imaging, drug profile, and clinical data are analyzed with four quantum computing technologies to capture varying aspects of cellular behavior. These technologies include: (top left) QCNNs to learn optimal CAR T-cell intracellular signalling domain design from limited experiment data; (center) hybrid classical-quantum GNNs to model tumor microenvironments from single cell spatial data; (top right) single cell perturbation response using Quantum Conditional OT; and (bottom) QTDA to identify topological signatures of single cell perturbation response.

ing Reiher et al. [129], Tubman et al. [143], Elfving et al. [65], and Goings et al. [82], who considered the application of quantum phase estimation to challenging instances. In QC/QI, quantum simulation of electronic structure problems of atoms and molecules is one of the most intensively studied realms.

Applications of quantum computing in quantum chemistry and material science involve solving problems related to the ground state energies of electrons and their wave functions. These simulations can lead to drug discovery, improving battery power and life in electric vehicles, and solving travel-related problems like the traveling salesman problem. To overcome this challenge, researchers are developing quantum algorithms that can simulate the electronic structure of atoms and molecules. One promising approach is the Variational Quantum Eigensolver (VQE) algorithm, which has been used to simulate the properties of various molecules.

VQE has been used to simulate the properties of various molecules, including water, methane, and large organic molecules. To perform quantum simulations of atoms and molecules on quantum computers, information on electronic wave functions should be mapped onto quantum registers. Several approaches for wave function mapping were proposed [18, 133, 135, 142], and the most fundamental one is direct mapping (DM) [18]. Obtaining accurate vibrational spectra of molecules is a costly task on conventional computers. While uncovering the electronic structure of molecules stands as a fundamental challenge in quantum chemistry and material design, to truly make an impact in both scientific research and practical applications, it is vital to go beyond the electronic structure. This requires creating a kinetic model that relies on a deep understanding of a molecule's vibrational structure. Knowing a molecule's vibrational structure enables the prediction of thermodynamic properties that are key in many fields, such as atmospheric science, catalysis, and fuel combustion modeling. Although classical computers often handle simulation of the electronic structure of small molecules reasonably well, they struggle with calculating vibrational structures beyond the harmonic approximation, even for small molecules. Accurately predicting the vibrational spectra of molecules is a challenging task that requires advanced computational methods, particularly when dealing with higher-order terms and complex interactions between bosonic modes.

The development of quantum algorithms for simulating excited states of quantum systems is crucial for understanding the behavior of many physical and chemical systems, including the electronic and magnetic properties of materials. Several algorithms have been developed for excited state simulations, including quantum subspace expansion (QSE), quantum equation-of-motion (qEOM), and quantum Lanczos algorithms.

Finance Quantum computing offers valuable tools and solutions in finance, improving tasks such as risk assessment, portfolio optimization, and derivative pricing. It has been applied to develop financial models like churn prediction and credit risk assessment, outperforming traditional methods in some cases. In article of Baaquie [20], there are two main points about this: first, defining and analyzing the subject of quantitative finance in the conceptual and mathematical framework of quantum theory, with special emphasis on its path-integral formulation, and, second, the introduction of the techniques and methodology of quantum field theory in the study of interest rates. No attempt is made to apply quantum theory in reworking the fundamental principles of finance. Instead, the term quantum refers to the abstract mathematical constructs of quantum theory that include probability theory, state space, operators, Hamiltonians, commutation equations, lagrangians, path integrals, quantized fields, bosons, fermions and so on. All these theoretical structures find natural and useful applications in finance. The path integral and Hamiltonian formulations of (random) quantum processes have been given special emphasis since they are equivalent to, as well as independent of, the formalism of stochastic calculus which currently is one of the cornerstones of mathematical finance. The starting point for the application of path integrals and Hamiltonians in finance is in stock option pricing. Path integrals are subsequently applied to the modelling of linear and nonlinear theories of interest rates as a two-dimensional quantum field, something that is beyond the scope of stochastic calculus.

In a review article [112] Orus et al., explore the potential applications of quantum computation in the field of finance. They review insights into current approaches and future prospects for utilizing quantum optimization algorithms, including quantum annealers, for tasks such as portfolio optimization, identifying arbitrage opportunities, and credit scoring. Additionally, they delve into the intersection of deep learning and finance, highlighting the potential for enhanced performance through quantum machine learning techniques. Furthermore, they examine the benefits of quantum amplitude estimation in speeding up Monte Carlo sampling, which can have significant implications for financial tasks such as derivative pricing and risk analysis. Overall, they shed light on the exciting possibilities that quantum computation offers for revolutionizing financial methods and decision-making processes.

5 Conclusion

In this article, we have reviewed the concepts of perfect state transfer and pretty good state transfer, examining the necessary and sufficient conditions for Cayley graphs. We have also demonstrated how a solid understanding of state transfer forms the foundation of a constructive technique with broad applicability, highlighting the significance of this protocol.

