Journal of Discrete Mathematics and Its Applications 9 (3) (2024) 211–248

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

A discussion of Feng-Liu operator and fixed point theorems on metric space

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Academic Editor: Vahid Mohammadi

Abstract. In this paper, a collection of various multi-valued fixed point results using Feng-Liu operator on metric space are examined. Comparative discussions on some of the important ideas, using this operators are presented. Thereafter, a handful of potential improvements on the existing literature are proposed.

Keywords: fixed point, multi-valued contraction, Hausdorff metric space, Feng-Liu operator.
Mathematics Subject Classification (2010): $47H10$, $54H25$, $37C25$.

1 Introduction

Banach contraction principle, which first emerged in Banach's thesis in 1922 and was utilized to prove the existence of a solution to an integral equation, is the central finding of metric fixed point (FP) theory. Since then, it has gained a lot of popularity as a method for resolving existence problems in various areas of mathematical analysis due to its ease of use and practicality. The following is how we put this foundational theorem forward.

Theorem 1.1. *([10]) If $T : \Omega \longrightarrow \Omega$ is a mapping from a complete metric space (CMS) (Ω, ρ) and if there is a constant $\alpha \in [0,1)$ such that for every $\mu, \nu \in \Omega$,*

$$\rho(T\mu, T\nu) \leq \alpha\rho(\mu, \nu), \tag{1.1}$$

holds, then there exists exactly one point $u \in \Omega$ such that $Tu = u$. Moreover, for each point $\mu_0 \in \Omega$,

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Received 4 August 2024 ; Revised 20 August 2024 ; Accepted 29 August 2024
First Publish Date: 1 september 2024

the iterative sequence $\{\mu_n\}$ *given by* $\mu_{n+1} = T\mu_n$ *converges to* u.

The mapping T satisfying (1.1) is called a contraction. A number of mathematicians have generalized this famous theorem (e.g., see Rhoades [53]).

Let (Ω, ρ) be a metric space (MS), $\eth(\Omega)$ denotes the collection of all nonempty subsets of Ω, $\complement(\Omega)$ is the collection of all nonempty closed subsets of Ω, $\Xi(\Omega)$ be the family of nonempty, closed and bounded subsets of Ω, and $¥(\Omega)$ the family of nonempty compact subsets of Ω. Let $£, Ł \in \Xi(\Omega)$. Then, the Hausdorff metric on $\Xi(\Omega)$ is the function $H : \Xi(\Omega) \times \Xi(\Omega) \longrightarrow [0, +\infty)$ given by

$$H(£, Ł) = \max\{\rho(£, Ł), \rho(Ł, £)\},$$

where $\rho(\mu, £) = \inf\{\rho(\mu, \nu) : \nu \in £\}$ and similarly, $\rho(£, Ł) = \sup\{\rho(\mu, Ł) : \mu \in £\}$. The pair $(\Xi(\Omega), H)$ is called the Hausdorff MS. We note that the metric H actually depends on the metric for Ω.

Definition 1.1. *Let* (Ω, ρ) *be MS. A mapping* $T : \Omega \longrightarrow \Xi(\Omega)$ *is said to be a multi-valued contraction on* Ω *if there exists* $\alpha \in [0, 1)$ *such that* $H(T\mu, T\nu) \leq \alpha\rho(\mu, \nu)$*, for all* $\mu, \nu \in \Omega$.

Definition 1.2. *A point* μ *is said to be a FP of a multi-valued mapping* T *if* $\mu \in T\mu$.

We denote by $Fix(T)$ the set of all FPs of T, that is,

$$Fix(T) = \{x \in \Omega : \mu \in T\mu\}.$$

The first generalization of the Banach contraction principle into multi-valued was presented by Nadler (1969) which states that a multi-valued contraction mapping of a CMS Ω into the nonempty closed and bounded subsets of Ω has a FP.

The following theorem is due to Nadler (1969).

Theorem 1.2. ([45]) *Let* (Ω, ρ) *be a CMS. If* $T : \Omega \longrightarrow \Xi(\Omega)$ *is a multi-valued contraction mapping, then* T *has a FP.*

The generalizations to the multi-valeud FP theory are enormous and can be found in [44, 61,63] and some references therein. For about four decades, research in metric FP theorem for set-valued mapping has been considered using the Hausdorff distance function. A hallmark result in this direction where FP theorems for multi-valued mapping can be studied without using the Hausdorff metric was initiated by Feng and Liu [18].

The purpose of this article is to carry out a comprehensive survey of important developments in the study of metric FP results using Feng-Liu operator. We highlight, analyze and suggest further improvements of Feng-Liu contractive-type inequalities. In what follows hereafter, the requisite preliminaries and the main results of [18] are discussed.

Definition 1.3. *A function* $f : \Omega \longrightarrow \mathbb{R}$ *is called lower semi-continuous (LSC), if for any* $\{\mu_n\} \subset \Omega$ *and* $\mu \in \Omega$, $\mu_n \longrightarrow \mu$ *implies* $f(\mu) \leq \liminf_{n \longrightarrow \infty} f(\mu_n)$. *A function* $f : \Omega \longrightarrow \mathbb{R}$ *is called upper semi-continuous, if for any* $\{\mu_n\} \subset \Omega$ *and* $\mu \in \Omega$, $\mu_n \longrightarrow \mu$ *implies* $f(\mu) \geq \limsup_{n \longrightarrow \infty} f(\mu_n)$.

A multi-valued mapping (MVM) $T : \Omega \longrightarrow 2^{\Omega}$ (collection of all nonempty subsets of Ω) is called upper semi-continuous, if for any $\mu \in \Omega$ and a neighborhood V of $T(\mu)$, there is a neighborhood U of μ such that for any $\nu \in U$, we have $T(\nu) \subset V$. A MVM $T : \Omega \longrightarrow 2^{\Omega}$ is called LSC, if for any $\mu \in \Omega$ and a neighborhood V, $V \cap T(\mu) \neq \emptyset$, there is a neighborhood U of μ such that for any $\nu \in U$, we have $T(\nu) \cap V \neq \emptyset$.

Let $T : \Omega \longrightarrow \eth(\Omega)$ be a MVM. Define a function $f : \Omega \longrightarrow \mathbb{R}$ as $f(\mu) = \rho(\mu, T\mu)$. For a positive constant $b(b \in (0,1))$, define the set $I_b^{\mu} \in \Omega$ as

$$I_b^{\mu} = \{\nu \in T\mu : b\rho(\mu,\nu) \leq \rho(\mu, T\mu)\}.$$

The following theorem is due to Feng and Liu [18].

Theorem 1.3. *Let (Ω,ρ) be a CMS, $T : \Omega \longrightarrow \complement(\Omega)$ be a MVM. If there exists a constant $c \in (0,1)$ such that for any $\mu \in \Omega$, there is $\nu \in I_b^{\mu}$ satisfying*

$$\rho(\nu, T\nu) \leq c\rho(\mu,\nu), \tag{1.2}$$

then T has a FP in Ω provided $c < b$ and f is LSC.

Proof. Since $T\mu \in \complement(\Omega)$ for any $\mu \in \Omega$, I_b^{μ} is nonempty for any constant $b \in (0,1)$. For any initial point $\mu_0 \in \Omega$, there exists $\mu_1 \in I_b^{\mu_0}$ such that

$$\rho(\mu_1, T\mu_1) \leq c\rho(\mu_0,\mu_1),$$

and, for $\mu_1 \in \Omega$, there is $\mu_2 \in I_b^{\mu_1}$ satisfying

$$\rho(\mu_2, T\mu_2) \leq c\rho(\mu_1,\mu_2).$$

Continuing this process, we can get an iterative sequence $\{\mu_n\}$, where $\mu_{n+1} \in I_b^{\mu_n}$ and

$$\rho(\mu_{n+1}, T\mu_{n+1}) \leq c\rho(\mu_n,\mu_{n+1}), \quad n = 0,1,2,\cdots.$$

In what follows, we will verify that $\{\mu_n\}$ is a Cauchy sequence. On the one hand,

$$\rho(\mu_{n+1}, T\mu_{n+1}) \leq c\rho(\mu_n,\mu_{n+1}), \quad n = 0,1,2,\cdots. \tag{1.3}$$

On the other hand, $\mu_{n+1} \in I_b^{\mu_n}$ implies

$$b\rho(\mu_n,\mu_{n+1}) \leq \rho(\mu_n, T\mu_n) \quad n = 0,1,2,\cdots. \tag{1.4}$$

By inequalities (1.3) and (1.4), we have

$$\rho(\mu_{n+1},\mu_{n+2}) \leq \frac{c}{b}\rho(\mu_n,\mu_{n+1}), \quad n = 0,1,2,\cdots.$$

$$\rho(\mu_{n+1}, T\mu_{n+1}) \leq \frac{c}{b}\rho(\mu_n, T\mu_n) \quad n = 0,1,2,\cdots.$$

It is easy to prove that

$$\rho(\mu_n,\mu_{n+1}) \leq \frac{c^n}{b^n}\rho(\mu_0,\mu_1), \quad n = 0,1,2,\cdots.$$

Then for $m,n \in N$ and $m > n$

$$\begin{aligned}\rho(\mu_m,\mu_n) &\leq \rho(\mu_m,\mu_{m-1}) + \rho(\mu_{m-1},\mu_{m-2}) + \cdots + \rho(\mu_{n+1},\mu_n)\\ &\leq a^{m-1}\rho(\mu_0,\mu_1) + a^{m-2}\rho(\mu_0,\mu_1) + +a^n\rho(\mu_0,\mu_1)\\ &\leq \frac{a^n}{1-a}\rho(\mu_0,\mu_1),\end{aligned}$$

where $a = c/b$. Due to $c < b$, we have $a_n \longrightarrow 0 \quad (n \longrightarrow \infty)$, which means that $\{\mu_n\}$ is a Cauchy sequence. According to the completeness of Ω, there exists $\mu \in \Omega$ such that $\{\mu_n\}$ converges to μ. We assert that μ is a FP of T.

In fact, from the above proof, we get that $\{\mu_n\}$ converges to μ. On the other hand, $\{f(\mu_n)\} = \{\rho(\mu_n, Te\mu_n)\}$ is decreasing and hence converges to 0. Since f is LSC, we have

$$0 \leq f(\mu) \leq \liminf_{n\longrightarrow\infty} f(\mu_n) = 0.$$

Hence, $f(\mu) = 0$. Finally, the closeness of $T\mu$ implies $\mu \in T\mu$. □

Remark 1.1. *Theorem 1.3 is a generalization of Theorem 1.2. In addition, if T satisfies the contractive condition of Theorem 1.2, then f being LSC, which follows from the fact that T, being a multi-valued contraction, is upper semi-continuous. Moreover, suppose that T is a Nadler contraction, then T is a Feng-Liu contraction (see Equation* (1.2)*). To see this, for any $\mu \in \Omega$,* and *$\nu \in T(\mu)$, we have*

$$\rho(\nu, T\nu) \leq H(T\mu, T\nu) \leq c\rho(\mu,\nu).$$

Hence, T satisfies conditions of Theorem 1.3. However, every Feng and Liu contraction is not necessarily a Nadler's contraction. As an example, see [18, Remark 1].

2 Some Variants of Feng and Liu Approach

In this section, we highlight some of the important extensions of the results of Feng and Liu [18]. One of the generalizations was given by Guran [19]. We first consider this result.

2.1 Guran (2007)

Guran [19] generalized Theorem 1.3 where the author proved a FP theorem for contractive type multi-valued operator on a CMS endowed with a w-distance.
The concept of w-distance was introduced by Kada *et al*. [29] as follows.

Definition 2.1. *Let (Ω,ρ) be a MS. Then the mapping $w : \Omega \times \Omega \longrightarrow [0,\infty)$ is called w-distance on Ω, if the following axioms are satisfied:*

(1) $w(\mu,\varsigma) \leq w(\mu,\nu) + w(\nu,\varsigma)$ *for any* $\mu,\nu,\varsigma \in \Omega$*;*

(2) *for any* $\mu \in \Omega$, $w(\mu,\cdot) : \Omega \longrightarrow [0,\infty)$ *is LSC; that is, if a sequence* $\{\nu_n\}$ *in* Ω *with* $\nu_n \longrightarrow \nu \in \Omega$*, then* $w(\mu,\nu) \leq \liminf_{n\longrightarrow\infty} w(\mu,\nu_n)$*;*

(3) *for any* $\epsilon > 0$*, there exists* $\delta > 0$ *such that* $w(\varsigma,\mu) \leq \delta$ *and* $w(\varsigma,\nu) \leq \delta$ *implies* $\rho(\mu,\nu) \leq \epsilon$*.*

Definition 2.2. *([19]) Let* $T : \Omega \longrightarrow \eth(\Omega)$ *be a multi-valued operator,* $w : \Omega \times \Omega \longrightarrow [0,\infty)$ *be a w-distance on* Ω*. Define the function* $f : \Omega \longrightarrow \mathbb{R}$ *as* $f(\mu) = D_w(\mu,T\mu)$*, where* $D_w(\mu,T\mu) = \inf\{w(\mu,\nu) : \nu \in T\mu\}$*. For a positive constant* $b \in (0,1)$*, define the set* $I_b^\mu \subset \Omega$ *as follows:*

$$I_b^\mu = \{\nu \in T\mu : bw(\mu,\nu) < D_w(\mu,T\mu)\}.$$

Theorem 2.1. *([19]) Let* (Ω,ρ) *be a CMS,* $T : \Omega \longrightarrow \complement(\Omega)$ *a multi-valued operator,* $w : \Omega \times \Omega \longrightarrow [0,\infty)$ *be a w-distance on* Ω *and* $b \in (0,1)$*. Suppose that:*

(i) *there exists* $c \in (0,1)$*, with* $c < b$*, such that for any* $\mu \in \Omega$ *there is* $\nu \in I_b^\mu$ *satisfying*

$$D_w(\mu,T\mu) \leq cw(\mu,\nu);$$

(ii) $f : \Omega \longrightarrow \mathbb{R}, f(\mu) = D_w(\mu,T\mu)$ *is LSC.*

Then T *has a FP in* Ω*.*

Remark 2.1. *Assumption (ii) in Theorem 2.1 is redundant in the proof of the Theorem, this is because the function* $w(\mu,\cdot)$ *is LSC. Here, we only prove the existence of the FP. To see this, we have*

$$\begin{aligned} 0 \leq D_w(\mu,T\mu) &\leq \inf\{w(\mu,\nu) : \nu \in T\mu\} \\ &\leq \liminf_{n\longrightarrow\infty} w(\mu,\nu_n) = 0. \end{aligned}$$

This implies $D_w(\mu,T\mu) = 0$*. From* $T\mu \in \complement(\Omega)$*, we have that* $\mu \in T\mu$*. Then* T *has a FP.*

2.2 Klim and Wardowski (2007)

Klim and Wardowski [33] established the conditions guaranteeing the existence of a FP for Feng-Liu-type operators, using Reich [52] and Mizoguchi-Takahashi [44] results.

