Journal of Discrete Mathematics and Its Applications 9 (3) (2024) 191–202

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Configuration sets; a right place for ping-pong arguments

Meisam Soleimani Malekan*

Department of Mathematics, Faculty of Science, Shahid Rajaee Teacher Training University, Tehran, I. R. Iran

Academic Editor: Alireza Abdollahi

Abstract. Giving a condition for the amenability of groups, Rosenblatt and Willis first introduced the concept of configuration. In this paper, we investigate the relationship between ping-pong lemma and configuration sets and show that only one configuration set is enough to ensure that several elements in a group generates a free subgroup of that group. Using only one two-sided configuration sets, we give, in a sense, a generalization of this result to polycyclic or FC-groups. Finiteness and paradoxical decompositions of groups, are other properties which can be characterized with only one configuration set.

Keywords: configuration, FC-group, finitely presented group, polycyclic group
Mathematics Subject Classification (2010): 20E05, 20F05, 20F24, 20F50.

1 Introduction

The ping-pong lemma ensures that several elements in a group acting on a set freely generates a free subgroup of that group. The ping-pong argument goes back to late 19th century and is commonly attributed to Felix Klein who used it to study subgroups of Kleinian groups, that is, of discrete groups of isometries of the hyperbolic 3-space or, equivalently Möbius transformations of the Riemann sphere ([5, Ch. II. B]). The ping-pong lemma was a key tool used by Jacques Tits in his 1972 paper containing the proof of a famous result now known as the Tits alternative ([13]). The result states that a finitely generated linear group is either virtually solvable or contains a free subgroup of rank two. The ping-pong lemma and

*Corresponding author(*Emial address*:msmalekan@gmail.com)
Received 6 August 2024 ; Revised 28 August 2024 ; Accepted 30 August 2024
First Publish Date: 1 September 2024

its variations are widely used in geometric topology and geometric group theory. Modern versions of the ping-pong lemma can be found in many books such as [5,6] or [4].

Let $\mathfrak{g} = (g_1, \ldots, g_n)$ and $\mathcal{E} := \{E_1, \ldots, E_m\}$ be an ordered subset and a collection of pairwise disjoint subsets of G, respectively. We call $(\mathfrak{g}, \mathcal{E})$ a configuration pair of G. A configuration corresponding to a configuration pair $(\mathfrak{g}, \mathcal{E})$ is an $(n+1)$-tuple $C = (c_0, c_1, \ldots, c_n)$, where $c_i \in \{1, \ldots, m\}$ for each i, such that there exists x in G with $x \in E_{c_0}$ and $g_i x \in E_{c_i}$ for each $i \in \{1, \ldots, n\}$.

The set of configurations corresponding to a configuration pair $(\mathfrak{g}, \mathcal{E})$ of G will be denoted by $\mathrm{Con}(\mathfrak{g}, \mathcal{E})$. Also, the set of all configuration sets is denoted by $\mathrm{Con}(G)$.

Rosenblatt and Willis [10] first introduce the concept of the configuration for characterizing the amenability of groups. It is worth noting that they worked with configuration sets in which $\mathcal{E}$ was a partition of the group. One of Willis' pursuits was to find properties of groups that could be characterized by only one configuration set. In [3], he somewhat addressed this idea. The works after [3] is more concerned with answering the questions posed by the authors of that paper.

In order to study quotients of a group, in addition to left translations of subsets, right translations must also be considered; so the concept of *two-sided configuration sets* was suggested in [9]; a *two-sided configuration* corresponding to a configuration pair $(\mathfrak{g}, \mathcal{E})$ is a $(2n+1)$-tuple $C = (c_0, c_1, \ldots, c_{2n})$ satisfying $c_i \in \{1, \ldots, m\}$, $i = 0, 1, \ldots, 2n$, and there exists $x \in E_{c_0}$ such that $g_i x \in E_{c_i}$ and $x g_i \in E_{c_{i+n}}$ for each $i \in \{1, \ldots, n\}$; the sets $x_0(C)$, $\mathrm{Con_t}(\mathfrak{g}, \mathcal{E})$ and $\mathrm{Con_t}(G)$ are defined as above.

There was a lack of proper notations and good foundations for the theory of configuration, these defects were partially fixed in [8] and [7]. What we need from these notations and foundations is in Section 2.