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References

[1] A. Abbas, D. Sutter, C. Zoufal, A. Lucchi, A. Figalli, S. Woerner, The power of quantum neural networks, Nature Computational Science 1 (2021) 403–409. https://doi.org/10.1038/s43588-021-00084-1

[2] A. A. Abushgra, Variations of QKD protocols based on conventional system measurements: A literature review, Cryptography 6 (2022). https://doi.org/10.3390/cryptography6010012

[3] E. Ackelsberg, Z. Berhm, A. Chan, J. Mundinger, C. Tamon, Laplacian state transfer in coronas, Linear Algebra and its Application 506 (2016) 154–167.https://doi.org/10.1016/j.laa.2016.05.018

[4] A. Acuaviva, A. Chan, S. Eldridge, C. Godsil, M. How-chun-Lun, C. Tamon, E. Wright, X. Zhang, State transfer in complex quantum walks, Int. J. of Quantum Information 7(8) (2023) 1417–1427.https://doi.org/10.48550/arXiv.2301.01473

[5] Y. Aharonov, L. Davidovich, N. Zagury, Quantum random walks, Physical Review A 48 (1993). https://doi.org/10.1103/physreva.48.1687

[6] K. Akash, B. Bhattacharjya, Perfect state transfer on Cayley graphs over a group of order 8n, arXiv preprint arXiv:2405.02122 (2024). https://doi.org/10.48550/arXiv.2405.02122

[7] E. Altman, K. R. Brown, G. Carleo, L. D. Carr, E. Demler, C. Chin, B. DeMarco, Sophia E Economou, Mark A Eriksson, Kai-Mei C Fu, et al., Quantum simulators: Architectures and opportunities, PRX Quantum 2 (2021). https://doi.org/10.1103/prxquantum.2.017003

[8] T. M. Apostol, Modular Functions and Dirichlet series in Number Theory, second ed., Springer-Verlag-New work, 1997.https://doi.org/10.1007/978-1-4612-0999-7

[9] A. Ambainis, E. Bach, A. Nayak, A. Vishwanath, and J. Watrous. One-dimensional quantum walks, In Proceedings of the 33rd ACM Symposium on The Theory of Computation (STOC 01) ACM, (2001) 37–49. https://doi.org/10.1145/380752.380757

[10] R. J. Angeles-Canul, R. Norton, M. Opperman, C. Paribello, M. Russell, C. Tamon, On quantum perfect state transfer in weighted join graphs, International Journal of Quantum Information 7 (2009) 1429–1445. https://doi.org/10.1142/s0219749909006103

[11] R. J. Angeles-Canul, R. Norton, M. Opperman, C. Paribello, M. Russell, C. Tamam, Perfect state transfer, integral circulants and join of graphs, Quantum Information and Computation, 10 (2010), 325–342. https://doi.org/10.26421/qic10.3-4-10

[12] V. Anishchenko, S. J. Pellock, T. M. Chidyausiku, T. A. Ramelot, S. Ovchinnikov, J. Hao, Khushboo Bafna, C. Norn, A. Kang, A. K. Bera, et al., De novo protein design by deep network hallucination, Nature 600 (2021), 547–552. https://doi.org/10.1038/s41586-021-04184-w

[13] E. R. Anschuetz, H.-Y. Hu, J.L. Huang, and X. Gao, Interpretable quantum advantage in neural sequence learning, arXiv preprint arXiv:2209.14353, (2022). https://doi.org/10.1103/prxquantum.4.020338

[14] T. Anuradha, A. Patra, R. Gupta, A. Sende, Perfect transfer of arbitrary continuous variable state across optical waveguide lattices, (2023) 2306–13068. https://doi.org/10.48550/arXiv.2306.13068

[15] M. Arezoomand, B. Taeri, On the characteristic polynomial of n-Cayley digraphs, Electronic Journal of Combinatorics 20 (2013). https://doi.org/10.37236/3105

[16] M. Arezoomand, F. Shafiei, M. Ghorbani, Perfect state transfer on Cayley graphs