Theorem 2.2. *Let* (Ω,ρ) *be a CMS and let* $T : \Omega \longrightarrow \Xi(\Omega)$*. Assume that there exists a mapping* $\varphi : (0,\infty) \longrightarrow [0,1)$ *such that*

$$\forall_{t\in[0,\infty)}, \quad \limsup_{r\longrightarrow t^+} \varphi(r) < 1, \tag{2.1}$$

and

$$\forall_{\mu,\nu\in\Omega,\mu\neq\nu}, \quad H(T\mu,T\nu) \leq \varphi(\rho(\mu,\nu))\rho(\mu,\nu).$$

Then T *has a FP.*

Two following result is due to Klim and Wardowski [33].

Theorem 2.3. *Let (Ω,ρ) be a CMS and let $T:\Omega \longrightarrow \Xi(\Omega)$. Assume that the following conditions hold:*

(i) the mapping $f:\Omega \longrightarrow \mathbb{R}$ given by $f(\mu)=\rho(\mu,T\mu)$, $\mu \in \Omega$ is LSC;

(ii) there exist $b \in (0,1)$ and $\varphi:(0,\infty) \longrightarrow [0,b)$ such that

$$\forall_{t\in[0,\infty)}, \quad \limsup_{r\longrightarrow t^+} \varphi(r) < b,$$

and

$$\forall_{\mu\in\Omega}\, \exists \nu \in I_b^{\mu} \quad \rho(\nu,T\nu) \leq \varphi(\rho(\mu,\nu))\rho(\mu,\nu). \tag{2.2}$$

Then T has a FP.

Theorem 2.4. *Let (Ω,ρ) be a CMS and let $T:\Omega \longrightarrow \complement(\Omega)$. Assume that the following conditions hold:*

(i) the mapping $f:\Omega \longrightarrow [0,\infty), f(\mu)=\rho(\mu,T\mu)$, $\mu \in \Omega$, is LSC;

(ii) there exists $b \in (0,1)$ and $\varphi:(0,\infty) \longrightarrow [0,b)$ such that for all $t \in (0,\infty)$, $\limsup_{r\longrightarrow t^+} \varphi(r) < 1$ and for all $\mu \in \Omega, \exists \nu \in I_b^{\mu}$ satisfying

$$\rho(\nu,T\nu) \leq \varphi(\rho(\mu,\nu))\rho(\mu,\nu).$$

Then T has a FP.

Remark 2.2. *It is easy to see from the work of [18] and [33] that the inequalities*

$$\rho(\nu,T\nu) \leq c\rho(\mu,\nu) \leq \varphi(\rho(\mu,\nu))\rho(\mu,\nu), \tag{2.3}$$

hold. But the converse of inequality (2.3) *is not always true (see [33, Example 3.1]).*

2.3 Ciric (2008)

Ciric [14] proved some existence theorems for FPs of Feng-Liu type contractions in a CMS.

Definition 2.3. *Let (Ω,ρ) be a metric space. A subset ¥ is called proximinal if for each $\mu \in \Omega$, there exists an element $k \in$ ¥ such that*

$$\rho(\mu,k) = D(\mu,¥) = \inf\{\rho(\mu,\nu) : \nu \in ¥\}.$$

Theorem 2.5. *([14]) Let (Ω,ρ) be a CMS and let $T:\Omega \longrightarrow \complement(\Omega)$ be a mapping. If there exists a function $\varphi:[0,\infty) \longrightarrow [0,1)$ satisfying (2.1), and is such that for any $\mu \in \Omega$, there is $\nu \in T\mu$ satisfying the following two conditions:*

$$\rho(\mu,\nu) \leq (2-(\rho(\mu,\nu)))D(\mu,T\mu),$$

and

$$D(\nu,T\nu) \leq \varphi(\rho(\mu,\nu))\rho(\mu,\nu), \tag{2.4}$$

then T has a FP in Ω provided $f(\mu)=D(\mu,T\mu)$ is LSC.

Theorem 2.6. *([14]) Let (Ω,ρ) be a CMS and let $T:\Omega \longrightarrow \complement(\Omega)$ be a mapping. If there exists a function $\varphi:(0,\infty)\longrightarrow[0,1)$ and a non-decreasing function $\eta:[0,\infty)\longrightarrow[b,1), b>0$, such that*

$$\varphi(t)<\eta(t),$$

and

$$\limsup_{t\longrightarrow r^+}\varphi(t)<\limsup_{t\longrightarrow r^+}\eta(t).$$

For all $t\in(0,\infty)$, and for any $\mu\in\Omega$ there is $\nu\in T\mu$ satisfying the following two conditions:

$$\eta(\rho(\mu,\nu))\rho(\mu,\nu)\leq D(\mu,T\mu),$$

and

$$D(\nu,T\nu)\leq\varphi(\rho(\mu,\nu))\rho(\mu,\nu), \tag{2.5}$$

then T has a FP in Ω provided $f(\mu)=D(\mu,T\mu)$ is LSC.

Theorem 2.7. *([14]) Let (Ω,ρ) be a CMS and T be a MVM of Ω into a collection of all non-empty proximinal subsets of Ω. If there exists a function $\varphi:(0,\infty)\longrightarrow[0,1)$ satisfying (2.1) and for any $\mu\in\Omega$, there is $\nu\in T\mu$ satisfying the following two conditions:*

$$\rho(\mu,\nu)=D(\mu,T\mu),$$

and

$$D(\nu,T\nu)\leq\varphi(\rho(\mu,\nu))\rho(\mu,\nu), \tag{2.6}$$

then T has a FP in Ω provided $f(\mu)=D(\mu,T\mu)$ is LSC.

Remark 2.3. *Theorem 2.7 is a proper generalisation of Theorem 2.4 of Klim and Wardowski [33] (see [14, Example 1]).*

2.4 Latif and Albar (2008)

Latif and Albar [38] using the concept of w-distance, proved the FP theorems for multi-valued contractive mappings. Let $T:\Omega\longrightarrow 2^{\Omega}$ (where 2^{Ω} denotes the set of all power sets of Ω) be a MVM. They considered the following:

(i) for $\mu\in\Omega$, define a set $J_b^{\mu}\subset\Omega$ as

$$J_b^{\mu}=\{\nu\in T\mu: bp(\mu,\nu)<w(\mu,T\mu), b\in(0,1)\};$$

(ii) A mapping T is weakly contractive if there exist a w-distance w on Ω and a constant $h\in(0,1)$ such that for any $\mu\in\Omega$, there is $\nu\in J_b^{\mu}$ satisfying

$$w(\nu,T\nu)\leq hw(\mu,\nu),\ h<b.$$

Theorem 2.8. *Let (Ω,ρ) be a CMS and let $T:\Omega\longrightarrow\complement(\Omega)$ be a weakly contractive mapping. If the real valued function f (given by $f(\mu)=w(\mu,T\mu)$) is LSC, then there exists $\varsigma\in\Omega$ such that $w(\varsigma,T\varsigma)=0$. Further, if $w(\varsigma,\varsigma)=0$, then ς is a FP of T.*

2.5 Berinde and Păcurar (2008)

Berinde and Păcurar [11] approximated both single-valued and multi-valued FPs of almost contractions by means of Picard iteration.

Definition 2.4. *A single valued mapping $T:\Omega \longrightarrow \Omega$ is called an almost contraction or (δ, L)-almost contraction if and only if there exist two constants, $\delta \in (0,1)$ and $L \leq 0$ such that*

$$\rho(T\mu, T\nu) \leq \delta \cdot \rho(\mu,\nu) + L\rho(\nu, T\mu), \quad \forall \mu,\nu \in \Omega. \tag{2.7}$$

Theorem 2.9. *Let (Ω,ρ) be a CMS and $T:\Omega \longrightarrow \complement(\Omega)$. Assume that the following conditions hold:*

(i) the mapping $f:\Omega \longrightarrow [0,\infty), f(\mu)=\rho(\mu,T\mu), \mu \in \Omega$, is LSC;

(ii) there exist $L \geq 0$, $b \in (0,1)$ and $\varphi:(0,\infty) \longrightarrow [0,b)$ such that for all $t \in (0,\infty)$,

$$\limsup_{r \longrightarrow t^+} \varphi(r) < b,$$

and for all $\mu \in \Omega$, $\exists \nu \in I_b^\mu$ satisfying

$$\rho(\nu, T\nu) \leq \varphi(\rho(\mu,\nu))\rho(\mu,\nu) + L\min\{\rho(\mu,T\mu), \rho(\nu,T\nu), \rho(\mu,T\nu), \rho(\nu,T\mu)\}. \tag{2.8}$$

Remark 2.4. *We notice that the inequality* (2.8) *in the sense of Berinde and Păcurar [11] implies the Inequalities* (2.4), (2.5) *and* (2.6) *due to Ciric [14]. To see this, it suffices to take $L=0$ in the inequality* (2.8).

Many of similar results (see, [15, 33, 34, 36]), can also be deduced from (2.8) *accordingly.*

2.6 Ciric (2009)

Ciric [15] introduced two concepts of nonlinear contractions for MVMs in complete MS and also proved three following FP theorems.

Theorem 2.10. *Let (Ω,ρ) be a CMS and T be a MVM from Ω into $\complement(\Omega)$. Suppose that the function $f:\Omega \longrightarrow \mathbb{R}$, given by $f(\mu)=\rho(\mu,T\mu)$, $\mu \in \Omega$, is LSCand that there exists a function $\varphi:[0,\infty) \longrightarrow [a,1)$, $0<a<1$, satisfying (2.1) and for any $\mu \in \Omega$, there is $\nu \in T\mu$ satisfying the following two conditions:*

$$\sqrt{\varphi(f(\mu))}\rho(\mu,\nu) \leq f(\mu) \quad \text{and} \quad f(\nu) \leq \varphi(f(\mu))\rho(\mu,\nu). \tag{2.9}$$

Then there exists $\varsigma \in \Omega$ such that $\varsigma \in T\varsigma$.

Theorem 2.11. *Let (Ω,d) be a CMS and T be a MVM from Ω into $\complement(\Omega)$. Suppose that the function $f:\Omega \longrightarrow \mathbb{R}$, given by $f(\mu)=\rho(\mu,T\mu)$, $\mu \in \Omega$, is LSC and that there exists a function $\varphi:[0,\infty) \longrightarrow [a,1)$, $0<a<1$, satisfying (2.1) and for any $\mu \in \Omega$, there is $\nu \in T\mu$ satisfying the following two conditions:*

$$\sqrt{\varphi(\rho(\mu,\nu))}\rho(\mu,\nu) \leq \rho(\mu,T\mu),$$

and

$$\rho(\nu,T\nu) \leq \varphi(\rho(\mu,\nu))\rho(\mu,\nu).$$

Then there exists $\varsigma \in \Omega$ such that $\varsigma \in T\varsigma$.

Theorem 2.12. *Let (Ω,ρ) be a CMS and T be a MVM from Ω into $\mathtt{C}(\Omega)$. Suppose that the function $f:\Omega\longrightarrow\mathbb{R}$, given by $f(\mu)=\rho(\mu,T\mu)$, $\mu\in\Omega$, is LSC and that there exists a function $\varphi:[0,\infty)\longrightarrow[a,1)$, $a>0$, satisfying Equation (2.1) and for any $\mu\in\Omega$ there is $\nu\in T\mu$ satisfying the following two conditions:*

$$\sqrt[\frac{n-1}{n}]{\varphi(\rho(\mu,T\mu))}\rho(\mu,\nu)\leq\rho(\mu,T\mu),$$

and

$$\rho(\nu,T\nu)\leq\varphi(\rho(\mu,T\mu))\rho(\mu,\nu).$$

Then there exists $\varsigma\in\Omega$ such that $\varsigma\in T\varsigma$.

2.7 Kamran (2009)

Kamran [30] generalized Mizoguchi-Takahashi's [44] FP theorem as well as improved the result by Klim and Wadowski [33], and also extended Hicks and Rhoades [20] FP theorem to MVMs.

Definition 2.5. *([30]) A sequence $\{\mu_n\}$ in Ω is called an orbit of T at $\mu_0\in\Omega$ if $\mu_n\in T(\mu_{n-1})$ for all $n\geq1$. A function $f:\Omega\longrightarrow\mathbb{R}$ is said to be T-orbitally LSC at ξ if there exist a mapping $T:\Omega\longrightarrow\Omega$ and a sequence $\{\mu_n\}$ in $O_T(\mu_0)$, for some $\mu_0\in\Omega$, such that $\lim\limits_{n\longrightarrow\infty}\mu_n=\xi$ implies*

$$f(\xi)\leq\liminf_{n\longrightarrow\infty}f(\mu_n),$$

where $O_T(\mu_0)=\{\mu_0,\mu_1,\mu_2,...\}$.

Theorem 2.13. *([30]) Let (Ω,ρ) be a CMS and $T:\Omega\longrightarrow\mathtt{C}(\Omega)$ satisfying*

$$\rho(\nu,T\nu)\leq\alpha(\rho(\mu,\nu))\rho(\mu,\nu)\ for\ each\ \mu\in\Omega\ and\ \nu\in T\mu,$$

where α is a function from $(0,\infty)$ into $[0,1)$ such that (2.1) is satisfied. Then

(i) For each $\mu_0\in\Omega$, there exists an orbit $\{\mu_n\}$ of T and $\xi\in\Omega$ such that $\lim_n\mu_n=\xi$;

(ii) ξ is FP of T if and only if the function $f(\mu)=\rho(\mu,T\mu)$ is T-orbitally LSC at ξ.

2.8 Pathak and Shahzad (2009)

Pathak and Shahzad [50] introduced a contraction condition by altering distances for set-valued maps in MS.

Theorem 2.14. *Let (Ω,ρ) be a CMS and $T:\Omega\longrightarrow\mathtt{C}(\Omega)$. Assume that the following conditions hold:*

(i) the mapping $f:\Omega\longrightarrow\mathbb{R}$, given by $f(\mu)=\rho(\mu,T\mu)$, $\mu\in\Omega$, is LSC;

(ii) *there exist* $b \in (0,1)$ *and* $\varphi : (0,\infty) \longrightarrow [0,b)$ *such that for all* $t \in (0,\infty)$*,*

$$\limsup_{r \longrightarrow t^+} \varphi(r) < b;$$

(iii) *there exists* $\theta \in \Theta[0,A)$ *satisfying the following condition (with b and* φ *of (ii)):*

for all $\mu \in \Omega$*,* $M(a,\mu:\theta)$ *is nonempty for any constant* $a \in (0,1)$ *and*

$$\forall_{\mu,\in\Omega}\exists_{\nu\in M(a,\mu:\theta)}, \quad \theta\rho(\nu,T\nu) \leq \varphi(\rho(\mu,\nu))\theta(\rho(\mu,\nu)).$$

Then T has a FP.