In the third section, we state and prove our version of Ping-Pong lemma. Also, we show that a polycyclic or FC-group G admits a configuration set which only groups with an isomorphic section with G have that set among their configuration sets. Then we show that the infiniteness of a non-locally finite group can also be determined with just one configuration set.

In the last section, we deal with paradoxical decomposition and, using configuration tools, we give a new proof for this fact that the Tarski number of a group is smaller than the Tarski number of its sections.

2 Definitions and Preliminaries

We devote this section to provide the preliminaries and notations needed in the following sections.

Let G and H be two groups with configuration pairs $(\mathfrak{g}, \mathcal{E})$ and $(\mathfrak{h}, \mathcal{F})$, respectively, such that equality $\mathrm{Con}_{\mathrm{t}}(\mathfrak{g},\mathcal{E}) = \mathrm{Con}_{\mathrm{t}}(\mathfrak{h},\mathcal{F})$, or $\mathrm{Con}(\mathfrak{g},\mathcal{E}) = \mathrm{Con}(\mathfrak{h},\mathcal{F})$, established. If

$$\mathfrak{g} = (g_1,\ldots,g_n), \quad \mathcal{E} = \{E_1,\ldots,E_m\},$$
$$\mathfrak{h} = (h_1,\ldots,h_n), \quad \mathcal{F} = \{F_1,\ldots,F_m\},$$

then we may say that g_i and E_j is *corresponding* to h_i and F_j, respectively. We use "$\leftrightsquigarrow$" to illustrate this correspondence, i.e. $g_i \leftrightsquigarrow h_i$ and $E_j \leftrightsquigarrow F_j$.

Let G be a group with $\mathfrak{g} = (g_1,\ldots,g_n)$ as its ordered subset. Let p be a positive integer, let J and ρ be p-tuple with components in $\{1,2,\ldots,n\}$ and $\{\pm 1\}$, respectively. We denote the product $\prod_{i=1}^{p} g_{J(i)}^{\rho(i)}$ by $W(J,\rho;\mathfrak{g})$. We call the pair (J,ρ) a *representative pair* on $\mathfrak{g}$ and $W(J,\rho;\mathfrak{g})$ a *word* corresponding to (J,ρ) comprised of elements of $\mathfrak{g}$. A representative pair (J,ρ) is called *reduced* if $\rho(i) = \rho(i+1)$, whenever $J(i) = J(i+1)$, $i = 1,\ldots,p-1$.

For an arbitrary tuple J, we denoted its components number by $\mathbf{l}(J)$.When we speak of a representative pair (J,ρ) we assume the same number of components for J and ρ.

For positive integers p_i, if J_i is a p_i-tuple, $i = 1,2$, $J_1 \oplus J_2$ is a $(p_1 + p_2)$-tuple that has J_1 as its first p_1 components and J_2 as its second p_2 components. it can easily be shown that

$$W(J_1,\rho_1;\mathfrak{g})W(J_2,\rho_2;\mathfrak{g}) = W(J_1 \oplus J_2,\rho_1 \oplus \rho_2;\mathfrak{g}).$$

For a finite algebra $\mathcal{A}$ on a subset of a group G, we define $\mathrm{Con}(\mathfrak{g},\mathcal{A})$ and $\mathrm{Con}_{\mathrm{t}}(\mathfrak{g},\mathcal{A})$ to be $\mathrm{Con}(\mathfrak{g},\mathbf{atom}(\mathcal{A}))$ and $\mathrm{Con}_{\mathrm{t}}(\mathfrak{g},\mathbf{atom}(\mathcal{A}))$, respectively, where $\mathbf{atom}(\mathcal{A})$ is the collection of atomic sets in $\mathcal{A}$.

For a finite collection $\mathcal{D}$ of subsets of G, the algebra on $\bigcup\{D \in \mathcal{D}\}$ generated by elements of $\mathcal{D}$ is denoted by $\sigma(\mathcal{D})$ and is finite.