over dicyclic groups, Linear Algebra and its Applications 639 (2022) 116–134. https://doi.org/10.1016/j.laa.2021.12.019
[17] M. Arezoomand, Perfect state transfer on semi-Cayley graphs over abelian groups, Linear and Multilinear Algebra 71 (2022) 2337–2353. https://doi.org/10.1080/03081087.2022.2101602
[18] A. Aspuru-Guzik, A. D. Dutoi, P. J. Love, M. Head-Gordon, Simulated quantum computation of molecular energies, Science 309 (2005) 1704–1707. https://doi.org/10.48550/arXiv.quant-ph/0604193
[19] I. Sh. Averbukh, N. Perelman, Fractional revivals: Universality in the longterm evolution of quantum wave packets beyond the correspondence principle dynamics. Physics Letters A 139 (1989) 449–453. https://doi.org/10.1016/0375-9601(89)90943-2
[20] B. Baaquie, Quantum finance: Path integrals and Hamiltonians for options and interest rates, Cambridge University Press (2007). https://api.DT0400.9780511262265_A23690789
[21] R. Bachman, E. Fredette, J. Fuller, M. Landry, M. C. Opperman, C. Tamon, A. Tollefson, perfect state transfer on quotient graphs, Quantum information and computation 12 (2012) 293–313. https://doi.org/10.26421/qic12.3-4-9
[22] A. Baiardi, M. Christandl, M. Reiher, Quantum Computing for Molecular Biology, ChemBioChem 24 (2023) e202300120. https://doi.org/10.48550/arXiv.2212.12220
[23] L. Banchi, G. Coutinho, C. Godsil, S. Severini, Pretty good state transfer in qubit chains - the Heisenberg Hamiltonian, Journal of Mathematical Physics 58 (2017) 032202. https://doi.org/10.1063/1.4978327
[24] M. Bašić, M. Petković, D. Stefanović, Perfect State transfer in integral circulant graphs, Applied Mathematics Letters 22 (2009), 1117–1121. https://doi.org/10.1016/j.aml.2008.11.005
[25] M. Bašić, M. Petković, Some classes of integral circulant graphs either allowing or not allowing Perfect state transfer, Applied Mathematics Letters 22 (2009) 1609–1615. https://doi.org/10.1016/j.aml.2009.04.007
[26] M. Bašić, M. Petković, Perfect state transfer in integral circulant graphs of non-square-free order, Linear Algebra and its Application 433 (2010) 149–163. https://doi.org/10.1016/j.laa.2010.01.042
[27] M. Bašić, Characterization of quantum circulant networks having perfect state transfer, Quantum Information Processing 12 (2011) 345–364. https://doi.org/10.48550/arXiv.1104.1825
[28] S. Basu, J. Born, A. Bose, S. Capponi, D. Chalkia, T. A. Chan, H. Doga, M. Goldsmith, T. Gujarati, A. Guzman-Saenz, et al., Towards quantum-enabled cell-centric therapeutics, arXiv preprint arXiv 2307.05734, (2023). https://doi.org/10.48550/arXiv.2307.05734
[29] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, W. K. Wootters, Teleporting an unknown quantum state via dual classical and einstein-Podolsky-Rosen channels, Physical Review Letters 70 (1993). https://doi.org/10.1103/PhysRevLett.70.1895
[30] A. Bernasconi, C. Godsil, S. Severini, Quantum networks on cubelike graphs, Physical Review A 78 (2008). https://doi.org/10.1103/PhysRevA.78.052320
[31] M. Berry, I. Marzoli, W. Schleich, Quantum carpets, Carpets of light, Physics World 14 (2001) 39. https://doi.org/10.1088/2058-7058/14/6/30
[32] J. Biamonte, P. Wittek, N. Pancotti, P. Rebentrost, N. Wiebe, and S. Lloyd, Quantum machine learning, Nature 549 (2017). https://doi.org/10.1038/nature23474
[33] S. Bose, Quantum communication through an unmodulated spin chain, Physical Review Letters 91 (2003). https://doi.org/10.1103/PhysRevLett.91.207901
[34] C. Bommel, Pretty good state transfer and minimal polynomials, arxiv preprint: 2010, (2020). https://doi.org/10.48550/arXiv.2010.06779
[35] P. Bryant, G. Pozzati, W. Zhu, A. Shenoy, P. Kundrotas, A. Elofsson, Predicting the structure of large protein complexes using alphafold and Monte Carlo tree search, Nature Communications, 13 (2022). https://doi.org/10.1038/s41467-022-33729-4
[36] S. Cameron, S. Fehrenbach, L. Granger, O. Hennigh, S. Shrestha, C. Tamon, Universal state transfer on graphs, Linear Algebra and its Applications 455 (2014) 115–142. https://doi.org/10.1016/j.laa.2014.05.004
[37] X. Cao, B. Chen, S. Ling, Perfect state transfer on Cayley graph over dihedral groups: the non-normal case, Electronic Journal of Combinatorics 27 (2020). https://doi.org/10.37236/9184
[38] X. Cao, K. Feng, Pretty good state transfer on Cayley graph over dihedral groups, Discrete Math-