Theorem 2.15. *Let* (Ω,ρ) *be a CMS and* $T : \Omega \longrightarrow \Xi(\Omega)$*. Assume that the following conditions hold:*

(i) *the mapping* $f : \Omega \longrightarrow \mathbb{R}, f(\mu) = \rho(\mu,T\mu), \mu \in \Omega$*, is LSC;*

(ii) *there exists* $\varphi : (0,\infty) \longrightarrow [0,1)$ *satisfying inequality (2.1);*

(iii) *there exists* $\theta \in \Theta[0,A)$ *satisfying the following condition (with b and* φ *of (ii)):*

for all $\mu \in \Omega$*,* $M(1,\mu:\theta)$ *is nonempty and*

$$\forall_{\mu,\in\Omega}\exists_{\nu\in M(1,\mu:\theta)}, \quad \theta\rho(\nu,T\nu) \leq \varphi(\rho(\mu,\nu))\theta(\rho(\mu,\nu)).$$

Then T has a FP.

Remark 2.5. *The ideas of [15, 30, 50] can be unified in the following theorem.*

Theorem 2.16. *Let* (Ω,ρ) *be a CMS and T be a MVM from* Ω *into* $\complement(\Omega)$*. Suppose that the function* $f : \Omega \longrightarrow \mathbb{R}$*, given by* $f(\mu) = \rho(\mu,T\mu)$*,* $\mu \in \Omega$*, is a LSC and that there exist functions* $\theta : [0,A) \longrightarrow \mathbb{R}$ *and* $\varphi : [0,\infty) \longrightarrow [a,1)$*,* $a > 0$ *satisfying (2.1) and for any* $\mu \in \Omega$*, there is* $\nu \in T\mu$ *satisfying the following two conditions:*

$$\sqrt[\frac{n-1}{n}]{\varphi(\rho(\mu,T\mu))}\rho(\mu,\nu) \leq \rho(\mu,T\mu)$$

and

$$\theta(\rho(\nu,T\nu)) \leq \varphi(\rho(\mu,\nu))\max\{\rho(\mu,\nu),\theta(\rho(\mu,\nu)\}.$$

Then there exists $\varsigma \in \Omega$ *such that* $\varsigma \in T\varsigma$*.*

Using Zamfirescu's approach, we can also establish the following result:

Theorem 2.17. *There exist* $c \in [0,1)$ *and functions* $\theta : [0,A) \longrightarrow \mathbb{R}$*,* $\varphi : [0,\infty) \longrightarrow [a,1)$*,* $a > 0$ *satisfying (2.1) such that for any* $\mu \in \Omega$*, there is* $\nu \in T\mu$*, at least one of the following is true:*

(i) $\theta(\rho(\nu,T\nu)) \leq c\rho(\mu,\nu)$*;*

(ii) $\theta(\rho(\nu,T\nu)) \leq \varphi(\rho(\mu,\nu))\rho(\mu,\nu)$;

(iii) $\theta(\rho(\nu,T\nu)) \leq \varphi(\rho(\mu,\nu))\max\{\rho(\mu,\nu),\theta(\rho(\mu,\nu)\}$.

The following example shows that Theorem 2.16 is more general than the ideas of [15,30, 50].

Example 2.1. *Let $\Omega = [0,1]$ and $\rho : \Omega \times \Omega \longrightarrow \mathbb{R}$ be the standard metric. Let $T : \Omega \longrightarrow \mathtt{C}(\Omega)$ be given by*

$$T\mu = \begin{cases} \left\{\frac{1}{2}\mu^2\right\}, & for\ \mu \in [0,\frac{15}{32}) \cup (\frac{15}{32},1], \\ \left\{\frac{17}{96},\frac{15}{32},\frac{1}{4}\right\}, & for\ \mu = \frac{15}{32}. \end{cases}$$

Let $\theta : [0,A) \longrightarrow \mathbb{R}$ be given by $\theta(t) = t^{\frac{1}{2}}$, $0 < A < \infty$ and $\varphi : [0,\infty) \longrightarrow [0,1)$ be of the form

$$\varphi(t) = \begin{cases} \max\left\{\frac{1}{17},(\frac{25}{16})^{\frac{1}{4}}t^{\frac{1}{4}}\right\}, & for\ t \in [0,\frac{1}{2}], \\ \frac{23}{24}, & for\ t \in (\frac{1}{2},\infty). \end{cases}$$

Since

$$f(\mu) = \rho(\mu,T\mu) = \begin{cases} \mu - \frac{1}{2}\mu^2, & for\ \mu \in [0,\frac{15}{32}) \cup (\frac{15}{32},1], \\ 0, & for\ \mu = \frac{15}{32}. \end{cases}$$

It follows that the mapping f is LSC. Moreover, for each $\mu \in [0,\frac{15}{32}) \cup (\frac{15}{32},1]$, we have $\nu = T(\mu) = \{\frac{1}{2}\mu^2\}$ and therefore, $\rho(\mu,\nu) = \rho(\mu,T(\mu)) = \mu - \frac{1}{2}\mu^2$. Further

$$\begin{aligned} \theta(\rho(\nu,T\nu)) &= \left|\frac{1}{2}\mu^2 - \frac{1}{8}\mu^4\right|^{\frac{1}{2}} \\ &= \left(\frac{1}{2}\right)^{\frac{1}{2}}\left|\mu^2 - \left(\frac{1}{2}\mu^2\right)^2\right|^{\frac{1}{2}} \\ &= \left(\frac{1}{2}\right)^{\frac{1}{2}}\left|\mu + \frac{1}{2}\mu^2\right|^{\frac{1}{2}}\left|\mu - \frac{1}{2}\mu^2\right|^{\frac{1}{2}} \\ &\leq \left(\frac{25}{16}\right)^{\frac{1}{4}}\left|\mu - \frac{1}{2}\mu^2\right|^{\frac{1}{4}}\left|\mu - \frac{1}{2}\mu^2\right|^{\frac{1}{2}} \\ &\leq \max\left\{\frac{1}{17},\left(\frac{25}{16}\right)^{\frac{1}{4}}\left|\mu - \frac{1}{2}\mu^2\right|^{\frac{1}{4}}\right\}\left|\mu - \frac{1}{2}\mu^2\right|^{\frac{1}{2}} \\ &\leq \varphi(\rho(\mu,\nu))\theta(\rho(\mu,\nu)) \\ &\leq \varphi(\rho(\mu,\nu))\max\{\rho(\mu,\nu),\theta(\rho(\mu,\nu))\}. \end{aligned}$$

Now for $\mu = \frac{15}{32}$, $\nu = \frac{15}{32}$, and in particular for $n = 2$, we have

$$\sqrt{\varphi(\rho(\mu,\nu))}\rho(\mu,\nu) = 0 = \rho(\mu,T\mu),$$

choose $\nu = \frac{17}{96}$, *we have*

$$\begin{aligned}\theta(\rho(\nu,T\nu)) &= \theta\left(\rho\left(\frac{17}{96},\frac{1}{2}\frac{17^2}{96^2}\right)\right)\\ &< \theta\left(\frac{17}{96}\cdot\frac{11}{12}\right)\\ &= \left(\frac{17}{96}\cdot\frac{11}{12}\right)^{\frac{1}{2}}\\ &\leq \left(\frac{25}{16}\right)^{\frac{1}{4}}\left(\frac{7}{24}\right)^{\frac{1}{4}}\left(\frac{7}{24}\right)^{\frac{1}{2}}\\ &= \varphi(\rho(\mu,\nu))\theta(\rho(\mu,\nu))\\ &\leq \varphi(\rho(\mu,\nu))\max\{\rho(\mu,\nu),\theta(\rho(\mu,\nu))\}.\end{aligned}$$

Therefore, all assumptions of Theorem 2.16 are satisfied. However, Theorem 2.11 in [15] (also the main ideas of [30, 50]) fails in this instance. To see this, let $\nu = \frac{1}{4}$ *and* $\mu = \frac{15}{32}$. *Then* $T\nu = \{\frac{1}{32}\}$, $\rho(\mu,\nu) = \frac{7}{32}$. *And so we have*

$$\rho(\nu,T\nu) = \frac{7}{32} = \rho(\mu,\nu) > \varphi(\rho(\mu,\nu))(\rho(\mu,\nu)).$$

2.9 Falset et al. (2009)

Falset et al. [17] obtained a generalization of some well known FP theorems for MVMs of contractive type in the framework of complete MS, by using the concept of w-distance.

Theorem 2.18. ([17]) *Let* Ω *be a CMS with a w-distance* w. *Let* $T : \Omega \longrightarrow \complement(\Omega)$ *be a MVM. Assume that the following conditions hold:*

(i) there exists $b \in (0,1)$ *and a function* $\varphi : [0,\infty) \longrightarrow [0,b)$ *such that for all* $t \in (0,\infty)$,

$$\limsup_{r\longrightarrow t^+} \varphi(r) < b,$$

and for each $\mu \in \Omega$, *there exists* $\nu \in I^{\mu}_{b,p}$ *such that*

$$D_w(\nu,T(\nu)) \leq \varphi(w(\mu,T\mu))w(\mu,\nu);$$

(ii) for every $\nu \in \Omega$ *with* $\nu \notin T\nu$,

$$\inf\{(\mu,\varsigma) + D_w(\mu,T\mu) : \mu \in \Omega\} > 0.$$

Then T has a FP.

2.10 Latif and Abdou (2009)

Latif and Abdou [34] used the concept of w-distance to establish FP results for multi-valued generalized w-contractive maps.

Definition 2.6. *([34]) A MVM $T : \Omega \longrightarrow 2^{\Omega}$ is called a generalized w-contractive if there exist a w-distance w on Ω and a constant $b \in (0,1)$ such that for any $\mu \in \Omega$, there is $\nu \in J_b^{\mu}$ such that*

$$w(\nu, T\nu) \leq \varphi(w(\mu,\nu))w(\mu,\nu),$$

where φ is a function from $[0,\infty)$ to $[0,b)$ with $\limsup_{r \longrightarrow t^+} \varphi(r) < b$, for every $t \in [0,\infty)$.

Theorem 2.19. *([34]) Let Ω be a CMS and let $T : \Omega \longrightarrow \mathtt{C}(\Omega)$ be a generalized w-contractive mapping. Suppose that a real-valued function f on Ω defined by $f(\mu) = w(\mu, T\mu)$ is LSC. Then there exists $\varsigma \in \Omega$ such that $f(\varsigma) = 0$. Further, if $w(\varsigma,\varsigma) = 0$, then $\varsigma \in Fix(T)$.*

Latif and Abdou [37] introduced the concept of generalized contractive multimaps in the setting of MS with w-distance, and guaranteed the existence of FPs for such maps under certain conditions.

Theorem 2.20. *([37]) Let Ω be a CMS with a w-distance w. Let $T : \Omega \longrightarrow \mathtt{C}(\Omega)$ be a MVM. Assume that the following conditions hold:*

(i) there exists a function $\varphi : (0,\infty) \longrightarrow [0,1)$ satisfying inequality (2.1);

(ii) for each $\mu \in \Omega$, there exists $\nu \in T\mu$ such that

$$\begin{aligned} w(\mu,\nu) &\leq (2 - \varphi(w(\mu,\nu)))w(\mu, T\mu), \\ w(\nu, T\nu) &\leq \varphi(w(\mu, T\mu))w(\mu,\nu); \end{aligned}$$

(iii) the mapping $f : \Omega \longrightarrow \mathbb{R}$, given by $f(\mu) = w(\mu, T\mu)$, $\mu \in \Omega$, is LSC.

Then there exists $\varsigma \in \Omega$ such that $f(\varsigma) = 0$. Further, if $w(\varsigma,\varsigma) = 0$, then $\varsigma \in T\varsigma$.

Theorem 2.21. *([37]) Let Ω be a CMS with a w-distance w. Let $T : \Omega \longrightarrow \mathtt{C}(\Omega)$ be a MVM. Assume that the following conditions hold:*

(i) there exist functions $\varphi : (0,\infty) \longrightarrow [0,1)$ and $\phi : (0,\infty) \longrightarrow [0,b)$ with $b > 0$ such that for all $t \in (0,\infty)$,

$$\varphi(t) < \phi(t), \quad \limsup_{r \longrightarrow t^+} \varphi(r) < \limsup_{r \longrightarrow t^+} \phi(r);$$

(ii) for each $\mu \in \Omega$, there exists $\nu \in T\mu$ such that

$$\begin{aligned} \phi(w(\mu,\nu)w(\mu,\nu) &\leq w(\mu, T\mu), \\ w(\nu, T\nu) &\leq \varphi(w(\mu, T\mu))w(\mu,\nu); \end{aligned}$$

(iii) the mapping $f:\Omega\longrightarrow\mathbb{R}$, given by $f(\mu)=w(\mu,T\mu)$, $\mu\in\Omega$, is LSC.

Then there exists $\varsigma\in\Omega$ such that $f(\varsigma)=0$. Further if $w(\varsigma,\varsigma)=0$, then $\varsigma\in T\varsigma$.

Latif and Abdou [37] presented a result which is a generalization of Theorem 2.4 due to Klim and Wardowski [33] and Theorem 2.7 of Ciric [14] as follows.

Theorem 2.22. *([37]) Let Ω be a CMS with a w-distance w. Let $T:\Omega\longrightarrow\mathtt{C}(\Omega)$ be a MVM. Assume that the following conditions hold:*

(i) there exists a function $\varphi:(0,\infty)\longrightarrow[0,1)$ satisfying inequality (2.1);

(ii) for each $\mu\in\Omega$, there exists $\nu\in T\mu$ such that

$$w(\mu,\nu)=w(\mu,T\mu),$$
$$w(\nu,T\nu)\leq\varphi(w(\mu,T\mu))w(\mu,\nu);$$

(iii) the mapping $f:\Omega\longrightarrow\mathbb{R}$, given by $f(\mu)=w(\mu,T\mu)$, $\mu\in\Omega$, is LSC.

Then there exists $\varsigma\in\Omega$ such that $f(\varsigma)=0$. Further, if $w(\varsigma,\varsigma)=0$, then $\varsigma\in T\varsigma$.

Latif and Abdou [35] proved some results on the existence of FPs for weakly contractive type MVMs in the setting of MS with w-distance.

Definition 2.7. *([35]) Let $A\in(0,+\infty]$ and $\eta:[0,A)\longrightarrow\mathbb{R}$ satisfies:*

(i) $\eta(0)=0$ and $\eta(t)>0$ for each $t\in(0,A)$;

(ii) η is nondecreasing on $[0,A)$;

(iii) η is subadditive; that is

$$\eta(t_1+t_2)\leq\eta(t_1)+\eta(t_2)\quad\forall t_1,t_2\in(0,A).$$

Define $\Omega[0,A)=\{\eta:\eta$ satisfies $(i)-(iii)$ above $\}$.