Let $\mathcal{E} := \{E_1,\ldots,E_m\}$ and $\mathcal{F} := \{F_1,\ldots,F_m\}$ be collections of pairwise disjoint subsets of G and H respectively, such that $E_i \leftrightsquigarrow F_i$, $i = 1,\ldots,m$. For $A \in \sigma(\mathcal{E})$ and $B \in \sigma(\mathcal{F})$, we say A is corresponding to B, written $A \leftrightsquigarrow B$, when

$$\{k : E_k \subseteq A\} = \{k : F_k \subseteq B\}.$$

If $A \leftrightsquigarrow B$ and $A = E_{i_1} \cup \cdots \cup E_{i_j}$, then $B = F_{i_1} \cup \cdots \cup F_{i_j}$. By an argument as used in Lemma 2.2 of [8], one can show that

Lemma 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two algebra on two subsets of groups G and H, respectively. Assume that $\mathfrak{g} = (g_1,\ldots,g_n)$ and $\mathfrak{h} = (h_1,\ldots,h_n)$ are ordered subsets of G and H, respectively, such that $\mathrm{Con}_t(\mathfrak{g},\mathcal{A}) = \mathrm{Con}_t(\mathfrak{h},\mathcal{B})$. Consider $A_1,A_2 \in \mathcal{A}$ and $B_1,B_2 \in \mathcal{B}$ with $A_i \leftrightsquigarrow B_i$, $i = 1,2$. We have:*

(1) If $g_r A_1 \subseteq A_2$, then $h_r B_1 \subseteq B_2$,

(2) If $A_1 g_r \subseteq A_2$, then $B_1 h_r \subseteq B_2$,

(3) If $g_r A_1 = A_2$, then $h_r B_1 = B_2$,

(4) If $A_1 g_r = A_2$, then $B_1 h_r = B_2$,

for $r \in \{1,\ldots,n\}$.

In Lemma 2.1, if we have the equality $\mathrm{Con}(\mathfrak{g},\mathcal{A}) = \mathrm{Con}(\mathfrak{h},\mathcal{B})$, then the implications (1) and (3) remain true.

Let G and H be two groups. Consider collections $\mathcal{E} = \{E_1,\ldots,E_r\}$ and $\mathcal{F} = \{F_1,\ldots,F_r\}$ of pairwise disjoint subsets of G and H, respectively. Assume that

$$\mathcal{E}' = \{E'_1,\ldots,E'_s\} \text{and } \mathcal{F}' = \{F'_1,\ldots,F'_s\},$$

are their refinements. We say that these two refinements $\mathcal{E}'$ and $\mathcal{F}'$ are **similar** and write $(\mathcal{E}',\mathcal{E}) \sim (\mathcal{F}',\mathcal{F})$, if

$$\{l : E_k \cap E'_l \neq \emptyset\} = \{l : F_k \cap F'_l \neq \emptyset\} \quad (k = 1,\ldots,r).$$

In other words, if $E_k = \bigcup_{j=1}^{t} E'_{i_j}$, then we have $F_k = \bigcup_{j=1}^{t} F'_{i_j}$.

Note that it is implicit in the definition of similarity that similar partitions have equal numbers of sets. An important feature of similar refinements is presented below:

Lemma 2.2. *Let G and H be two groups. Assume that $(\mathfrak{g},\mathcal{E})$ and $(\mathfrak{h},\mathcal{F})$ are two configuration pairs for G and H, respectively and let $\mathcal{E}'$ and $\mathcal{F}'$ be their similar refinements such that* $\mathrm{Con}(\mathfrak{g},\mathcal{E}') = \mathrm{Con}(\mathfrak{h},\mathcal{F}')$. *Then* $\mathrm{Con}(\mathfrak{g},\mathcal{E}) = \mathrm{Con}(\mathfrak{h},\mathcal{F})$. *The result remains true when replacing* Con *with* Con_t.

To prove this lemma, it suffices to slightly improve the proof of [8, Lemma 4.3].

3 Group properties that are only characterized by one configuration set

First, we show how with only one configuration set we can identify a free subgroup. Let's start with the following definition.

Definition 1. Let $\mathfrak{f} := (f_1,\ldots,f_n)$ be a free generating set of the free group $\mathbf{F}_n$. Assume that $F_0 := \{e\}$ and for $k = 1,\ldots,n$,

$$F_k := \{\text{reduced words starting with } f_k\},$$
$$F_{-k} := \left\{\text{reduced words starting with } f_k^{-1}\right\}.$$

Put $\mathcal{F} := \{F_0, F_{\pm k} : k = 1,\ldots,n\}$. The configuration set $\mathrm{Con}(\mathfrak{f},\mathcal{F})$ is called *the free configuration set of rank n*.

In the following case, we will see how the free configuration sets are connected to the Ping-Pong lemma:

Theorem 3.1. *Let G be a group. Then the following are equivalent:*

(i) G contains a non Abelian free subgroup of rank n.