ematics, 343 (2020). https://doi.org/10.1016/j.disc.2019.111636
[39] X. Cao, K. Feng, Perfect state transfer on Cayley graphs over dihedral groups, Linear and Multilinear Algebra 69 (2021) 343–360. https://doi.org/10.1080/03081087.2019.1599805
[40] X. Coa, K. Feng, Y-Y. Tan, Perfect state transfer on weighted abelian Cayley graphs, Chinese Annals of Mathematics, Series B 42 (2027) 625–642. https://doi.org/10.1007/s11401-021-0283-4
[41] X. Cao, D. Wang, Pretty good state transfer on Cayley graphs over semi-dihedral groups, Linear and Multilinear Algebra 70 (2022) 5716–5731. https://doi.org/10.1080/03081087.2021.1926414
[42] A. Casaccino, S. Lioyd, S. Mancini, S. Severini, Quantum state transfer through a qubit network with energy shifts and fluctuations, International Journal of Quantum information 7 (2009) 1417–1427. https://doi.org/10.48550/arXiv.0904.4510
[43] P. A. Casares, R. Campos, M. A. Martin-Delgado, QFold: quantum walks and deep learning to solve protein folding, Quantum Science and Technology (2021). https://doi.org/10.1088/2058-9565/ac4f2f
[44] M. Cerezo, A. Arrasmith, R. Babbush, S. C. Benjamin, S. Endo, K. Fujii, J. R. McClean, K. Mitarai, X. Yuan, L. Cincio, and P. J. Coles, Variational quantum algorithms, Nature Reviews Physics 3 (2021) 625–644. https://doi.org/10.1038/s42254-021-00348-9
[45] M. Cerezo, G. Verdon, H.-Y. Huang, L. Cincio, and P. J. Coles, Challenges and opportunities in quantum machine learning, Nature Computational Science (2022) 567–576. https://doi.org/10.1038/s43588-022-00311-3
[46] A. Chan, W. Drazen, O. Eisenberg, M. Kempton, G. Lippner, Pretty good quantum fractional revival in paths and cycles, Algebraic Combinatorics 4 (2021) 989–1004. https://doi.org/10.5802/alco.189
[47] A. Chan, H. Zhan, Pretty good state transfer in discrete-time quantum walks, Journal of Physics A: Mathematical and Theoretical 56 (2023) 1751–8121. https://doi.org/10.1088/1751-8121/acc4f5
[48] W. Cheung, C. Godsil, Perfect state transfer in cubelike graphs, Linear Algebra and its Applications 435 (2011) 2468–2474. https://doi.org/10.1016/j.laa.2011.04.022
[49] M. Christandl, N. Datta, A. Ekeyt, A. Landahl, Perfect state transfer in quantum spin networks, Physical Review letters 92 (2004). https://doi.org/10.1103/PhysRevLett.92.187902
[50] M. Christandl, N. Datta, T. Dorlas, A. Ekert, A. Kay, A. Landahl, Perfect transfer of arbitrary states in quantum spin networks, Physical Review A 71 (2005). https://doi.org/10.1103/PhysRevA.71.032312
[51] I. Cong, S. Choi, M. D. Lukin, Quantum convolutional neural networks, Nature Physics 15 (2019) 1273–1278. https://doi.org/10.1038/s41567-019-0648-8
[52] E. Connelly, N. Grammel, M. Kraut, L. Serazo, C. Tamon, Universality in perfect state transfer, Linear Algebra and its Applications 531 (2017) 576–532. https://doi.org/10.1016/j.laa.2017.06.015
[53] G. Coutinho, Quantum state transfer in graphs, PhD dissertation, University of Waterloo, 2014. https://208a19-a827-448b-bbc3-b12116d3eada/content
[54] G. Coutinho, C. Godsil, K. Guo, F. Vanhove, Perfect state transfer On distance regular graphs and association schemes, Linear Algebra and its Applications 478 (2015) 108–130. https://doi.org/10.1016/j.laa.2015.03.024
[55] G. Coutinho, H. Liu, No Laplacian perfect state transfer in trees, SIAM Journal on Discrete Mathematics 29 (2015) 2179–2188. https://doi.org/10.1137/140989510
[56] G. Coutinho, C. Godsil, Perfect state transfer in products and covers of graphs, Linear and Multi Linear Algebra 64 (2016) 235–246. https://doi.org/10.1080/03081087.2015.1033381
[57] G. Coutinho, K. Guo, C. Bommel, Pretty good state transfer between internal nodes of paths, Quantum information and computation 17 (2017) 825–830. https://doi.org/10.48550/arXiv.1611.09836
[58] G. Coutinho, E. Juliano, T. J. Spier, No perfect state transfer in trees with more than 3 vertices, arxiv preprint arxiv:2305-10199 (2023). https://doi.org/10.48550/arXiv.2305.10199
[59] A. J. Daley, I. Bloch, C. Kokail, S. Flannigan, N. Pearson, M. Troyer, P. Zoller, Practical quantum advantage in quantum simulation, Nature 607 (2022) 667–676. https://doi.org/10.1038/s41586-022-04940-6
[60] J. Dauparas, I. Anishchenko, N. Bennett, H. Bai, R. J. Ragotte, L. F. Milles, B. I. M. Wicky, A. Courbet, R. J. deHaas, N. Bethel, et al., Robust deep learning-based protein sequence design using proteinmpnn, Science 378 (2022) 49–56. https://doi.org/10.1126/science.add2187