Theorem 2.23. *([35]) Let (Ω,ρ) be a CMS with a w-distance w. Let $T:\Omega\longrightarrow\mathtt{C}(\Omega)$ be a MVM. Assume that the following conditions hold:*

(i) there exist $b\in(0,b)$ and $\varphi:(0,\infty)\longrightarrow[0,b)$ such that for each $t\in[0,\infty)'$

$$\limsup_{r\longrightarrow t^+}\varphi(r)<b.$$

(ii) there exists a function $\eta\in\Omega[0,A)$ such that for any $\mu\in\Omega$, there exists $\nu\in T\mu$ satisfying

$$b\eta(w(\mu,\nu))\leq\eta(w(\mu,T\mu)),$$
$$\eta(w(\nu,T\nu))\leq\varphi(w(\mu,\nu))\eta(w(\mu,\nu)).$$

(iii) *the mapping* $f : \Omega \longrightarrow \mathbb{R}$ *given by* $f(\mu) = w(\mu, T\mu), \mu \in \Omega$ *is LSC.*

Then there exists $\varsigma \in \Omega$ *such that* $f(\varsigma) = 0$*. Further, if* $w(\varsigma, \varsigma) = 0$*, then* $\varsigma \in T\varsigma$*.*

Corollary 2.24. *([35]) Let* $\Omega, \rho)$ *be a CMS with a* w*-distance* w*. Let* $T : \Omega \longrightarrow \complement(\Omega)$ *be a MVM satisfying that for any constants* $b, h \in (0, 1), h < b$ *and for each* $\mu \in \Omega$*, there is* $\nu \in J_b^\mu$ *such that*

$$\eta(w(\nu, T\nu)) \leq h\eta(w(\mu, \nu)),$$

where $J_b^\mu = \{\nu \in T\mu : b\eta(w(\mu, \nu) \leq \eta(w(\mu, T\mu))\}$*. Suppose that a real-valued function* f *on* Ω *defined by* $f(\mu) = w(\mu, T\mu)$ *is LSC. Then there exists* $\varsigma \in \Omega$ *such that* $f(\varsigma) = 0$*. Further, if* $w(\varsigma, \varsigma) = 0$*, then* $\varsigma \in Fix(T)$*.*

Theorem 2.25. *([35]) Let* (Ω, ρ) *be a CMS with a* w*-distance* w*. Let* $T : \Omega \longrightarrow \complement(\Omega)$ *be a MVM. Assume that the following conditions hold:*

(i) *there exists* $\varphi : (0, \infty) \longrightarrow [0, b)$ *satisfying (2.1);*

(ii) *there exists a function* $\eta \in \Omega[0, A)$ *such that for any* $\mu \in \Omega$*, there exists* $\nu \in T\mu$ *satisfying*

$$\eta(w(\mu, \nu)) \leq \eta(w(\mu, T\mu)),$$
$$\eta(w(\nu, T\nu)) \leq \varphi(w(\mu, \nu))\eta(w(\mu, \nu));$$

(iii) *the mapping* $f : \Omega \longrightarrow \mathbb{R}$ *given by* $f(\mu) = w(\mu, T\mu), \mu \in \Omega$ *is a LSC.*

Then there exists $\varsigma \in \Omega$ *such that* $f(\varsigma) = 0$*. Further, if* $w(\varsigma, \varsigma) = 0$*, then* $\varsigma \in T\varsigma$*.*

Remark 2.6. *Using the technique of Theorem 2.16, we can also unify the ideas of [17, 34, 35, 37].*

2.11 Liu et al. (2010)

Liu *et al.* [42] introduced three concepts of multi-valued contractions in complete MS and also proved the existence of FPs for the multi-valued contractions under weaker conditions than the ones in [14, 15, 33].

Definition 2.8. *For any* $\pounds, Ł \in \complement(\Omega)$ *and* $\mu \in \Omega$*, let* $\rho(\mu, \pounds) = \inf_{\nu \in \pounds} \rho(\mu, \nu)$ *and*

$$H(\pounds, Ł) = \begin{cases} \max\left\{\sup_{\mu \in \pounds} \rho(\mu, Ł), \sup_{\nu \in Ł} \rho(\nu, \pounds)\right\}, & \text{if the maximum exists,} \\ +\infty & \text{otherwise.} \end{cases}$$

Such a mapping H *is called a generalized Hausdorff metric in* $\complement(\Omega)$ *induced by* ρ*.*

Theorem 2.26. *Let* T *be a MVM from a CMS* (Ω, ρ) *into* $\complement(\Omega)$ *such that for each* $\mu \in \Omega$*, there exists* $\nu \in T\mu$ *satisfying*

$$\alpha(f(\mu))\rho(\mu, \nu) \leq f(\mu) \quad and \quad f(\nu) \leq \beta(f(\mu))\rho(\mu, \nu)),$$

where

$$U = \begin{cases} [0, \sup f(\Omega)], & if \ \sup f(\Omega) < +\infty, \\ [0, +\infty), & if \ \sup f(\Omega) = +\infty. \end{cases}$$

$\alpha : U \longrightarrow (0,1]$ *and* $\beta : U \longrightarrow [0,1)$ *satisfy that*

$$\liminf_{r \longrightarrow 0^+} \alpha(r) > 0, \quad \limsup_{r \longrightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in [0, \sup f(\Omega)).$$

Then

(A1) for each $\mu_0 \in \Omega$*, there exists an orbit* $\{\mu_n\}$ *of* T *and* $\varsigma \in \Omega$ *such that* $\lim\limits_{n \longrightarrow 0} \mu_n = \varsigma$*;*

(A2) ς *is a FP of* T *in* Ω *if and only if the function* f *is T-orbitally LSC at* ς*.*

Theorem 2.27. *Let* T *be a MVM from a CMS* (Ω, ρ) *into* $\complement(\Omega)$ *such that*

$$f(\nu) \leq \varphi(f(\mu))\rho(\mu, \nu), \qquad \forall (\mu, \nu) \in \Omega \times T\mu,$$

where $\varphi : Ł \longrightarrow [0,1)$ *satisfies that*

$$\liminf_{r \longrightarrow 0^+} \varphi(r) > 0, \quad \limsup_{r \longrightarrow t^+} \varphi(r) < 1, \quad \forall t \in [0, \sup f(\Omega)).$$

Then

(A1) for each $\mu_0 \in \Omega$*, there exists an orbit* $\{\mu_n\}$ *of* T *and* $\varsigma \in \Omega$ *such that* $\lim\limits_{n \longrightarrow 0} \mu_n = \varsigma$*;*

(A2) ς *is a FP of* T *in* Ω *if and only if the function* f *is T-orbitally LSC at* ς*.*

Theorem 2.28. *Let* T *be a MVM from a CMS* (Ω, ρ) *into* $\complement(\Omega)$ *such that for each* $\mu \in \Omega$ *there exists* $\nu \in T\mu$ *satisfying*

$$\alpha(\rho(\mu, \nu))\rho(\mu, \nu) \leq f(\mu) \quad and \quad f(\nu) \leq \beta(\rho(\mu, \nu))\rho(\mu, \nu)),$$

where

$$V = \begin{cases} [0, diam(\Omega)], & if \ diam(\Omega) < +\infty, \\ [0, +\infty), & if \ diam(\Omega) = +\infty. \end{cases}$$

$\alpha : V \longrightarrow (0,1]$ *and* $\beta : V \longrightarrow [0,1)$ *satisfy that*

$$\liminf_{r \longrightarrow 0^+} \alpha(r) > 0, \quad \limsup_{r \longrightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in [0, \operatorname{diam} f(\Omega)),$$

and one of α *and* β *is nondecreasing. Then*

(A1) for each $\mu_0 \in \Omega$*, there exists an orbit* $\{\mu_n\}$ *of* T *and* $\varsigma \in \Omega$ *such that* $\lim\limits_{n \longrightarrow 0} \mu_n = \varsigma$*;*

(A2) ς *is a FP of* T *in* Ω *if and only if the function* f *is T-orbitally LSC at* ς*.*

Remark 2.7. *The idea of Lui et al. (2010) is a proper extension of the work of [15, 18] and [33] as demonstrated by Examples 3.2, 3.4 and 3.10 respectively in [42].*

2.12 Nicolae (2011)

Nicolae [48] presented some FP theorems for multi-valued contraction mappings that generalized the results proved by Feng and Liu [18], Klim and Wardowski [33], or Ciric [14].

Theorem 2.29. *([48]) Let* (Ω,ρ) *be a CMS and* T *be a MVM and* $f:\Omega\longrightarrow\mathbb{R}$*, given by* $f(\mu)=D(\mu,T\mu)$*,* $\mu\in\Omega$*, is LSC. Suppose that there exist functions* $\varphi:[0,\infty)\longrightarrow[0,1)$*,* $\eta:[0,\infty)\longrightarrow[0,b]$*,* $b\in(0,1)$*, such that*

$$\varphi(t)\leq\eta(t),\quad \limsup_{r\longrightarrow t^{+}}\frac{\varphi(r)}{\eta(r)}<1,\quad \forall t\in[0,\infty),$$

and for any $\mu\in\Omega$*, there is* $y\in T\mu$ *satisfying*

$$\eta(f(\mu))\rho(\mu,\nu)\leq f(\mu),$$

and

$$f(\nu)\leq\varphi(f(\mu))\rho(\mu,\nu)).$$

Then T *has a FP.*

Theorem 2.30. *([48]) Let* (Ω,ρ) *be a CMS and* T *be a MVM and* $f:\Omega\longrightarrow\mathbb{R}$*, defined* $f(\mu)=D(\mu,T\mu)$*,* $\mu\in\Omega$*, is a LSC. Suppose that there exist functions* $\varphi:[0,\infty)\longrightarrow[0,1)$*,* $\eta:[0,\infty)\longrightarrow[0,b]$*,* $b\in(0,1)$*, such that*

$$\varphi(t)\leq\eta(t),\quad \limsup_{r\longrightarrow t^{+}}\frac{\varphi(r)}{\eta(r)}<1,\quad \forall t\in[0,\infty),$$

and for any $\mu\in\Omega$*, there is* $\nu\in T\mu$ *satisfying*

$$\eta((\rho(\mu,\nu))\rho(\mu,\nu)\leq\rho(\mu,\nu),$$

and

$$f(\nu)\leq\varphi(\rho(\mu,\nu))\rho(\mu,\nu)).$$

Then T *has a FP.*

2.13 Latif and Abdou (2011)

Latif and Abdou [36] introduced some notions of generalized nonlinear contractive mappings on a CMS with w-distance and proved some FP results for such maps.

Theorem 2.31. *Let* (Ω,ρ) *be a CMS and* $T:\Omega\longrightarrow\mathtt{C}(\Omega)$ *be a MVM. Assume that the following conditions hold:*

(i) there exist $b\in(0,1)$ *and a function* $\varphi:(0,\infty)\longrightarrow[b,1)$ *satisfying (2.1);*

(ii) *the mapping* $f : \Omega \longrightarrow \mathbb{R}$*, given by* $f(\mu) = w(\mu, T\mu)$*,* $\mu \in \Omega$*, is LSC;*

(iii) *for any* $\mu \in \Omega$*, there is* $\nu \in T\mu$ *satisfying*

$$\sqrt{\varphi(f(\mu))}w(\mu,\nu) \leq f(\mu),$$

and

$$f(\nu) \leq \varphi(f(\mu))w(\mu,\nu).$$

Then there exists $\varsigma \in \Omega$ *such that* $f(\varsigma) = 0$*. Further, if* $w(\varsigma,\varsigma) = 0$*, then* $\varsigma \in T\varsigma$*.*

Theorem 2.32. *Let* (Ω, ρ) *be a CMS and* $T : \Omega \longrightarrow \complement(\Omega)$ *be a MVM. Assume that the following conditions hold:*

(i) *there exist* $b \in (0,1)$ *and a function* $\varphi : (0,\infty) \longrightarrow [b,1)$ *satisfying inequality (2.1);*

(ii) *the mapping* $f : \Omega \longrightarrow \mathbb{R}$*, given by* $f(\mu) = w(\mu, T\mu)$*,* $\mu \in \Omega$*, is a LSC;*

(iii) *for any* $\mu \in \Omega$*, there is* $\nu \in T\mu$ *satisfying*

$$\sqrt{\varphi(w(\mu,\nu))}w(\mu,\nu) \leq w(\mu, T\mu),$$

and

$$w(\nu, T\nu) \leq \varphi(w(\mu,\nu))w(\mu,\nu).$$

Then there exists $\varsigma \in \Omega$ *such that* $f(\varsigma) = 0$*. Further, if* $w(\varsigma,\varsigma) = 0$*, then* $\varsigma \in T\varsigma$*.*

2.14 Lin and Chuang (2011)

Lin and Chuang [39] presented some new FP theorems for generalized nonlinear contractive MVMs.

Lin and Du [40] introduced the concept of the τ-function on a MSas follows.