(ii) $\mathrm{Con}(G)$ *contains the free configuration set of rank n.*

(iii) There is a finitely generated subgroup $\langle g_1, \ldots, g_n \rangle$ of G with an action on a set X along with pairwise disjoint subsets $\{E_{\pm k} : k = 1, \ldots, n\}$ of X such that $E_k^c \subset g_k E_{-k}$, $k = 1, \ldots, n$.

Proof. (i) ⇒ (ii): It is clear.

(ii) ⇒ (iii): With $(\mathfrak{f}, \mathcal{F})$ as in Definition 1, we have $f_k F_{-k}^c = F_k$, $k = 1, \ldots, n$. So, Lemma 2.1 implies (3).

(iii) ⇒ (i): Let $\mathfrak{g} = (g_1, \ldots, g_n)$. It suffices to prove that if (J, ρ) is a reduced representative word on $\mathfrak{g}$ then

$$W(J, \rho; \mathfrak{g}) E^c_{-\rho(p)J(p)} \subset E_{\rho(1)J(1)}, \tag{1}$$

where p is the number of components of J. Using induction on p, we prove this claim. By (iii), it is easy to see that $g_k^{-1} E_k^c \subset E_{-k}$ and $g_k E_{-k}^c \subset E_k$. This proves (1) for $p = 1$. Assume that (1) holds for $p \in \mathbb{N}$. If (J, ρ) is a reduced representative word on $\mathfrak{g}$ where J is a $(p+1)$-tuple, then using the hypothesis of the induction, we get

$$W(J, \rho; \mathfrak{g}) E^c_{-\rho(p+1)J(p+1)} \subset g_{J(1)}^{\rho(1)} E_{\rho(2)J(2)}.$$

We consider two cases:

Case 1: $J(2) = J(1)$, so $\rho(1) = \rho(1)$ and therefore

$$\begin{aligned} g_{J(1)}^{\rho(1)} E_{\rho(2)J(2)} &= g_{J(1)}^{\rho(1)} E_{\rho(1)J(1)} \\ &\subset g_{J(1)}^{\rho(1)} E^c_{-\rho(1)J(1)} \\ &\subset E_{\rho(1)J(1)}. \end{aligned}$$

Case 2: $J(2) \neq J(1)$, hence

$$g_{J(1)}^{\rho(1)} E_{\rho(2)J(2)} \subset g_{J(1)}^{\rho(1)} E^c_{-\rho(1)J(1)} \subset E_{\rho(1)J(1)}.$$

□

Theorem 3.1 makes it clear how a configuration set ensures that the group contains a non Abelian free subgroup. So, it seems natural to ask:

Question 3.1. *Which group properties can be characterized with only one (two-sided) configuration set?*

Below we have listed some of the group properties that respond positively to the question above.

We now turn to two-sided configuration sets to generalize Theorem 3.1 to some finitely presented groups. First we need the following definition (see [7, Definition 3.2]).

Definition 2. Let G be a group with a generating set $\mathfrak{g}$ and a partition $\mathcal{E}$. We say that $(\mathfrak{g}, \mathcal{E})$ is *golden*, if $\mathcal{E}$ contains $\{e_G\}$ and there exist representative pairs (J_g, ρ_g), $g \in G^\#$, such that

(1) $g = W(J_g, \rho_g; \mathfrak{g})$,

(2) If $\mathrm{Con_t}(\mathfrak{g}, \mathcal{E}) = \mathrm{Con_t}(\mathfrak{h}, \mathcal{F})$, for a configuration pair $(\mathfrak{h}, \mathcal{F})$ of a group H, then

$$W(J_g, \rho_g; \mathfrak{h})M \cap M = \emptyset,$$

where $M \in \mathcal{F}$ is corresponding to N.

We now state and prove the following generalization of Theorem 3.1. Before that, recall that a group G *is involved* in a group H if G is isomorphic to K/N for some subgroups K, N of G with N normal in K. The quotient K/N is called a *section* of the group G.

Theorem 3.2. *Let G be a finitely presented group with a golden configuration pair $(\mathfrak{g}, \mathcal{E})$. Then G admits a configuration pair $(\mathfrak{g}, \mathcal{D})$, such that $\mathrm{Con}_t(\mathfrak{g}, \mathcal{D}) \in \mathrm{Con}_t(H)$ for a group H, if and only if G is involved in H.*

Proof. As a consequence of Lemma 2.2, note that for each refinement $\mathcal{E}'$ of $\mathcal{E}$, the configuration pair $(\mathfrak{g}, \mathcal{E}')$ is golden, too.