[61] P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions, Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete 57 (1981) 159–179. https://doi.org/10.1007/bf00535487
[62] K. A. Dill, J. L. MacCallum, The protein-folding problem, 50 years on, Science 338 (2012) 1042–1046. doi:10.1126/science.1219021. https://doi.org/10.1126/science.1219021
[63] O. Eisenberg, M. Kempton, G. Lippner, Pretty good quantum state transfer in asymmetric graphs via potential, Discrete Mathematics 342 (2019) 2821–2833. https://doi.org/10.1016/j.disc.2018.10.037
[64] A. Elaraby, Quantum medical images processing foundations and applications, IET Quantum Communication, 3 (2022) 201–213. https://doi.org/10.1049/qtc2.12049
[65] V. E. Elfving, B. W. Broer, M. Webber, J. Gavartin, M. D. Halls, K. P. Lorton, A. Bochevarov, How will quantum computers provide an industrially relevant computational advantage in quantum chemistry, arXiv preprint arXiv:2009.12472, (2020). https://doi.org/10.48550/arXiv.2009.12472
[66] P. S. Emani, J. Warrell, A. Anticevic, et al., Quantum computing at the frontiers of biological sciences, Nature Methods 18 (2021) 701–709. https://doi.org/10.48550/arXiv.1911.07127
[67] C. Facer, J. Twamley, J. Cresser, Quantum Cayley networks of the hypercube, Physical Review A (3) 77 (2008). https://doi.org/10.1103/PhysRevA.77.012334
[68] E. Farhi, S. Gutmann, Quantum computation and decision trees, Physical Review A (3) 58 (1998) 915–928. https://doi.org/10.48550/arXiv.quant-ph/9706062
[69] X. Fan, C. Godsil, Pretty good state transfer on double stars, Linear Algebra and its Applications 438 (2013) 2346–2358. https://doi.org/10.1016/j.laa.2012.10.006
[70] M. F. S. Ferreira, A. N. Pinto, H. Hübel, Quantum communications, Fiber and Integrated Optics (2020) 1–2. https://doi.org/10.1080/01468030.2020.1712536
[71] R. P. Feynman, Simulating physics with computers, International Journal of Theoretical Physics 21 (1982) 467–488. https://doi.org/10.1007/BF02650179
[72] F. F. Flother, The state of quantum computing applications in health and medicine. (2023). https://doi.org/10.48550/arXiv.2301.09106
[73] F. F. Flother, P. F. Griffin, How can quantum technologies be applied in healthcare, medicine and the life sciences, Research Directions: Quantum Technologies 1 (2023). https://doi.org/10.1017/qut.2023.1
[74] Y. Ge, B. Greenberg, O. Perez, C. Tamon, Perfect state transfer, graph products and equitable partitions, Quantum physics (2010). https://doi.org/10.1142/S0219749911007472
[75] C. Godsil, Periodic graphs, The Electronic Journal of Combinatorics 18 (2011). https://doi.org/10.37236/510
[76] C. Godsil, State transfer on graphs, Discrete Mathematics 312 (2012) 129–147. https://doi.org/10.1016/j.disc.2011.06.032
[77] C. Godsil, When can perfect state transfer occur, Electronic Journal of Linear Algebra 23 (2012) 877–890. https://doi.org/10.13001/1081-3810.1563
[78] C. Godsil, Real state transfer, (2017), arXiv:1710:04042. https://doi.org/10.48550/arXiv.1710.04042
[79] C. Godsil, S. Kirkland, S. Severini, J. Smith, Number-theoretic Nature of communication in Quantum Spin Systems, Physical Review Letters 109 (2012). https://doi.org/10.1103/PhysRevLett.109.050502
[80] C. Godsil, K. Guo, M. Kempton, G. Lippner, F. Munch, State transfer in strongly regular graphs with an edge perturbation, Journal of Combinatorial Theory, Series A 172 (2020). https://doi.org/10.1016/j.jcta.2019.105181
[81] C. Godsil, S. Late, Perfect state transfer on oriented graphs, Linear Algebra and its Applications 604 (2020) 278-292.
https://doi.org/10.1016/j.laa.2020.06.025
[82] J. J. Goings, A. White, J. Lee, C. S. Tautermann, M. Degroote, C. Gidney, T. Shiozaki, R. Babbush, N. C. Rubin, Reliably assessing the electronic structure of cytochrome p450 on today's classical computers and tomorrow's quantum computers, Proceedings of the National Academy of Sciences 119 (2022). https://doi.org/10.48550/arXiv.2202.01244
[83] S. Gupta, S. Modgil, P. C. Bhatt, C. J. C. Jabbour, S. Kamble, Quantum computing led innovation for achieving a more sustainable COVID-19 healthcare industry, Technovation 120 (2022).

https://doi.org/10.1016/j.technovation.2022.102544

[84] Y. T. Huang, J. D. Lin, H. Y. Ku, Y. N. Chen, Benchmarking quantum state transfer on quantum devices, Quantum Physics (2020). https://doi.org/10.1103/PhysRevResearch.3.023038

[85] H. Y. Huang, R. Kueng, and J. Preskill, Predicting many properties of a quantum system from very few measurements, Nature Physics 16 1050 (2020). https://doi.org/10.1038/s41567-020-0932-7

[86] I. R. Humphreys, J. Pei, M. Baek, A. Krishnakumar, I. Anishchenko, S. Ovchinnikov, J. Zhang, T. J. Ness, S. Banjade, S. R. Bagde, et al., Computed structures of core eukaryotic protein complexes, Science 374 (2021). https://www.science.org/doi/10.1126/science.abm4805

[87] B. Huppert, Character Theory of Finite Groups, De Gruyter Expositions in Mathematics, Walter de Gruyter, Berlin, 25 (1998). https://doi.org/10.1515/9783110809237

[88] M. A. Jafarizadeh, R. Sufiani, Prefect state transfer over distance - regular spin networks, Physical Review A 77 (2008). https://doi.org/10.48550/arXiv.0709.0755

[89] M. A. Jafarizadeh, R. Sufiani, M. Azimi, F. Eghbalifam, perfect state transfer over interacting boson networks associated with group schemes, Quantum information processing 11 (2012) 171-187. https://doi.org/10.1007/s11128-011-0237-y

[90] A. Jamali, Theory of Finite Groups, Mobtakeran Publications-Tehran (2011).

[91] N. Johnston, S. Kirkland, S. Plosker, R. Storey, X. Zhang, Perfect quantum state transfer using Hadamard diagonalizable graphs, Linear Algebra and Its Applications 531 (2017) 375-398. https://doi.org/10.1016/j.laa.2017.05.037

[92] J. Jumper, R. Evans, A. Pritzel, T. Green, M. Figurnov, O. Ronneberger, K. Tunyasuvunakool, R. Bates, A. Žídek, A. Potapenko, et al., Highly accurate protein structure prediction with AlphaFold, Nature 596 (2021) 583-589. https://doi.org/10.1038/s41586-021-03819-2

[93] M. Larocca, N. Ju, D. García-Martín, P. J. Coles, M. Cerezo, Theory of overparametrization in quantum neural networks, Nature Computational Science 3 542 (2023). https://doi.org/10.1038/s43588-023-00467-6

[94] J. Li, M. Alam, C. M. Sha, J. Wang, N. V. Dokholyan and S. Ghosh, Invited: Drug Discovery Approaches using Quantum Machine Learning, 58th ACM/IEEE Design Automation Conference (DAC), San Francisco, CA, USA, (2021),1356-1359, doi: 10.1109/DAC18074.2021.9586268. https://doi.org/10.1109/DAC18074.2021.9586268