Definition 2.9. *Let* (Ω, ρ) *be a MS. A function* $\tau : \Omega \times \Omega \longrightarrow [0,\infty)$ *is called a* τ*-function if the following conditions hold:*

(τ_1) *for each* $\mu,\nu,\varsigma \in \Omega$*,* $\tau(\mu,\varsigma) \leq \tau(\mu,\nu) + \tau(\nu,\varsigma)$*;*

(τ_2) *If* $\mu \in \Omega$ *and* $\{\nu_n\} \subseteq \Omega$ *with* $\lim_{n\longrightarrow\infty} \nu_n = \nu$ *such that* $\tau(\mu,\nu_n) \leq M$ *for some* $M = M(\mu) > 0$*, then* $\tau(\mu,\nu) \leq M$*;*

(τ_3) *for any sequence* $\{\mu_n\}$ *in* Ω *with* $\limsup_{n\longrightarrow\infty}\{\tau(\mu_n,\mu_m) : m > n\} = 0$*, if there exists a sequence* $\{\nu_n\}$ *such that* $\lim_{n\longrightarrow\infty} \tau(\mu_n,\nu_n) = 0$*, then* $\lim_{n\longrightarrow\infty} \rho(\mu_n,\nu_n) = 0$*;*

(τ_4) *for* $\mu,\nu,\varsigma \in \Omega$*,* $\tau(\mu,\nu) = 0$ *and* $\tau(\mu,\varsigma) = 0$ *imply* $\nu = \varsigma$*.*

Theorem 2.33. *([39]) Let (Ω,ρ) be a CMS and $\tau : \Omega \times \Omega \longrightarrow [0,\infty)$ be a function with properties $(\tau 1)$,$(\tau 2)$, and $(\tau 3)$. Let $T : \Omega \longrightarrow \mathtt{C}(\Omega)$ be a MVM and $f : \Omega \longrightarrow \mathbb{R}$ be given by $f(\mu) = \tau(\mu, T\mu)$ for each $\mu \in \Omega$. Let $0 < a < 1$, and let $\phi : [0,\infty) \longrightarrow [0,1)$ and $\varphi : [0,\infty) \longrightarrow [a,1)$ be two functions with the properties:*

$$\limsup_{r \longrightarrow t^+} \frac{\varphi(r)}{\phi(r)} < 1 \text{ for each } t \in [0,\infty).$$

For each $\mu \in \Omega$, there exists $\nu \in T\mu$ such that

$$\phi(\tau(\mu, T\mu)) \cdot \tau(\mu,\nu) \leq \tau(\mu, T\mu) \text{ and } \tau(\nu, T(\nu)) \leq \varphi(\tau(\mu, T\mu)) \cdot \tau(\mu,\nu).$$

Then there exists $\overline{\mu} \in \Omega$ such that

(A1) if T is closed (i.e., $Gr(T) = \{(\mu,\nu) \in \Omega \times \Omega : \nu \in T\mu\}$ is a closed set), then $\overline{\mu} \in T\overline{\mu}$;

(A2) if f is LSC, then $\tau(\overline{\mu}, T\overline{\mu}) = 0$. Furthermore, if $\tau(\overline{\mu},\overline{\mu}) = 0$, then $\overline{\mu} \in T\overline{\mu}$;

(A3) $\inf\{\tau(\mu,\varsigma) + \tau(\mu, T\mu) : \mu \in \Omega\} > 0$ for each $\varsigma \in \Omega$ with $\varsigma \notin T\varsigma$ implies that $\overline{\mu} \in T\overline{\mu}$.

Theorem 2.34. *([39]) Let (Ω,ρ) be a CMS and $\tau : \Omega \times \Omega \longrightarrow [0,\infty)$ be a function with properties $(\tau 1)$,$(\tau 2)$, and $(\tau 3)$. Let $T : \Omega \longrightarrow \mathtt{C}(\Omega)$ be a MVM and $f : \Omega \longrightarrow \mathbb{R}$ be given by $f(\mu) = \tau(\mu, T\mu)$ for each $\mu \in \Omega$. Let $\phi : [0,\infty) \longrightarrow [0,1)$ and $\varphi : [0,\infty) \longrightarrow [a,1)$, $0 < a < 1$, be two functions with the properties:*

$$\varphi(t) \leq (\phi(t))^2 \text{ for each } t \in [0,\infty),$$

and

$$\limsup_{r \longrightarrow t^+} \phi(r) < 1 \text{ for each } t \in [0,\infty).$$

For each $\mu \in \Omega$, there exists $\nu \in T\mu$ such that

$$\phi(\tau(\mu, T\mu)) \cdot \tau(\mu,\nu) \leq \tau(\mu, T\mu), \text{ and } \tau(\nu, T(\nu)) \leq \varphi(\tau(\mu, T\mu)) \cdot \tau(\mu,\nu).$$

Then there exists $\overline{\mu} \in \Omega$ such that

(A1) if T is closed, then $\overline{\mu} \in T\overline{\mu}$;

(A2) if f is LSC, then $\tau(\overline{\mu}, T\overline{\mu}) = 0$. Furthermore, if $\tau(\overline{\mu},\overline{\mu}) = 0$, then $\overline{\mu} \in T\overline{\mu}$;

(A3) $\inf\{\tau(\mu,\varsigma) + p(\mu, T\mu) : \mu \in \Omega\} > 0$ for each $\varsigma \in \Omega$ with $\varsigma \notin T\varsigma$ implies that $\overline{\mu} \in T\overline{\mu}$.

Theorem 2.35. *([39]) Let (Ω,ρ) be a CMS and $\tau : \Omega \times \Omega \longrightarrow [0,\infty)$ be a function with properties $(\tau 1)$,$(\tau 2)$, and $(\tau 3)$. Let $T : \Omega \longrightarrow \mathtt{C}(\Omega)$ be a MVM and $f : \Omega \longrightarrow \mathbb{R}$ be given by $f(\mu) = \tau(\mu, T\mu)$ for each $\mu \in \Omega$. Let $\varphi : [0,\infty) \longrightarrow [0,1)$, be a function with the properties:*

$$\limsup_{r \longrightarrow t^+} \varphi(r) < 1 \quad \text{for each } t \in [0,\infty).$$

For each $\mu \in \Omega$, there exists $\nu \in T\mu$ such that

$$\tau(\mu,\nu) \leq (2 - \varphi(\tau(\mu,T\mu))) \cdot \tau(\mu,T\mu), \text{ and } \tau(\nu,T\nu) \leq \varphi(\tau(\mu,T\mu)) \cdot \tau(\mu,\nu).$$

Then there exists $\overline{\mu} \in \Omega$ such that

(A1) if T is closed, then $\overline{\mu} \in T\overline{\mu}$;

(A2) if f is LSC, then $\tau(\overline{\mu},T\overline{\mu}) = 0$. Furthermore, if $\tau(\overline{\mu},\overline{\mu}) = 0$, then $\overline{\mu} \in T\overline{\mu}$;

(A3) $\inf\{\tau(\mu,\varsigma) + \tau(\mu,T\mu) : \mu \in \Omega\} > 0$ for each $\varsigma \in \Omega$ with $\varsigma \notin T\varsigma$ implies that $\overline{\mu} \in T\overline{\mu}$.

Remark 2.8. *The work of Lin and Chuang [39] is a proper generalization of the work of Nicolae [48]. Due to the fact that τ-function is not a metric, it should be noticed that the work of Nicolae [48] can be taken to quasi-metric. In addition, Theorem 2.35 generalized Theorem 2.5 of Ciric [14] and Theorem 2.20 of Latif and Abdou [37] (see [39, Remark 3.2 and Remark 3.3]).*

2.15 Bin Dehaish and Latif (2012)

Bin Dehaish and Latif [12] proved some results on the existence of FP for generalized contractive MVMs with respect to u-distance.

Ume [62] introduced the notion of u-distance as a generalization of τ-distance. A function $u : \Omega \times \Omega \longrightarrow \mathbb{R}_+$ is called a u-distance on Ω if there exists a function $\theta : \Omega \times \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that the following hold for $\mu,\nu,\varsigma \in \Omega$:

(u_1) $u(\mu,\varsigma) \leq u(\mu,\nu) + u(\nu,\varsigma)$;

(u_2) $\theta(\mu,\nu,0,0) = 0$ and $\theta(\mu,\nu,s,t) \geq \min\{s,t\}$ for each $s,t \in \mathbb{R}_+$, and for every $\epsilon > 0$, there exists $\delta > 0$ such that $|s - s_0| < \delta, |t - t_0| < \delta, s, s_0, t, t_0 \in \mathbb{R}_+$ and $\nu \in \Omega$ imply

$$|\theta(\mu,\nu,s,t) - \theta(\mu,\nu,s_0,t_0)| < \epsilon;$$

(u_3) $\lim\limits_{n\longrightarrow\infty} \mu_n = \mu$, $\lim\limits_{n\longrightarrow\infty} \sup\{\theta(w_n,\varsigma_n,u(w_n,\mu_m),u(\varsigma_n,\mu_m)) : m \geq n\} = 0$ imply

$$u(\nu,\mu) \leq \liminf_{n\longrightarrow\infty} u(\nu,\mu_n);$$

(u_4) $\lim\limits_{n\longrightarrow\infty} \sup\{u(\mu_n,w_m)) : m \geq n\} = 0$, $\lim\limits_{n\longrightarrow\infty} \sup\{u(\nu_n,\varsigma_m)) : m \geq n\} = 0$, $\lim\limits_{n\longrightarrow\infty} \theta(\mu_n,w_n,s_n,t_n) = 0$, $\lim\limits_{n\longrightarrow\infty} \theta(\nu_n,\varsigma_n,s_n,t_n) = 0$ imply

$$\lim_{n\longrightarrow\infty} \theta(w_n,\varsigma_n,s_n,t_n) = 0;$$

or
$\lim\limits_{n\longrightarrow\infty} \sup\{u(w_n,\mu_m)) : m \geq n\} = 0$, $\lim\limits_{n\longrightarrow\infty} \sup\{u(\varsigma_n,\nu_m)) : m \geq n\} = 0$, $\lim\limits_{n\longrightarrow\infty} \theta(\mu_n,w_n,s_n,t_n) = 0$, $\lim\limits_{n\longrightarrow\infty} \theta(\nu_n,\varsigma_n,s_n,t_n) = 0$ imply

$$\lim_{n\longrightarrow\infty} \theta(w_n,\varsigma_n,s_n,t_n) = 0;$$

(u_5) $\lim_{n\longrightarrow\infty}\theta(w_n,\varsigma_n,p(w_n,\mu_n),p(\varsigma_n,\mu_n))=0$, $\lim_{n\longrightarrow\infty}\theta(w_n,\varsigma_n,u(w_n,\nu_n),u(\varsigma_n,\nu_n))=0$ imply

$$\lim_{n\longrightarrow\infty}\rho(\mu_n,\nu_n)=0;$$

or

$\lim_{n\longrightarrow\infty}\theta(a_n,b_n,u(\mu_n,a_n),u(\mu_n,b_n))=0$, $\lim_{n\longrightarrow\infty}\theta(a_n,b_n,u(\nu_n,a_n),u(\nu_n,b_n))=0$ imply

$$\lim_{n\longrightarrow\infty}\rho(\mu_n,\nu_n)=0.$$

Definition 2.10. *([12]) A MVM $T:\Omega\longrightarrow \complement(\Omega)$ is called a generalized u-contractive if there exist a u-distance u on Ω and a constant $b\in(0,1)$ such that, for any $\mu\in\Omega$, there is $\nu\in J_b^{\mu}$ satisfying*

$$u(\nu,T\nu)\leq\varphi(u(\mu,\nu))u(\mu,\nu), \tag{2.10}$$

where $J_b^{\mu}=\{\nu\in T\mu : bu(\mu,\nu)\leq u(\mu,T\mu)\}$ and φ is a function from $[0,\infty)$ to $[0,b)$ with $\limsup_{r\longrightarrow t^+}\varphi(r)<b$ for each $t\in[0,\infty)$.

Theorem 2.36. *([12]) Let (Ω,ρ) be a CMS and let $T:\Omega\longrightarrow\complement(\Omega)$ be a generalized u-contractive mapping. Suppose that a real valued function f on Ω defined by $f(\mu)=u(\mu,T\mu)$ is T-orbitally LSC. Then there exists $\varsigma\in\Omega$ such that $\varsigma\in Fix(T)$ and $u(\varsigma,\varsigma)=0$.*

Remark 2.9. *In accordance with the result of Ciric [14,15], the main idea of Bin Dehaish [12] can be improved in the following forms:*

Theorem 2.37. *Let (Ω,ρ) be a CMS and T be a generalized u-contractive mapping from Ω into $\complement(\Omega)$. Suppose that the function $f:\Omega\longrightarrow\mathbb{R}$, given by $f(\mu)=u(\mu,T\mu)$, $\mu\in\Omega$, is T-orbitally LSCand that there exists a function $\varphi:[0,\infty)\longrightarrow[a,1)$, $a>0$, satisfying Equation (2.1) and for any $\mu\in\Omega$ there is $\nu\in T\mu$ satisfying the following two conditions:*

$$u(\mu,\nu)\leq(2-(u(\mu,\nu)))u(\mu,T\mu),$$

and

$$u(\nu,T\nu)\leq\varphi(u(\mu,T\mu))u(\mu,\nu).$$

Then there exists $\varsigma\in\Omega$ such that $\varsigma\in T\varsigma$.

Theorem 2.38. *Let (Ω,ρ) be a CMS and T be a generalized u-contractive mapping from Ω into $\complement(\Omega)$. Suppose that the function $f:\Omega\longrightarrow\mathbb{R}$, given by $f(\mu)=u(\mu,T\mu)$, $\mu\in\Omega$, is T-orbitally LSC and that there exists a function $\varphi:[0,\infty)\longrightarrow[a,1)$, $a>0$, satisfying Equation (2.1) and for any $\mu\in\Omega$ there is $\nu\in T\mu$ satisfying the following two conditions:*

$$\sqrt[\frac{n-1}{n}]{\varphi(u(\mu,T\mu))u(\mu,\nu)}\leq u(\mu,T\mu),$$

and

$$u(\nu,T\nu)\leq\varphi(u(\mu,T\mu))u(\mu,\nu).$$

Then there exists $\varsigma\in\Omega$ such that $\varsigma\in T\varsigma$.

Taking $\rho=u$ in Example 2.1, this observation can also demonstrated.

2.16 Kamran (2012)

Kamran [31] generalized the results of Feng and Liu [18] and Klim and Wardowski [33] in the setting of single-valued and MVMs.

Definition 2.11. *([32]) Let $T : \Omega \longrightarrow \complement(\Omega)$ and $f : \Omega \longrightarrow \Omega$. Then T is called f-contraction if there exists $\alpha \in (0,1)$ such that $H(T\mu, T\nu) \leq \alpha\rho(f\mu, f\nu)$. A point $\mu \in \Omega$ is called a coincidence point of f and T if $f\mu = T\mu$.*

Definition 2.12. *Let $f : \Omega \longrightarrow \Omega$ be a mapping. We say that a function $g : \Omega \longrightarrow \mathbb{R}$ is lower f -semi-continuous if for any sequence $\{\mu_n\} \subset \Omega$ and $\mu \in \Omega$*

$$f\mu_n \longrightarrow f\mu \quad implies \quad g\mu \leq \liminf_{n \longrightarrow \infty} g\mu_n.$$

Let $T : \Omega \longrightarrow \eth(\Omega)$ and $f : \Omega \longrightarrow \Omega$ be mappings. Define $g : \Omega \longrightarrow \mathbb{R}$ by $g(\mu) = \rho(f\mu, T\mu)$. Further, for a positive constant $b \in (0,1)$ and $TX \subseteq fX$, define the set $I_b^{f(\mu)}$ by $I_b^{f(\mu)} = \{\nu \in \Omega : f\nu \in T\mu : b\rho(f\mu, f\nu) \leq \rho(f\mu, T\mu)\}$.

Theorem 2.39. *Let (Ω, ρ) be a MS, f be a mapping from Ω into Ω and T be a mapping from Ω into $\complement(\Omega)$. Assume that the following conditions hold:*

(i) There exist $b \in (0,1)$ and $\varphi : [0,1) \longrightarrow (0.b)$ such that for all $t \in [0,1)$,

$$\limsup_{r \longrightarrow t^+} \varphi(r) < b,$$

(ii) for any $\mu \in \Omega$, there is $\nu \in I_b^{f(\mu)}$ satisfying $\rho(f\nu, T\nu) \leq \varphi(\rho(f\mu, f\nu))(\rho(f\mu, f\nu)$.