We will prove this theorem in two steps:

Step 1. We, first, show that there exists a refinement $\mathcal{D}$ of $\mathcal{E}$, such that if $\mathrm{Con_t}(\mathfrak{g}, \mathcal{D}) = \mathrm{Con_t}(\mathfrak{h}, \mathcal{F})$, then the relation

$$W(J, \rho; \mathfrak{g}) \mapsto W(J, \rho; \mathfrak{h})M, \text{for representative pairs} \, (J, \rho) \, \text{in} \, \mathfrak{g}$$

will be a well-defined function on G, where $M \in \mathcal{F}$ is corresponding to $\{e_G\}$. Suppose that $\mathfrak{g} = (g_1, \ldots, g_n)$. Let $\{W(J_i, \sigma_i; \mathfrak{g}) : i = 1, \ldots, m\}$ be a set of defining relators of G and

$$\mathfrak{F} := \{(J_i, \sigma_i) : i = 1, \ldots, m\}.$$

Assume that $k := \max\{\mathbf{l}(J) : (J, \rho) \in \mathfrak{F}\}$ and set

$$S := \{(J, \rho) : \mathbf{l}(J) \leq 2k\},$$
$$S_0 := \{(J, \rho) : \mathbf{l}(J) \leq k\}.$$

Now, consider a refinement $\mathcal{D}$ which contains singleton sets $\{W(J, \rho; \mathfrak{g})\}$, $(J, \rho) \in S$. Let $(\mathfrak{h}, \mathcal{F})$ be a configuration pair for a group H, such that $\mathrm{Con_t}(\mathfrak{g}, \mathcal{E}') = \mathrm{Con_t}(\mathfrak{h}, \mathcal{F})$ and consider $M \in \mathcal{F}$ to be a set that $\{e_G\} \leftrightsquigarrow M$. By (3) and (4) in Lemma 2.1,

$$Mh = hM \quad (h \in \langle \mathfrak{h} \rangle). \tag{2}$$

Also, by induction on $\mathbf{l}(J)$, one can show that

$$W(J, \rho; \mathfrak{h})M = M(J, \rho), \quad (J, \rho) \in S_0,$$

where $M(J, \rho)$ is an element of $\mathcal{F}$ corresponding to $\{W(J, \rho; \mathfrak{g})\}$. So, in particular, we have

$$W(J_i, \rho_i; \mathfrak{h})M = M, \quad i = 1, \ldots, m.$$

Let (J,ρ) and (I,δ), be two representative pair and take $W_i = W(J_i,\rho_i;\mathfrak{h})$, $i = 1,\ldots,m$. By (2), $W_iM = M$, so, it can be seen that

$$W(J,\rho;\mathfrak{h})W_i^{\pm}W(I,\delta;\mathfrak{h})M = W(J \oplus I,\rho \oplus \delta;\mathfrak{h})M. \tag{3}$$

This shows that the above function is well-defined; indeed, $\{W(J_i,\sigma_i;\mathfrak{g}) : i = 1,\ldots,m\}$ is the set of defining relators of G, this along with (3) implies $W(J,\rho;h)M = M$.

Step 2. Let $\mathcal{D}$ and M be as in Step 1. Put

$$K := \{h \in \langle\mathfrak{h}\rangle : hM = M\}.$$

By the previous step and considering that $(\mathfrak{g},\mathcal{D})$ is golden, we have K is a normal subgroup of $\langle\mathfrak{h}\rangle$. Also, for all representative pairs (J,ρ) in $\mathfrak{g}$, $W(J,\rho;\mathfrak{h})M \cap M \neq \emptyset$ if and only if $W(J,\rho;\mathfrak{g}) = \{e_G\}$ and the last one is equivalent to $W(J,\rho;\mathfrak{h}) \in K$. Therefore, the map

$$G \to \langle\mathfrak{h}\rangle/K, \quad W(J,\rho;\mathfrak{g}) \mapsto W(J,\rho;\mathfrak{h})K$$

introduces an isomorphism between G and $\langle\mathfrak{h}\rangle/K$.

The other direction can be proved by Corollary 4.4 below. □

Now let's move on to the question of whether there is a group with a golden configuration or not? The answer is "yes, there is" and the class of groups with golden configuration pair include polycyclic and FC-groups.