[95] J. Liu, F. Tacchino, J. R. Glick, L. Jiang, A. Mezzacapo, Representation learning via quantum neural tangent kernels, PRX Quantum (2022). https://doi.org/10.48550/arXiv.2111.04225

[96] A. Kay, Perfect state transfer: Beyond nearest-neighbor couplings, Physical Review A 73 (2006). https://doi.org/10.48550/arXiv.quant-ph/0509065

[97] A. Kay, Basics of communication through quantum networks, Physical Review A 84 (2011). https://doi.org/10.48550/arXiv.1102.2338

[98] A. Kay, A Review of perfect, efficient, state transfer and its application as a constructive tool, Quantum Information 8 (2010) 641-676. https://doi.org/10.48550/arXiv.0903.4274

[99] J.Kempe. Quantum random walk - an introductory overview, Contemporary Physics 44 (2003). https://doi.org/10.48550/arXiv.quant-ph/0303081

[100] M. Kempton, G. Lippner, and S.-T. Yau, Pretty good quantum state transfer in symmetric spin networks via magnetic field, Quantum Information Processing 16 (23) (2017). https://doi.org/10.1007/s11128-017-1658-z

[101] V. Kendom, C. Tamon, perfect state transfer in quantum walks on graphs, Computational and Theoretical Nanoscience 8(3) (2011) 422-433. chrome-extension://efaidnbmnnnibpcajpcglclefindmkaj/https://lin-web.clarkson.edu/ ctamon/ps-dir/kt.pdf

[102] A. Khalilipour, M. Ghorbani, perfect state transfer on Cayley graphs over groups, Discrete Mathematics and Its Applications 9 49-62 (2024). https://doi.org/10.22061/jdma.2024.10551.1066

[103] S. Kirkland, Sensitivity analysis of perfect state transfer in quantum spin networks, Linear Algebra and its Applications 472 (2015) 1-30. https://doi.org/10.1016/j.laa.2015.01.013

[104] S. Kubota and E. Segawa, perfect State transfer in Grover walks between state associated to vertices of a graph, Linear Algebra and Its Applications 646 (2022) 238-251. https://doi.org/10.1016/j.laa.2022.04.006

[105] Y. Li, X. Liu, S. Zhang, S. Zhou, perfect State transfer in NEPS of complete graphs, Discrete Applied

Mathematics 289 (2021) 98-114. https://doi.org/10.48550/arXiv.2010.03198
[106] G. Lou, X. Cao, D. Wang, X. Wu, perfect quantum state transfer on Cayley graphs over semi-dihedral groups, Linear and Multilinear Algebra (2021). https://doi.org/10.1080/03081087.2021.1954585
[107] S. Mazurenko, Z. Prokop, J. Damborsky, Machine learning in enzyme engineering, ACS Catalysis 10 (2019) 1210-1223. https://doi.org/10.1021/acscatal.9b04321
[108] G. Mograby, M. Derevyagin, G. Dunne, A. Teplyaev, Spectra of perfect state transfer Hamiltonians on fractal-like graphs, arxiv preprint arxiv: 2003. 11190 (2020). https://doi.org/10.48550/arXiv.2003.11190
[109] H. Miki, S. Tsujimoto, L. Vinet, perfect state transfer in two dimensions and the bivariate dual-Hahn Polynomials, Progress of Theoretical and Experimental Physics 2022 (2022). https://doi.org/10.1093/ptep/ptac069
[110] D. Niraula, J. Jamaluddin, M. M. Matuszak, R. K. T. Haken, I. E. Naqa, Quantum deep reinforcement learning for clinical decision support in oncology: Application to adaptive radiotherapy, Scientific Reports 11 (2021). https://doi.org/10.1038/s41598-021-02910-y
[111] J. Pakela, Quantum Inspired Machine Learning Algorithms for Adaptive Radiotherapy. Ph.D. Thesis, University of Michigan, Ann Arbor, MI, USA, (2021). https://dx.doi.org/10.7302/2999
[112] R. Orús, S. Mugel, E. Lizaso, Quantum computing for finance: Overview and prospects, Reviews in Physics 4 (2019) 2405-4283.
https://doi.org/10.1016/j.revip.2019.100028
[113] D. T. Paik, S. Cho, L. Tian, H. Y. Chang, J. C. Wu, Single-cell RNA sequencing in cardiovascular development, disease and medicine, Nature Reviews Cardiology 17 (2020) 457-473. https://doi.org/10.1038/s41569-020-0359-y
[114] H. Pal, B. Bhattacharjya, perfect state transfer on gcd - graphs, Linear MultiLinear Algebra 65 (2017) 2245 - 2256. https://doi.org/10.1080/03081087.2016.1267105
[115] H. Pal, B. Bhattacharjya, perfect state transfer on NEPS of the Path on three vertices, Discrete Mathematics 339 (2016) 831 - 838. https://doi.org/10.1016/j.disc.2015.10.027
[116] H. Pal, B. Bhattacharjya, Pretty good state transfer on some NEPS, arxiv preprint arxiv: 1604. 08858, (2016). https://doi.org/10.1016/j.disc.2016.11.026
[117] H. Pal, B. Bhattacharjya, A class of gcd - graphs having perfect state transfer, Electron, Notes on Discrete Mathematics 53 (2016) 319-329. https://doi.org/10.48550/arXiv.1601.07398
[118] H. Pal, B. Bhattacharjya, Pretty good state transfer on circulant graphs, Electronic Journal of Combinatorics 24 (2017). https://doi.org/10.37236/6388
[119] H. Pal, More circulant graphs exhibiting Pretty good state transfer, Discrete Mathematics 341 (2018) 889 - 895. https://doi.org/10.1016/j.disc.2017.12.012
[120] H. Pal, Laplacian state transfer on graphs with an edge Perturbation between twin vertices, Discrete Mathematics 345 (2022). https://doi.org/10.48550/arXiv.2109.05306
[121] A. Parakh, Quantum teleportation with one classical bit, Scientific Reports 12 (2022). https://doi.org/10.1038/s41598-022-06853-w
[122] J. Parker, C. Stroud Jr, Coherence and decay of Rydberg wave packets, Physical Review Letters 56 1986 716. https://doi.org/10.1103/PhysRevLett.56.716
[123] A. P. Patel, I. Tirosh, J. J. Trombetta, A. K. Shalek, S. M. Gillespie, H. Wakimoto, D. P. Cahill, B. V. Nahed, W. T. Curry, R. L. Martuza, et al., Single-cell rna-seq highlights intratumoral heterogeneity in primary glioblastoma, Science 344 (2014) 1396-401. https://doi.org/10.1126/science.1254257
[124] A. Pesah, M. Cerezo, S. Wang, T. Volkoff, A. T. Sornborger, P. J. Coles, Absence of barren plateaus in quantum convolutional neural networks, Physical Review X 11 (2021). https://doi.org/10.48550/arXiv.2011.02966
[125] M. Petković, M. Bašić, Further results on the perfect state transfer in integral circulant graphs, Computers and Mathematics with Applications 61 (2011) 300 - 312. https://doi.org/10.1016/j.camwa.2010.11.005
[126] A. N. Pinto, N. A. Silva, Á. J. Almeida and N. J. Muga, Using quantum technologies to improve fiber optic communication systems, IEEE Communications Magazine 51 (2013). https://doi.org/10.1109/MCOM.2013.6576337
[127] M. Riebe, H. Häffner, C. Roos, W. Hänsel, J. Benhelm, G. Lancaster, T. Körber, C. Becher, F. Schmidt-