Then f and T have a coincidence point in Ω provided that fX is complete and g is lower f-semi-continuous.

Theorem 2.40. *Let (Ω, ρ) be a MS, f be a mapping from Ω into Ω and T be a mapping from Ω into $\complement(\Omega)$. If there exists a constant $\alpha \in (0,1)$ such that for any $\mu \in \Omega$, there is $\nu \in I_b^{f(\mu)}$ satisfying*

$$\rho(f\nu, T\nu) \leq \alpha(\rho(f\mu, f\nu)),$$

then f and T have a coincidence point in Ω provided that fX is complete, $\alpha < b$ and g is lower f-semi-continuous.

Remark 2.10. *Clearly, if the mapping f is reduced to identity mapping on Ω, then Theorem 2.39 due to Kamran [31] reduced to Theorem 2.3 of Klim and Wardoski [33]. Similarly, if $\varphi(t) = \lambda$, $\lambda \in (0,1)$ is a constant mapping, then Theorem 2.39 reduces to Theorem 1.2 of Feng and Liu. However, Theorem 2.39 is a proper generalisation of Theorem 1.2 (see [31, Example 2.5]).*

2.17 Amini-Harandi et al. (2013)

Amini-Harandi et al. [8] obtained some FP theorems for new set-valued contractions in complete MS. Then, by using these results and the scalarization method, they present some FP theorems for set-valued contractions in complete cone MS without the normality assumption.

Definition 2.13. *Let P be a nonempty closed convex cone of a real Banach space E. A partial ordering $\preceq$ with respect to P is defined by:*

$$\mu \preceq \nu \Longleftrightarrow \nu - \mu \in P.$$

We shall write $\mu \prec \nu$ if $\mu \preceq \nu$ and $\mu \neq \nu$. If $intP \neq \emptyset$, then we write $\mu \ll \nu$ if and only if $\nu - \mu \in intP$. A convex subset Ł of a nonempty closed convex cone P is said to be a base of P if $0 \notin \overline{Ł}$ and $P = \cup_{t \geq 0} tB$. Let E^* be the topological dual space of E. Set

$$P^* = \{\varphi \in E^* : \varphi(\mu) \geq 0, \ \forall \mu \in P\}.$$

P^* is called the dual cone of P.

Definition 2.14. *[22] Let Ω be a nonempty set. Suppose that the mapping $\rho : \Omega \times \Omega \longrightarrow E$ satisfies:*

(i) $0 \preceq \rho(\mu,\nu)$ *for all* $\mu,\nu \in \Omega$ *and* $\rho(\mu,\nu) = 0$ *if and only if* $\mu = \nu$;

(ii) $\rho(\mu,\nu) = \rho(\nu,\mu)$ *for all* $\mu,\nu \in \Omega$;

(iii) $\rho(\mu,\varsigma) \preceq \rho(\mu,\nu) + \rho(\nu,\varsigma)$ *for all* $\mu,\nu,\varsigma \in \Omega$.

Then, ρ is called a cone metric on Ω and (Ω,ρ) is called a cone MS.

Amini-Harandi et al. [8] assumes (Ω,ρ) to be a complete cone MS. Define a function $f_\rho : \Omega \longrightarrow \mathbb{R}$ as $f_\rho(\mu) = \rho(\mu,T\mu)$, where $\rho(\mu,T\mu) = \inf_{\nu \in T\mu} \rho(\mu,\nu)$ and assume that f_ρ is LSC on Ω. For a scalar $b \in (0,1)$, set

$$I_b^\mu = \{\nu \in \Omega : b\rho(\mu,\nu) \leq \rho(\mu,T\mu)\},$$

and

$$I_b^\mu = \{\nu \in T\mu : b\rho(\mu,\nu) \leq \rho(\mu,T\mu)\}.$$

Theorem 2.41. *Assume that there exist $b \in (0,1)$ and $\varphi : [0,\infty) \longrightarrow [0,b)$ such that*

$$\limsup_{r \longrightarrow t} \varphi(r) < b, \ for\ each\ t \in [0,\infty).$$

If for any $\mu \in \Omega$ there is $\nu \in I_b^\mu$ such that

$$f_\rho(\nu) \leq \varphi(\rho(\mu,\nu)) \max\{\rho(\mu,\nu), f\rho(\mu)\}, \tag{2.11}$$

then T has a FP.

Remark 2.11. *Following the work of Kamran [31], one can obtain the following theorem.*

Theorem 2.42. *Assume that there exist $b \in (0,1)$ and $\varphi : [0,\infty) \longrightarrow [0,b)$ such that*

$$\limsup_{r \longrightarrow t} \varphi(r) < b, \ for\ each\ t \in [0,\infty).$$

If for any $\mu \in \Omega$, there is $\nu \in I_b^{f(\mu)}$ such that

$$\rho(f\nu, T\nu) \leq \varphi(\rho(f\mu, f\nu)) \max\{\rho(f\mu, f\nu), \rho(f\mu, T\mu)\}, \tag{2.12}$$

then T has a FP.

We noticed that if f is an identity mapping, inequality (2.12) *reduced to inequality* (2.11) *of Amini-Harandi.*

2.18 Liu et al. (2013)

Liu et al. [41] proved some FP theorems for nonlinear multi-valued contraction mappings in complete MS with w-distance.

Theorem 2.43. *Let (Ω, ρ) be a CMS with w-distance w and let T be a MVM from Ω into $\complement(\Omega)$ such that for each $\mu \in \Omega$, there exists $\nu \in T\mu$ satisfying*

$$\alpha(f_w(\mu))\rho(\mu,\nu) \leq f_w(\mu) \quad and \quad f_w(\nu) \leq \beta(f_w(\mu))\rho(\mu,\nu)),$$

where

$$U_p = \begin{cases} [0, \sup f_w(\Omega)], & if\ \sup f_w(\Omega) < +\infty, \\ [0, +\infty), & if\ \sup f_w(\Omega) = +\infty, \end{cases}$$

$\alpha : U_w \longrightarrow (0,1]$ and $\beta : U_w \longrightarrow [0,1)$ satisfy that

$$\liminf_{r \longrightarrow 0^+} \alpha(r) > 0, \quad \limsup_{r \longrightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in U_w.$$

Then

(A1) for each $\mu_0 \in \Omega$ there exists an orbit $\{\mu_n\}$ of T and $\varsigma \in \Omega$ such that $\lim_{n \longrightarrow 0} \mu_n = \varsigma$ for some $\varsigma \in \Omega$;

(A2) $f_w(\varsigma) = 0$ if and only if the function f is T-orbitally LSC at ς;

(A3) $\varsigma \in T\varsigma$ provided $w(\varsigma,\varsigma) = 0 = f_w(\varsigma)$.

Theorem 2.44. *Let (Ω, ρ) be a CMS with w-distance w and let T be a MVM from Ω into $\complement(\Omega)$ such that for each $\mu \in \Omega$, there exists $\nu \in T\mu$ satisfying*

$$\alpha(w(\mu,\nu))w(\mu,\nu) \leq f_w(\mu) \quad and \quad f_w(\nu) \leq \beta(w(\mu,\nu))w(\mu,\nu)),$$

where

$$V_w = \begin{cases} [0, diam(\Omega_w)], & if\ diam(\Omega_w) < +\infty, \\ [0, +\infty), & if\ diam(\Omega_w) = +\infty, \end{cases}$$

$\alpha : V_w \longrightarrow (0,1]$ *and* $\beta : V_w \longrightarrow [0,1)$ *satisfy that*

$$\liminf_{r \longrightarrow 0^+} \alpha(r) > 0, \quad \limsup_{r \longrightarrow t^+} \frac{\beta(r)}{\alpha(r)} < 1, \quad \forall t \in V_w,$$

and one of α *and* β *is nondecreasing. Then conditions* $(A_1), (A_2) and (A_3) of Theorem 2.43 hold$.

Theorem 2.45. *Let* (Ω, ρ) *be a CMS with w-distance w and let T a MVM from* Ω *into* $\complement(\Omega)$ *such that*

$$f_w(\nu) \leq \varphi(w(\mu, \nu)) w(\mu, \nu))\ \forall (\mu, \nu) \in \Omega \times T\mu,$$

where $\varphi : U_w \longrightarrow (0,1)$ *satisfies that* $\liminf_{r \longrightarrow 0^+} \varphi(r) > 0$ *and* $\limsup_{r \longrightarrow t^+} \varphi(r) < 1, \forall t \in U_w$.
Then conditions $(A_1), (A_2) and (A_3) of Theorem 2.43 hold$.

Remark 2.12. *The main contribution of Liu et al. [41] is the introduction of w-distance in the work of Liu et al. [42]. It is interesting to note that Theorem 2.43 extends the main ideas of [15, 18, 33] and [42]. For some examples in this regards, see [41, Examples 4.1-4.7].*

2.19 Hussain et al. (2014)

Hussain et al. [27] used the concept of w-distance and obtained FP results for multi-valued generalized w-contractive mappings.

Theorem 2.46. *([37]) Let* Ω *be a CMS with a w-distance p. Let* $T : \Omega \longrightarrow \complement(\Omega)$ *be a MVM. Suppose that the function* $f : \Omega \longrightarrow \mathbb{R}$*, given by* $f(\mu) = w(\mu, T\mu)$*,* $\mu \in \Omega$*, is LSCand there exists a function* $\varphi : (0, \infty) \longrightarrow [a, 1)$*,* $0 < a < 1$*, satisfying*

$$\limsup_{r \longrightarrow t^+} \varphi(r) < 1,\ for\ all\ t \in (0, \infty),$$

such that for any $\mu \in \Omega$*, there is* $\nu \in T\mu$ *satisfying the following conditions:*

$$[\varphi(f(\mu))]^r w(\mu, \nu) \leq f(\mu),\ \text{where } 0 < r < 1,$$

and

$$f(\nu) \leq \varphi(f(\mu)) w(\mu, \nu).$$

Then there exists $\varsigma \in \Omega$ *such that* $f(\varsigma) = 0$*. Further, if* $w(\varsigma, \varsigma) = 0$*, then* $\varsigma \in T\varsigma$*.*

Remark 2.13. *Theorem 2.46 properly contains the main ideas of Ciric [15] and that of Latif and Abdou [36]. In fact if* $w = \rho$ *and* $r \longrightarrow k$ *for* $k \in (0,1)$*, then the above mentioned results can be deduce from Theorem 2.46.*

From the surveyed literature, we notice that the result which is more unifying is the work of Hussain et al. [27].

2.20 Minak et al. (2015)

Minak et al. [43] considered the Wardowski's technique and gave many FP results for multi-valued maps on CMS.

Wardoski [64] introduced the concept of F-contraction as follows:

Let $F : (0, \infty) \longrightarrow \mathbb{R}$ be a function. For the sake of completeness, we will consider the following conditions:

(F_1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

(F_2) for each sequence $\{\alpha_n\}$ of positive numbers $\lim_{n \longrightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \longrightarrow \infty} F(\alpha_n) = -\infty$;

(F_3) there exists $k \in (0, 1)$ such that $\lim_{\alpha \longrightarrow 0^+} \alpha^k F(\alpha) = 0$;

(F_4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$. The set of all functions F satisfying (F1)-(F3) and (F1)-(F4) are denoted by $\mathfrak{F}$ and $\mathfrak{F}_*$, respectively .

Definition 2.15. *([64]) Let (Ω, ρ) be a MS and $T : \Omega \longrightarrow \Omega$ be a mapping. Then, we say that T is an F-contraction if $F \in \mathfrak{F}$ and there exists $\tau > 0$ such that*

$$\forall \mu, \nu \in \Omega \, [\rho(T\mu, T\nu) > 0 \Longrightarrow \tau + F(\rho(T\mu, T\nu)) \leq F(\rho(\mu, \nu))].$$

Let $T : \Omega \longrightarrow \eth(\Omega)$ be a MVM, $F \in \mathfrak{F}$ and $\sigma > 0$. For $\mu \in \Omega$, with $\rho(\mu, T\mu) > 0$, define the set $F_\sigma^\mu \subseteq \Omega$ as

$$F_\sigma^\mu = \{\nu \in T\mu : F(\rho(\mu, \nu)) \leq F(\rho(\mu, T\mu)) + \sigma\}.$$

Theorem 2.47. *([43]) Let (Ω, ρ) be a CMS, $T : \Omega \longrightarrow \yen(\Omega)$ be a MVM and $F \in \mathfrak{F}$. If there exists $\tau > 0$ such that for any $\mu \in \Omega$ with $\rho(\mu, T\mu) > 0$, there exists $\nu \in F_\sigma^\mu$ satisfying*

$$\tau + F(\rho(\nu, T\nu)) \leq F(\rho(\mu, \nu)).$$

Then T has a FP in Ω provided $\sigma < \tau$ and $\mu \longrightarrow \rho(\mu, T\mu)$ is LSC.

Theorem 2.48. *([43]) Let (Ω, ρ) be a CMS, $T : \Omega \longrightarrow \complement(\Omega)$ be a MVM and $F \in \mathfrak{F}_*$. If there exists $\tau > 0$ such that for any $\mu \in \Omega$ with $\rho(\mu, T\mu) > 0$, there exists $\nu \in F_\sigma^\mu$ satisfying*

$$\tau + F(\rho(\nu, T\nu)) \leq F(\rho(\mu, \nu)),$$

then T has a FP in Ω provided $\sigma < \tau$ and $\mu \longrightarrow \rho(\mu, T\mu)$ is LSC.

2.21 Altun et al. (2015)

Altun et al. [6] demonstrated that multi-valued nonlinear F-contractions of Ciric type are weakly Picard operators on complete MS.

Theorem 2.49. *Let (Ω, ρ) be a CMS, $T : \Omega \longrightarrow \complement(\Omega)$ and $F \in \mathfrak{F}$. Assume that the following conditions hold:*

(i) the mapping $\mu \longrightarrow \rho(\mu, T\mu)$ *is LSC;*

(ii) there exists a function $\tau : (0, \infty) \longrightarrow (0, \sigma]$, $\sigma > 0$ *such that*

$$\liminf_{t \longrightarrow s^{+}} \tau(t) > 0, \ \forall s \geq 0;$$

(iii) for any $\mu \in \Omega$ *with* $\rho(\mu, T\mu) > 0$*, there is* $\nu \in T\mu$ *satisfying*

$$F(\rho(\mu,\nu)) \leq F(\rho(\mu,T\mu)) + \frac{\tau(\rho(\mu,T\mu))}{2} \quad \text{and} \quad \tau(\rho(\mu,T\mu)) + F(\rho(\nu,T\nu)) \leq F(\rho(\mu,\nu)). \tag{2.13}$$

Then T *is a weakly Picard operator.*

Remark 2.14. *It has been examplified in [6] that Theorem 2.49 is a proper generalization of the main ideas of [6,15,18,33]. In the following, we demonstrate some of the inclusiveness of these inequalities.*

Corollary 2.50. *The inequalities* (1.2), (2.2) *and* (2.9) *imply inequality* (2.13).