Consider a finite-range function ς on G. Assume that $\varsigma(G) = \{\varsigma_1,\ldots,\varsigma_k\}$ and set

$$E(\varsigma_i) := \varsigma^{-1}(\varsigma_i),$$

$i = 1,\ldots,k$. The partition $\mathcal{E} := \{E(\varsigma_i) : i = 1,\ldots,k\}$ of G is called the *ς-partition* of G. We continue with the following definition (see [7, Definition 4.1]).

Definition 3. We say that a group G admits a *golden system*, if there exist a generating set $\mathfrak{g}$ of G, a set of representative pairs, $\{(J_g,\rho_g) : g \in G^{\#}\}$ and a finite-range function ς on G with following properties:

1. $g = W(J_g,\rho_g;\mathfrak{g})$, $g \in G^{\#}$,
2. $E(\varsigma(e_G)) = \{e_G\}$,
3. Let $\mathcal{E}$ be the ς-partition of G. If we have $\mathrm{Con}_{\mathfrak{t}}(\mathfrak{g},\mathcal{E}) = \mathrm{Con}_{\mathfrak{t}}(\mathfrak{h},\mathcal{F})$, for a configuration pair $(\mathfrak{h},\mathcal{F})$ of a group H and we denote by $F(\varsigma_i)$ the element in $\mathcal{F}$ corresponding to $E(\varsigma_i)$, $i = 1,\ldots,k$, then
$$W(J_g,\sigma_g;\mathfrak{h})F(\varsigma(e_G)) \subseteq F(\varsigma(g)).$$

We call the triple $(\mathfrak{g},\varsigma,\{(J_g,\rho_g)\}_g)$ *the golden system* and (J_g,ρ_g) *the golden representative pair* of g.

The importance of this definition will appear in the following obvious proposition:

Proposition 3.3. *Let G be a group which admits a golden system $(\mathfrak{g}, \varsigma, \{(J_g, \rho_g)\}_g)$. If $\mathcal{E}$ is the ς-partition of G, then $(\mathfrak{g}, \mathcal{E})$ will be a golden configuration pair.*

Following the material in the fourth section of [7], we see that a polycyclic or an FC-group admits a golden configuration system. This fact, together with Theorem 3.2, leads us to the following theorem:

Theorem 3.4. *Let G be a a polycyclic or an FC-group. Then G admits a configuration pair $(\mathfrak{g}, \mathcal{E})$ such that $\mathrm{Con}_t(\mathfrak{g}, \mathcal{E}) \in \mathrm{Con}_t H$ for a group H if and only if G is involved in H.*

Another feature that can be characterized with only one configuration set is the infinitness of a non-locally finite group.

Theorem 3.5. *Every infinite non-locally finite group has a configuration set which is not a configuration set of any finite group.*

Some preparations are needed to prove this theorem. Let's start with the following definition (see [11, Definition 0.1.6]).

Definition 4. Let G be a group. Two subsets A, B of G are said to be equidecomposable if there are $A_1, \ldots, A_n \subset A$, $B_1, \ldots, B_n \subset B$ and $g_1, \ldots, g_n \in G$ such that

(i) $A = \bigcup_{j=1}^{n} A_j$ and $B = \bigcup_{j=1}^{n} B_j$;

(ii) $A_j \cap A_k = \emptyset = B_j \cap B_k$ for $j, k \in \{1, \ldots, n\}$, $j \neq k$;

(iii) $g_j A_j = B_j$, for $j = 1, \ldots, n$.

It is obvious from the above definition that no finite group can have equidecomposable subsets A and B such that $B \subsetneq A$.

We continue with some notions from graph theory: A (undirected) graph is an ordered pair $\mathcal{G} = (V, E)$, comprising a set V of *vertices* together with a set E of *edges* (i.e. an edge is associated with two vertices and the association takes the form of the unordered pair of the vertices). A graph $\mathcal{G} = (V, E)$ is called infinite if V is an infinite set. If $e \in E$ is associated with two vertices $v, w \in V$, we say that e joins v and w or that v and w are the endpoints of e. A *path* in a graph $\mathcal{G} = (V, E)$ is alternating sequence of vertices and edges $v_0, e_0, v_1, e_1, v_2, e_2, \ldots, v_{k-1}, e_{k-1}, v_k$ in which e_j is the edge with endpoints v_j, v_{j+1}, $j = 1, \ldots, k$. We say that v_0 and v_k are joined by the path. A *ray* in an infinite graph $\mathcal{G} = (V, E)$ is alternating sequence of vertices and edges $v_0, e_0, v_1, e_1, v_2, e_2, \ldots$ which contains no repeated vertices and for $n \in \mathbb{N} \cup \{0\}$, e_n is the edge with endpoints v_n, v_{n+1}. A graph is *connected* when there is a path between every pair of vertices. A graph is *locally finite* if each vertex is endpoint of a finite number edges.