Kaler, D. James, R. Blatt, Deterministic quantum teleportation with atoms, Nature 429 (2004) 734-737. https://doi.org/10.1038/nature02570
[128] H. de Riedmatten, I. Marcikic, W. Tittel, H. Zbinden, D. Collins, N. Gisin, Long distance quantum teleportation in a quantum relay configuration, Physical Review Letters 92 (2004). https://doi.org/10.1103/PhysRevLett.92.047904
[129] M. Reiher, N. Wiebe, K. M. Svore, D. Wecker, M. Troyer, Elucidating reaction mechanisms on quantum computers, Proceedings of the National Academy of Sciences 114 (2017). https://doi.org/10.48550/arXiv.1605.03590
[130] N. Saxena, S. Severini, I. Shparlinski, Parameters of Integral circulant graphs and Periodic quantum dynamics, International Journal of Quantum Information 5 (2007) 417 - 430. https://doi.org/10.48550/arXiv.quant-ph/0703236
[131] M. Schuld and F. Petruccione, Machine Learning with Quantum Computers, Springer, 2021.
[132] M. Schuld and N. Killoran, Is quantum advantage the right goal for quantum machine learning, PRX Quantum (2022). https://doi.org/10.48550/arXiv.2203.01340
[133] J. T. Seeley, M. J. Richard, P. J. Love, The Bravyi-Kitaev transformation for quantum computation of electronic structure, Chemical physics 137 (2012) 224109. https://doi.org/10.48550/arXiv.1208.5986
[134] A. Segerstolpe, A. Palasantza, P. Eliasson, E. M. Andersson, A. C. Andréasson, X. Sun, S. Picelli, A. Sabirsh, M. Clausen, M. K. Bjursell, et al., Single-cell transcriptome profiling of human pancreatic islets in health and type 2 diabetes, Cell metabolism 24 (2016) 593-607. https://doi.org/10.1016/j.cmet.2016.08.020
[135] K. Setia, J. D. Whitfield, Bravyi-Kitaev superfast simulation of electronic structure on a quantum computer, Journal of Chemical Physics 148 (2018). https://doi.org/10.48550/arXiv.1712.0044
[136] J. E. Shin, A. J. Riesselman, A. W. Kollasch, C. McMahon, E. Simon, C. Sander, A. Manglik, A. C. Kruse, D. S. Marks, Protein design and variant prediction using autoregressive generative models, Nature communications 12 (2021). https://doi.org/10.1038/s41467-021-22732-w
[137] J. F. Sherson, H. Krauter, E. S. Polzik, Quantum teleportation between light and matter, Nature 443 (2006) 557-560. https://doi.org/10.1038/nature05136
[138] B. Steinberg, Representation theory of finite Groups, University, Springer, New York, 2012.
[139] F. Tacchino, A. Chiesa, S. Carretta, D. Gerace, Quantum computers as universal quantum simulators: state-of-the-art and perspectives, Advanced Quantum Technologies, 3 (2020). https://doi.org/10.48550/arXiv.1907.03505
[140] Y-Y. Tan, K. Feng, X-Cao, perfect state transfer on abelian Cayley graphs, Linear Algebra and its Applications, 563 (2019) 331-352. https://doi.org/10.1016/j.laa.2018.11.011
[141] I. Tirosh, B. Izar, S. M. Prakadan, M. H. Wadsworth, D. Treacy, J. J. Trombetta, A. Rotem, C. Rodman, C. Lian, G. Murphy, et al., Dissecting the multicellular ecosystem of metastatic melanoma by single-cell rna-seq, Science 352 (2016) 189-96. https://doi.org/10.1126/science.aad0501
[142] A. Tranter, S. Sofia, J. Seeley, M. Kaicher, J. McClean, R. Babbush, P.V. Coveney, F. Mintert, F. Wilhelm, P.J. Love, The Bravyi-Kitaev transformation: Properties and applications, International Journal of Quantum Chemistry 115 (2015) 1431-1441. https://doi.org/10.1002/qua.24969
[143] N. M. Tubman, C. Mejuto-Zaera, J. M. Epstein, D. Hait, D. S. Levine, W. Huggins, Z. Jiang, J. R. Mc-Clean, R. Babbush, M. Head-Gordon, K. B. Whaley, Postponing the orthogonality catastrophe: efficient state preparation for electronic structure simulations on quantum devices, arXiv e-prints, arXiv:1809.05523 (2018). https://doi.org/10.48550/arXiv.1809.05523
[144] L. Vinet, A. Zhedanov, Dual - 1 Hahn Polynomials and perfect state transfer, Journal of Physics 343 (2011). https://doi.org/10.48550/arXiv.1110.6477
[145] L. Vinet, A. Zhedanov, How to construct spin chain with perfect state transfer, Physical Review A 85 (2012). https://doi.org/10.48550/arXiv.1110.6474
[146] L. Vinet, A. Zhedanov, Almost perfect state transfer in quantum spin chains, Physical Review A 86 (2012). https://doi.org/10.48550/arXiv.1205.4680
[147] M. J. Vrakking, D. Villeneuve, A. Stolow, Observation of fractional revivals of a molecular wave packet, Physical Review A 54 (1996). https://doi.org/10.1103/PhysRevA.54.R37
[148] J. Wagner, M. A. Rapsomaniki, S. Chevrier, T. Anzeneder, C. Langwieder, A. Dykgers, M. Rees, A. Ramaswamy, S. Muenst, S. D. Soysal, et al., A single cell atlas of the tumor and immune ecosystem