Proof. Suppose (2.9) holds and taking the natural logarithm of both sides, we have

$$\ln(\sqrt{\varphi(\rho(\mu,T\mu))}\rho(\mu,\nu)) \leq \ln(\rho(\mu,T\mu)),$$

and

$$\ln(\rho(\nu,T\nu)) \leq \ln(\varphi(\rho(\mu,T\mu))\rho(\mu,\nu)),$$

from which we have

$$\ln(\varphi(\rho(\mu,T\mu))^{\frac{1}{2}} + \ln(\rho(\mu,\nu))) \leq \ln(\rho(\mu,T\mu)),$$

and

$$\ln(\rho(\nu,T\nu)) \leq \ln(\varphi(\rho(\mu,T\mu)) + \ln(\rho(\mu,\nu))),$$

which implies

$$\frac{1}{2}\ln(\varphi(\rho(\mu,T\mu)) + \ln(\rho(\mu,\nu))) \leq \ln(\rho(\mu,T\mu)),$$

and

$$\ln(\rho(\nu,T\nu)) \leq \ln(\varphi(\rho(\mu,T\mu)) + \ln(\rho(\mu,\nu))),$$

and hence we have

$$\ln(\rho(\mu,\nu))) \leq \ln(\rho(\mu,T\mu)) - \frac{\ln(\varphi(\rho(\mu,T\mu))}{2},$$

and

$$-\ln(\varphi(\rho(\mu,T\mu)) + \ln(\rho(\nu,T\nu)) \leq \ln(\rho(\mu,\nu))),$$

putting $-\ln(\varphi(t)) = \tau(t)$ and $\ln(t) = F(t)$, $t > 0$, we obtain

$$F(\rho(\mu,\nu))) \leq F(\rho(\mu,T\mu)) + \frac{(\tau(\rho(\mu,T\mu))}{2},$$

and

$$\tau(\rho(\mu,T\mu)) + F(\rho(\nu,T\nu)) \leq F(\rho(\mu,\nu))).$$

□

Using the same argument above, we can show that the inequalities (1.2) *and* (2.2) *implies* (2.13). *The converse of Corollary 2.50 is not necessary true (see Altun et al. [6, Example 1]).*

2.22 Altun et al. (2016)

Altun et al. [7] unified the main ideas of [43] and obtained the following theorem.

Theorem 2.51. *Let* (Ω,ρ) *be a CMS,* $T:\Omega\longrightarrow \complement(\Omega)$ *and* $F\in\mathfrak{F}$. *Assume that the following conditions hold:*

(i) the mapping $\mu\longrightarrow\rho(\mu,T\mu)$ *is LSC;*

(ii) there exist $\sigma>0$ *and a function* $\tau:(0,\infty)\longrightarrow(\sigma,\infty)$ *such that*

$$\liminf_{t\longrightarrow s^+}\tau(t)>\sigma \textit{ for all } s>0$$

and for any $\mu\in\Omega$ *with* $\rho(\mu,T\mu)>0$, *there exists* $\nu\in F_\sigma^\mu$ *satisfying*

$$\tau(\rho(\mu,\nu))+F(\rho(\nu,T\nu))\leq F(\rho(\mu,\nu)). \tag{2.14}$$

Then T has a FP.

Remark 2.15. *The conctractive inequality* (2.14) *can be generalized via some of the methods used above. However, the inequality can be refined via the following inequalities:*

$$\tau(\rho(\mu,\nu))+F(\rho(\nu,T\nu))\leq F(\max\{\rho(\mu,\nu),\rho(\mu,T\mu),\rho(\nu,T\nu),\rho(\mu,T\nu),\rho(\nu,T\mu)\});$$
$$\tau(\rho(\mu,\nu))+F(\rho(\nu,T\nu))\leq F(\alpha_1\rho(\mu,\nu)+\alpha_2\rho(\mu,T\mu)+\alpha_3\rho(\nu,T\nu)+\alpha_4\rho(\mu,T\nu)),$$

with $\sum_{i=1}^{3}\alpha_i+\alpha_4<1$.

2.23 Iqbal and Hussain (2016)

Iqbal and Hussain [28] introduced the concept of α-η-semi-continuous MVMs and proved some FP theorem for multi-valued nonlinear F-contractions.

In 2012, Samet et al. [57] introduced the class of α-admissible mappings.

Definition 2.16. *Let* $\alpha:\Omega\times\Omega\longrightarrow[0,\infty)$ *be given mapping. A self mapping T is called* α-*admissible, if for all* $\mu,\nu\in\Omega$, *we have*

$$\alpha(\mu,\nu)\geq 1 \textit{ implies that } \alpha(T\mu,T\nu)\geq 1.$$

Asl et al. [9] extended these concepts to MVMs by introducing the notion of α_*-admissible mappings as follows.

Definition 2.17. *Let $T : \Omega \longrightarrow 2^{\Omega}$ be a MVM on a MS (Ω, ρ), $\alpha : \Omega \times \Omega \longrightarrow \mathbb{R}^{+}$ be a function. Then T is an α_*-admissible mapping, if*

$$\alpha(\nu, \varsigma) \geq 1 \text{ implies that } \alpha_*(T\nu, T\varsigma) \geq 1,\ \nu, \varsigma \in \Omega,$$

where

$$\alpha_*(£, Ł) = \inf_{\nu \in £, \varsigma \in Ł} \alpha(\nu, \varsigma).$$

Hussain et al. [26] modified the notion of α_*-admissible as follows:

Definition 2.18. *Let $T : \Omega \longrightarrow 2^{\Omega}$ be a MVM on a MS(Ω, ρ), $\alpha, \eta : \Omega \times \Omega \longrightarrow \mathbb{R}^{+}$ be two functions where η is bounded. Then T is an α_*-admissible mapping with respect to η, if*

$$\alpha(\nu, \varsigma) \geq \eta(\nu, \varsigma) \text{ implies that } \alpha_*(T\nu, T\varsigma) \geq \eta_*(T\nu, T\varsigma), \qquad \nu, \varsigma \in \Omega,$$

where

$$\alpha_*(£, Ł) = \inf_{\nu \in £, \varsigma \in Ł} \alpha(\nu, \varsigma), \qquad \eta_*(£, Ł) = \sup_{\nu \in £, \varsigma \in Ł} \eta(\nu, \varsigma).$$

Further, Ali et al. [1] generalized Definition 2.18 in the following way.

Definition 2.19. *Let $T : \Omega \longrightarrow 2^{\Omega}$ be a MVM on a MS (Ω, ρ), $\alpha, \eta : \Omega \times \Omega \longrightarrow \mathbb{R}^{+}$ be two functions. We say that T is a generalized α_*-admissible mapping with respect to η, if*

$$\alpha(\nu, \varsigma) \geq \eta(\nu, \varsigma) \text{ implies that } \alpha(u, v) \geq \eta(u, v), \text{ for all } u \in T\nu,\ v \in T\varsigma.$$

In 2014, Hussain et al. [25] introduced the notion of α-η-continuous mappings as follows:

Definition 2.20. *Let (Ω, ρ) be a MS, $\alpha, \eta : \Omega \times \Omega \longrightarrow [0, \infty)$ and $T : \Omega \longrightarrow \Omega$ be functions. Then T is an α-η-continuous mapping on Ω, if for given $\varsigma \in \Omega$ and sequence $\{\varsigma_n\}$ with*

$$\varsigma_n \longrightarrow \varsigma \text{ as } n \longrightarrow \infty, \alpha(\varsigma_n, \varsigma_{n+1}) \geq \eta(\varsigma_n, \varsigma_{n+1}), \text{ for all } n \in \mathbb{N} \text{ implies } T\varsigma_n \longrightarrow T\varsigma.$$

Hussain et al. [24] generalized Definition 2.20 to MVMs.

Definition 2.21. *Let $T : \Omega \longrightarrow 2^{\Omega}$ be a MVM on a MS(Ω, ρ), $\alpha, \eta : \Omega \times \Omega \longrightarrow [0, \infty)$ and $T : \Omega \longrightarrow \Omega$ be two functions. We say that T is α-η continuous MVM, if for given $\varsigma \in \Omega$ and sequence $\{\varsigma_n\}$ with $\varsigma_n \longrightarrow \varsigma$ as $n \longrightarrow \infty$, $\alpha(\varsigma_n, \varsigma_{n+1}) \geq \eta(\varsigma_n, \varsigma_{n+1})$, for all $n \in \mathbb{N}$ we have $T\varsigma_n \longrightarrow T\varsigma$. That is, $\lim_{n \longrightarrow \infty} \rho(\varsigma_n, \varsigma) = 0$ and $\alpha(\varsigma_n, \varsigma_{n+1}) \geq \eta(\varsigma_n, \varsigma_{n+1})$ implies $\lim_{n \longrightarrow} H(T\varsigma_n, T\varsigma) = 0$.*

Definition 2.22. *([28]) Let $T : \Omega \longrightarrow 2^{\Omega}$ be a MVM on a MS(Ω, d), $\alpha, \eta : \Omega \times \Omega \longrightarrow \mathbb{R}^{+}$ be two functions. We say that T is α-η LSCmulti-valued mapping on Ω, if for given $\varsigma \in \Omega$ and sequence $\{\varsigma_n\}$ with*

$$\lim_{n \longrightarrow \infty} \rho(\varsigma_n, \varsigma) = 0, \alpha(\varsigma_n, \varsigma_{n+1}) \geq \eta(\varsigma_n, \varsigma_{n+1}), \quad \text{for all } n \in \mathbb{N},$$

implies

$$\lim_{n \longrightarrow \infty} \inf D(\varsigma_n, T\varsigma_n) \geq D(\varsigma, T\varsigma).$$

Definition 2.23. *([28]) Let $T : \Omega \longrightarrow 2^{\Omega}$ be a MVM on a MS (Ω, ρ), $\alpha, \eta : \Omega \times \Omega \longrightarrow \mathbb{R}^+$ be two functions. We say that T is α-η upper semi-continuous MVM on Ω, if for given $\varsigma \in \Omega$ and sequence $\{\varsigma_n\}$ with*

$$\lim_{n \longrightarrow \infty} \rho(\varsigma_n, \varsigma) = 0, \alpha(\varsigma_n, \varsigma_{n+1}) \geq \eta(\varsigma_n, \varsigma_{n+1}), \quad for\ all\ n \in \mathbb{N},$$

implies

$$\lim_{n \longrightarrow \infty} \sup D(\varsigma_n, T\varsigma_n) \leq D(\varsigma, T\varsigma).$$

Theorem 2.52. *[28] Let (Ω, ρ) be a CMS and $\alpha, \eta : \Omega \times \Omega \longrightarrow \mathbb{R}^+$ be two functions. Let $T : \Omega \longrightarrow \yen(\Omega)$, $F \in \mathfrak{F}$ and $G \in \mathfrak{G}$ fulfilling the following assertions:*

(i) if for any $\varsigma \in \Omega$ with $D(\varsigma, T\varsigma) > 0$, there exists $\nu \in F_{\sigma}^{\varsigma}$ with $\alpha(\varsigma, \nu) \geq \eta(\varsigma, \nu)$ satisfying $G(D(\varsigma, T\varsigma), D(\nu, T\nu), D(\varsigma, T\nu), D(\nu, T\varsigma)) + F(D(\nu, T\nu)) \leq F(\rho(\varsigma, \nu))$;

(ii) T is generalized α^*-admissible mapping with respect to η;

(iii) T is α-η LSCmapping;

(iv) there exists $\varsigma_0 \in \Omega$ and $\nu_0 \in T\varsigma_0$ such that $\alpha(\varsigma_0, \nu_0) \geq \eta(\varsigma_0, \nu_0)$.

Then T has a FP in Ω provided $\sigma < \tau$.

In 2017, [4] obtained some new FP results for nonlinear proximinal multi-valued contractions on some kind of complete quasi-MS, and also presented some results for MVMs with closed values by adding a condition on the function ensuring the contraction. Samet et al. [58] considered an approximate multi-valued FP problem under two constraint inequalities. The authors provided sfficient conditions for the existence of at least one solution and also presented some consequences and related results.

2.24 Durmaz and Altun (2018)

Durmaz and Altun [16] proved some FP results for set-valued mappings on CMS using the concept of set-valued θ-contraction.

Definition 2.24. *Let Θ be the set of all functions $\theta : (0, \infty) \longrightarrow (1, \infty)$ satisfying the following conditions:*

(θ1) θ is nondecreasing;

(θ2) for each sequence $\{t_n\} \subset (0, \infty)$, $\lim_{n \longrightarrow \infty} \theta(t_n) = 1$ and $\lim_{n \longrightarrow \infty} t_n = 0^+$ are equivalent;

(θ3) there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that $\lim_{t \longrightarrow 0^+} \dfrac{\theta(t) - 1}{t^r} = l$.

Definition 2.25. *Let (Ω,ρ) be a MS and let $\theta \in \Theta$. A mapping $T : \Omega \longrightarrow \Omega$ is said to be a θ-contraction if there exists $k \in (0,1)$ such that*

$$\theta(\rho(T\mu, T\nu)) \leq [\theta(\rho(\mu,\nu))]^k,$$

for all $\mu,\nu \in \Omega$ with $\rho(T\mu,T\nu) > 0$.

Let $T : \Omega \longrightarrow \eth(\Omega), \theta \in \Theta$ and $s \in (0,1]$. Define a set $\theta_s^\mu \subseteq \Omega$ by

$$\nu \in T\mu : [\theta(\rho(\mu,\nu))]^s \leq \theta(\rho(\mu, T\mu)),$$

$\mu \in \Omega$ with $\rho(\mu, T\mu) > 0$.

Theorem 2.53. *Let (Ω,ρ) be a metric space, $T : \Omega \longrightarrow \complement(\Omega)$ and $\theta \in \Theta$. If there exists $k \in (0,1)$ such that there is $\nu \in \theta_s^\mu$, $s \in (0,1]$ and $k < s$, satisfying*

$$\theta(\rho(\nu, T\nu)) \leq [\theta(\rho(\mu,\nu))]^k, \tag{2.15}$$

for each $\mu \in \Omega$ with $\rho(\mu, T\mu) > 0$, then T has a FP in Ω provided that function $\mu \longrightarrow \rho(\mu, T\mu)$ is LSC.