Lemma 3.6 (Konig's lemma)**.** *Let $\mathcal{G}$ be a connected, locally finite, infinite graph. Then $\mathcal{G}$ contains a ray.*

Appealing to Konig's lemma, we can obtain equidecomposable subsets in an infinite finitely generated group.

Theorem 3.7. *Let G be an infinite finitely generated group.Then there are equidecomposable subsets B, A such that $B \subsetneq A$.*

Proof. Let $\mathfrak{g} = (g_1, \ldots, g_n)$ be an ordered and symmetric set of generators for G (i.e. $g_k^{-1} \in \mathfrak{g}$ for $k = 1, \ldots, n$). Then there is a Cayley graph, denoted by $\Gamma = \Gamma(G, \mathfrak{g})$, with vertices being the elements of G and the edges being $\{x, y\}$, for $x, y \in G$, if $x = g_j y$ for some $j = 1, \ldots, n$. The graph Γ is obviously, connected, locally finite and infinite, so by König's lemma it contains a ray. That means there exists an infinite subset $A := \{x_0, x_1, x_2, \ldots\}$ of G such that $x_{j+1}x_j^{-1} \in \mathfrak{g}$ for all $j \in \mathbb{N} \cup \{0\}$. Put $B = A \setminus \{x_0\}$ and

$$A_k := \left\{x_j \in A : x_{j+1}x_j^{-1} = g_k\right\}, \quad (k = 1, \ldots, n).$$

Then A_k and $B_k := g_k A_k$, $k = 1, \ldots, n$, fulfills the conditions of Definition 4, so $B \subsetneq A$ and A, B are equidecomposable. □

Remark 1. The statement of Theorem 3.7 is proposed by the author in [1]. The proof given above is also the answer to the mentioned question.

We are now in a position to prove Theorem 3.5.

Proof of Theorem 3.5. By Theorem 3.7, there are equidecomposable subsets $B \subsetneq A$. Let $\{A_j\}_{j=1}^n$, $\{B_j\}_{j=1}^n$ and $\{g_j\}_{j=1}^n$ be as in Definition 4. Let $\mathfrak{g}$ be the ordered set $(g_1, \ldots, g_n)$ and $\mathcal{A}$ be an algebra on G generated by

$$A_j, B_j, \quad j = 1, \ldots, n.$$

Assume that $\mathrm{Con}(\mathfrak{g}, \mathcal{A}) \in \mathrm{Con}\, H$ for a group H. Hence there are a configuration set $\mathrm{Con}(\mathfrak{h}, \mathcal{B})$ of H such that $\mathrm{Con}(\mathfrak{g}, \mathcal{A}) = \mathrm{Con}(\mathfrak{h}, \mathcal{B})$. Let C_j and D_j be elements of $\mathcal{B}$ such that $C_j \leftrightsquigarrow A_j$ and $D_j \leftrightsquigarrow B_j$, $j = 1, \ldots, n$, so $C_j \cap C_k = \emptyset = B_j \cap B_k$, $j, k \in \{1, \ldots, n\}$. By Lemma 2.1, $D_j = h_j C_j$, for some $h_j \in \mathfrak{h}$. Put $C := \bigcup_j C_j$ and $D = \bigcup_j D_j$, so $D \subsetneq C$ and C, D are equidecomposable subsets of H, whence H is infinite. □

Remark 2. The result of Theorem 3.5 does not hold when the group G is locally finite. In the other words, if G is an infinite locally finite group, then for all configuration pair $(\mathfrak{g}, \mathcal{E})$ of G, there exists a finite group F such that $\mathrm{Con}(\mathfrak{g}, \mathcal{E}) \in \mathrm{Con}(F)$; for all $C \in \mathrm{Con}(\mathfrak{g}, \mathcal{E})$, select an element in $x_0(C)$, say x_C. Let F be a finite group generated by elements

$$x_C, g x_C \,\text{and}\, g \quad (g \in \mathfrak{g}, C \in \mathrm{Con}(\mathfrak{g}, \mathcal{E})).$$

Then for $\mathcal{F} = \{F \cap E : E \in \mathcal{E}\}$ we have $\mathrm{Con}(\mathfrak{g}, \mathcal{E}) = \mathrm{Con}(\mathfrak{g}, \mathcal{F})$ and $\mathrm{Con}(\mathfrak{g}, \mathcal{F}) \in \mathrm{Con}(F)$.