of human breast cancer. Cell, 177 (2019) 1330-1345. https://doi.org/10.1016/j.cell.2019.03.005
[149] Sh. Wang, T. Feng, perfect state transfer on bi-Cayley graphs over abelian groups, Discrete Mathematics 346 (2023). https://doi.org/10.1016/j.disc.2023.113362
[150] Sh. Wang, M. Arezoomand, T. Feng, Perfect state transfer on quasi-abelian semi-Cayley graphs, Journal of Algebraic Combinatorics 59 (2024) 179-211. https://doi.org/10.1007/s10801-023-01288-6
[151] J. Wang, S. Lisanza, D. Juergens, D. Tischer, J. L. Watson, K. M. Castro, R. Ragotte, A. Saragovi, L. F. Milles, M. Baek, et al., Scaffolding protein functional sites using deep learning, Science3 77 (2022) 387-394. https://doi.org/ 10.1126/science.abn2100
[152] W. Wang, X. Liu, J. Wang, Laplacian pretty good edge state transfer in path, arxiv: 2209 - 04630, (2022). https://doi.org/10.48550/arXiv.2209.04630
[153] J.Watrous, Quantum simulation of classical random walks and undirected graph connectivity, Journal of Computer and System Sciences 62 (2001) 376-391. https://doi.org/10.48550/arXiv.cs/9812012
[154] B. I. M. Wicky, L. F. Milles, A. Courbet, R. J. Ragotte, J. Dauparas, E. Kinfu, S. Tipps, R.D. Kibler, M. Baek, F. DiMaio, et al., Hallucinating symmetric protein assemblies, Science 378 (2022) 56-61. https://doi.org/ 10.1126/science.add1964
[155] K. K. Yang, Z. Wu, F. H. Arnold, Machine-learning-guided directed evolution for protein engineering, Nature methods 16 (2019) 687-694. https://doi.org/10.1038/s41592-019-0496-6
[156] J. A. Yeazell, C. Stroud. Jr, Observation of spatially localized atomic electron wave packets, Physical Review Letters 60 (1988). https://doi.org/10.1103/PhysRevLett.60.1494
[157] P. H. Zieschang, Cayley graphs of finite groups, Journal of Algebra and Its Applications 118 (1988) 56-61. https://doi.org/10.1016/0021-8693(88)90033-6
[158] X-Q. Zhang, S-Y. Cui, G-X. Tian, Signless Laplacian state transfer Q-graphs, Applied Mathematics and Computation 423 (2022). https://doi.org/10.1016/j.amc.2022.127070
[159] J. Zhou, C. Bu, J. Shen, Some results for the periodicity and perfect state transfer, Electron, Journal of Combinatorial Theory, Series A 18 (2011). https://doi.org/10.37236/671
[160] J. Zhou, C. By, State transfer and star complements in graphs, Discrete Applied Mathematics 175 (2014) 130-134. https://doi.org/10.1016/j.dam.2013.08.028
[161] Z. Zimboras, M. Faccin, Z. Kádár, J. Whitfield, B. Lanyon, J. Biamonte, Quantum Transport Enhancement by Time-Reversal Symmetry Breaking, Scientific Reports 3 2361 (2013). https://doi.org/10.1038/srep02361

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