Remark 2.16. *The inequality* (2.15) *can be reformulated via any of the following:*

(i) $\theta(\rho(\nu, T\nu)) \leq [\theta(M(\mu,\nu))]^k$;

(ii) $\phi(\mu,\nu)\theta(\rho(\nu, T\nu)) \leq [\psi(\theta(\rho(\mu,\nu)))]^k$, *where*

$$M(\mu,\nu) = \max\left\{\rho(\mu,\nu), \rho(\mu, T\mu), \rho(\nu, T\nu), \frac{\rho(\mu, T\nu) + \rho(\nu, T\mu)}{2}\right\}.$$

Since θ is nondecreasing, the inequality (2.15) *implies (i).*

In 2019, several work has been done in this regards as a generalization of Feng-Liu approach. Some of which include Hancer et al. [21], Sahin et al. [56], Sahin et al. [54] and some references therein.

2.25 Altun et al. (2020)

Altun et al. [3] presented a nonlinear version of the concept of set-valued P-contractions by taking the MT-function into account and then presented a FP theorem for such mappings.

Definition 2.26. *Let (Ω,ρ) be a MS and $T : \Omega \longrightarrow P(\Omega)$ be a set-valued mapping. If there exist $b \in (0,1)$ and $\theta : [0,\infty) \longrightarrow (0,b)$ such that*

$$\limsup_{r \longrightarrow t^+} \theta(r) < b,$$

for all $t \in [0,\infty)$ and there exists $\nu \in I_b^{\mu}$ such that

$$\rho(\nu,T\nu) \leq \varphi(\rho(\mu,\nu))[\rho(\mu,\nu) + |\rho(\mu,T\mu) - \rho(\nu,T\nu)|], \tag{2.16}$$

for any $\mu \in \Omega$, where $\varphi(t) = \frac{\theta(t)}{2-\theta(t)}$ for $t \geq 0$, then T is called a set-valued nonlinear P-contraction mapping.

Theorem 2.54. *Let (Ω,ρ) be a CMS and $T : \Omega \longrightarrow \complement(\Omega)$ be a set-valued nonlinear P-contraction mapping. If the function $f(\mu) = \rho(\mu,T\mu)$ is LSC, then T has a FP.*

Remark 2.17. *The contractive inequality* (2.16) *can be generalized in the following:*

$$\rho(\nu,T\nu) \leq \varphi(\rho(\mu,\nu))[\rho(\mu,\nu) + |\rho(\mu,T\mu) - \max\{\rho(\mu,T\mu),\rho(\nu,T\nu)|]\}. \tag{2.17}$$

The inequality (2.16) *implies inequality* (2.17)*. To see this, from inequality* (2.16) *we have that*

$$\begin{aligned}\rho(\nu,T\nu) &\leq \varphi(\rho(\mu,\nu))[\rho(\mu,\nu) + |\rho(\mu,T\mu) - \rho(\nu,T\nu)|] \\ &\leq \varphi(\rho(\mu,\nu))[\rho(\mu,\nu) + |\rho(\mu,T\mu) - \max\{\rho(\mu,T\mu),\rho(\nu,T\nu)|]\}.\end{aligned}$$

Similarly, if $\max\{\rho(\mu,T\mu),\rho(\nu,T\nu)\} = \rho(\nu,T\nu)$*, we obtain inequality* (2.16)*. but, if*

$$\max\{\rho(\mu,T\mu),\rho(\nu,T\nu)\} = \rho(\mu,T\mu)$$

we obtain the inequality due to Klim and Wardowski [33].

2.26 Hussain et al. (2020)

Hussain et al. [23] generalized the works of Klim and Wardowski [33], Ciric [14, 15], and also refined and extended the results of Feng and Liu [18].

Let $T : \Omega \longrightarrow 2^{\Omega}$ be the MVM, $F \in \mathfrak{F}$, $\sigma : (0,\infty) \longrightarrow (\sigma,\infty), \sigma > 0$ and $\mu \in \Omega$ with $D(\mu,T\mu) > 0$. Define the set

$$F_{\sigma}^{\mu} = \{\nu \in T\mu : F(\rho(\mu,\nu)) \leq F(M(\mu,\nu)) + \sigma(\rho(\mu,\nu))\},$$

where

$$M(\mu,\nu) = \max\left\{\rho(\mu,\nu), D(\mu,T\mu), D(\nu,T\nu), \frac{D(\mu,T\nu) + D(\nu,T\mu)}{2}.\right\} \tag{2.18}$$

Definition 2.27. *([23]) Let $T : \Omega \longrightarrow 2^{\Omega}$ be a MVM on a MS (Ω,ρ). Then T is said to be modified-F-contraction on Ω, if there exist $\tau : (0,\infty) \longrightarrow (0,\infty)$, $\sigma : (0,\infty) \longrightarrow (\sigma,\infty)$, $\sigma > 0$ and a function $F \in \mathfrak{F}$ such that, for all $\mu \in \Omega$ with $D(\mu,T\mu) > 0$, there exists $\nu \in F_{\sigma}^{\mu}$ satisfying*

$$\tau(\rho(\mu,\nu)) + F(D(\nu,T\nu)) \leq F(M(\mu,\nu)), \tag{2.19}$$

where $M(\mu,\nu)$ is defined in Equation (2.18)*.*

Theorem 2.55. *[23] Let (Ω,ρ) be a CMS and $T:\Omega\longrightarrow \yen(\Omega)$ be a MVM satisfying the following assertions:*

(i) T is a modified-F-contraction for $F\in\mathfrak{F}$;

(ii) $\mu\longrightarrow D(\mu,T\mu)$ is LSCmapping; and

(iii) $\tau:(0,\infty)\longrightarrow(0,\infty)$, $\sigma:(0,\infty)\longrightarrow(\sigma,\infty)$, $\sigma>0$ satisfy

$$\tau(t)>\sigma(t) \text{ and } \liminf_{t\longrightarrow s^+}\tau(t)>\liminf_{t\longrightarrow s^+}\sigma(t) \text{ for all } s\geq 0.$$

Then T has a FP in Ω.

Remark 2.18. *Following Definition 2.27, we propose the following concept. T is said to be modified-F-α-contraction on Ω, if there exist $\tau,\alpha:(0,\infty)\longrightarrow(0,\infty)$, $\sigma:(0,\infty)\longrightarrow(\sigma,\infty)$, $\sigma>0$ and a function $F\in\mathfrak{F}$ such that, for all $\mu\in\Omega$ with $D(\mu,T\mu)>0$, there exists $\nu\in F^{\mu}_{\sigma}$ satisfying*

$$\tau(\rho(\mu,\nu))+F(\alpha(\mu,\nu)D(\nu,T\nu))\leq F(M(\mu,\nu), \tag{2.20}$$

where $M(\mu,\nu)$ is defined in Equation (2.18).

Accordingly, if we take $\alpha(\mu,\nu)=1$, we recover the inequality (2.19) from which Definition 2.27 holds and hence Theorem 2.55 as well.

2.27 Nashine And Kadelburg (2020)

Nashine and Kadelburg [47] presented some Wardowski-Feng-Liu type FP theorems for MVMs in complete (ordered) MS.

Let $T:\Omega\longrightarrow \complement(\Omega)$ be a MVM, $F\in\mathfrak{F}$,and $\sigma:(0,\infty)\longrightarrow(\sigma,\infty)$. For $\mu\in\Omega$ with $D(\mu,T\mu)>0$. Define the set $F^{\mu}_{\sigma}\subseteq\Omega$ as

$$F^{\mu}_{\sigma}=\{\nu\in T\mu: F(\rho(\mu,\nu))\leq F(\max\{\rho(\mu,T\mu),\rho(\nu,T\nu))\}+\sigma(M(\mu,\nu))\},$$

where $M(\mu,\nu)$ is defined by Equation (2.18).

Theorem 2.56. *[47] Let (Ω,ρ) be a CMS and $T:\Omega\longrightarrow \complement(\Omega)$ be a MVM, $F\in\mathfrak{F}_*$. Assume that the following conditions hold:*

(i) $\mu\longrightarrow D(\mu,T\mu)$ is LSCmapping;

(ii) $\tau,\sigma:(0,\infty)\longrightarrow(0,\infty)$ such that

$$\tau(t)>\sigma(t) \text{ and } \liminf_{t\longrightarrow s^+}\tau(t)>\liminf_{t\longrightarrow s^+}\sigma(t) \text{ for all } s\geq 0;$$

(iii) for all $\mu\in\Omega$ with $D(\mu,T\mu)>0$, there exists $\nu\in F^{\mu}_{\sigma}$ satisfying

$$\tau(M(\mu,\nu))+F(D(\nu,T\nu))\leq F(\rho(\mu,\nu)). \tag{2.21}$$

Then T has a FP in Ω.

Remark 2.19. *Following our previous modification, the inequality* (2.21) *can be refined in several ways, in particular, for example see inequality* (2.20).

In 2021, a number of generalizations of Feng and Liu approach in different spaces has been investigated. Some example in this regards, see [46,49,60].

2.28 Choudhury et al. (2022)

Choudhury et al. [13] introduced a new multi-valued contraction of Feng-Liu-type and admissibility condition, and also utilized these concepts to proved a new result in FP theory. Further, they investigated the associated data dependence and stability problem of the FP sets.

Definition 2.28. *([13]) Let Ω be a nonempty set and $\alpha,\beta:\Omega\longrightarrow[0,+\infty)$. $T:\Omega\longrightarrow P(\Omega)$ is called a cyclic $(\alpha-\beta)$-admissible mapping if*

(i) $\alpha(\mu)\geq 1$, for some $\mu\in\Omega$, implies $\beta(u)\geq 1$, for $u\in T\mu$;

(ii) $\beta(\mu)\geq 1$, for some $\mu\in\Omega$, implies $\alpha(u)\geq 1$, for $u\in T\mu$.

Definition 2.29. *([13]) Let (Ω,ρ) be a MS, $T:\Omega\longrightarrow\Xi(\Omega)$ be a MVM and $\alpha,\beta:\Omega\longrightarrow[0,+\infty)$. Let $b,c\in(0,1)$ with $c<b$. Then T is said to be a generalized Feng-Liu-type contraction if for $\mu\in\Omega$ with $\Delta(\mu)>1$ there is a $\nu\in T\mu$, such that*

$$b\rho(\mu,\nu)\geq f\mu \text{ and } f\nu\geq cM(\mu,\nu), \tag{2.22}$$

where

$$M(\mu,\nu)=max\left\{\rho(\mu,\nu),\frac{D(\mu,T\mu)+D(\nu,T\nu)}{2},\frac{D(\mu,T\nu)}{2}\right\},$$

$\Delta(\mu)=\alpha(\mu)$ or $\beta(\mu)$ and $f:\Omega\longrightarrow\mathbb{R}$ is given by $f\mu=D(\mu,T\mu)$, $\mu\in\Omega$.

Definition 2.30. *Let (Ω,ρ), (Y,ρ_1) be two MS and H be the Hausdorff metric on $\Xi(Y)$. A MVM $T:\Omega\longrightarrow\Xi(Y)$ is said to be continuous at $\varsigma\in\Omega$ if for any sequence $\{\varsigma_n\}$ in Ω, $\lim_{n\longrightarrow+\infty}H(T\varsigma,T\varsigma_n)=0$, whenever $\rho(\varsigma,\varsigma_n)\longrightarrow 0$ as $n\longrightarrow+\infty$.*

Theorem 2.57. *([13]) Let $T:\Omega\longrightarrow\Xi(\Omega)$ be a MVM and $\alpha,\beta:\Omega\longrightarrow[0,+\infty)$, where (Ω,ρ) is a CMS. Suppose that:*

(i) there exists $\mu_0\in\Omega$ such that $\alpha(\mu_0)>1$ or $\beta(\mu_0)>1$;

(ii) T is cyclic $(\alpha-\beta)$-admissible;

(iii) T is continuous, or f, where $f(\mu)=D(\mu,T\mu)$ for all $\mu\in\Omega$, is LSC;

(iv) *there exist* $b,c \in (0,1)$ *with* $c < b$*, such that* T *is a generalized Feng-Liu-type contraction. Then* F_T *is non-empty.*

Remark 2.20. *Some possible modifications of inequality* (2.22) *using MT-function techniques are as follows:*

(i) $\varphi(\mu,\nu)\rho(\mu,\nu) \geq f\mu$ *and* $f\nu \geq cM(\mu,\nu)$*;*

(ii) $\varphi(\mu,\nu)\rho(\mu,\nu) \geq f\mu$ *and* $f\nu \geq \phi M(\mu,\nu)$*;*

(iii) $[\varphi(\mu,\nu)]^r\rho(\mu,\nu) \geq f\mu$ *and* $f\nu \geq \phi M(\mu,\nu)$*,*

where φ *is as defined in inequality (2.1). If* $\varphi(\mu,\nu) = b$*,* $b \in (0,1)$*, we obtain the inequality* (2.22) *from (i) above.*

2.29 Altun et al. (2023)

Altun et al. [5] presented some FP theorems for both single-valued and MVMs by considering P-contraction together with the w-distance in metric space.

Definition 2.31. *Let* (Ω,ρ) *be a metric space,* w *be a* w*-distance on* Ω *and* $T : \Omega \longrightarrow P(\Omega)$ *be a MVM and* $b \in (0,1)$*. If for all* $\mu \in \Omega$*, there exists* $\nu \in J_b^\mu$ *satisfying*

$$w(\nu,T\nu) \leq c[w(\mu,\nu) + |w(\mu,T\mu) - w(\nu,T\nu)|],$$

where c *is a nonnegative real number satisfying* $\frac{2c}{b(1+c)} < 1$*. Then* T *is said to be multi-valued* P_w*-contraction.*

Theorem 2.58. *Let* (Ω,ρ) *be a CMS,* w *be a* w*-distance on* Ω *and* $T : \Omega \longrightarrow \complement(\Omega)$ *be a multi-valued* P_w*-contraction. Assume that* $f(\mu) = w(\mu,T\mu)$ *is LSC. Then there exists* $\varsigma \in \Omega$ *such that* $f(\varsigma) = 0$*. Further, if* $w(\varsigma,\varsigma) = 0$*, then* ς *is a FP of* T*.*

Remark 2.21. *Using the technique of Remark 2.17, we can also unify the ideas of Altun et al. [5].*

Further generalization of Feng-Liu approach in 2023 include spaces with additional structure. For example, see [5,51,55,59] and some references therein.

Conflicts of Interests

The authors declare that they have no competing interests.

Acknowledgement

The authors are grateful to the editors and the anonymous reviewers for their useful comments to improve this manuscript.

Funding

This research received no external funding.

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Citation: A. A. Muhammad, M. S. Shagari, I. A. Fulatan, A discussion of Feng-Liu operator and fixed point theorems on metric space, J. Disc. Math. Appl. 9(3) (2024) 211–249.
https://doi.org/10.22061/jdma.2024.11180.1087