There are some other group properties which can be characterized by one configuration set; for example, being non-Abelian and in a more general way not satisfying in a group law ([8, Proposition 2.1]) or, having non-zero kth derived subgroup ([8, Proposition 2.2]) are such properties.

4 Configuration sets and Tarski numbers

In this section, we study the relation between the configuration sets of a group and the ones of its subgroups and quotients. By following the proof of [2, Lemma 4.1.] we obtain:

Lemma 4.1. *Let $\phi : G \to H$ be an epimorphism of groups. Then for all ordered sets $\mathfrak{g}$ of G and all disjoint collection $\mathcal{F}$ of H, we have* $\mathrm{Con}(\mathfrak{g},\phi^{-1}(\mathcal{F})) = \mathrm{Con}(\phi(\mathfrak{g}),\mathcal{F})$*. In particular* $\mathrm{Con}(G)$ *contains all the configuration sets of quotients of G.*

In the next two statement, we consider a subgroup H of a group G. Also, let T be a right transversal of H in G, that is, a subset of G which contains precisely one element from each right coset of H.

Proposition 4.2. *Let A and B be subsets of H and $S \subset T$. Then $AS \cap BS = (A \cap B)S$. In particular, AS and BS are disjoint if A and B are so.*

Proof. Let $a \in A$, $b \in B$ and $x, y \in S$ be such that $ax = by$. Then $Hx = Hy$ and so $x = y$, whence $a = b$. □

Lemma 4.3. *Let $\mathcal{F}$ be a disjoint collection of H. Then $\mathcal{F}T := \{FT : F \in \mathcal{F}\}$ is a disjoint collection of G and* $\mathrm{Con}(\mathfrak{h},\mathcal{F}) = \mathrm{Con}(\mathfrak{h},\mathcal{F}T)$*. In particular* $\mathrm{Con}(G)$ *contains all the configuration sets of subgroups of G*

As a corollary of Lemmas 4.1 and 4.3, we get:

Corollary 4.4. *Let G be a group. Then* $\mathrm{Con}(G)$ *contains all the configuration sets of its sections.*

We now turn to the paradoxical decompositions and the Tarski numbers; a group G admits a $(m+n)$-*paradoxical decomposition* if there exist positive integers m and n, a partition $\{P_1, \ldots, P_m, Q_1, \ldots, Q_n\}$ of G and elements $\mathfrak{x} = (x_1, \ldots, x_m)$, $\mathfrak{y} = (y_1, \ldots, y_n)$ of G such that

$$G = \bigcup_{i=1}^{m} x_i P_i = \bigcup_{j=1}^{n} y_j Q_i.$$

We can assume without loss of generality that $x_m = y_n = 1$ and we assume so, from now on.

Now, let $\mathcal{A}$ be an algebra generated by

$$P_i,\, x_i P_i,\, Q_j,\, y_j Q_j \quad (i = 1, \ldots, m,\, j = 1, \ldots, n),$$

and put $\mathfrak{g} = \mathfrak{x} \oplus \mathfrak{y} = (x_1, \ldots, x_{m-1}, y_1, \ldots, y_{n-1})$. We may call $\mathrm{Con}(\mathfrak{g},\mathcal{A})$ *a $(m+n)$-paradoxical configuration.*

The minimal possible value of $m+n$ in a paradoxical decomposition of G is the *Tarski number* of G and denoted by $\tau(G)$. If a group G doesn't have a paradoxical decomposition, it means that G is amenable; In this case we will define $\tau(G)$ to be ∞.

Proposition 4.5. $\mathrm{Con}(G)$ *contains a $(m+n)$-paradoxical configuration if and only if G admits a $(m+n)$-paradoxical decomposition.*

Proof. Only one direction needs to be proved. Assume that $\mathrm{Con}(G)$ contains a $(m+n)$-paradoxical configuration $\mathrm{Con}(\mathfrak{g},\mathcal{A})$, as described above. Then, there exists an ordered set $\mathfrak{h}=\mathfrak{u}\oplus\mathfrak{v}=(u_1,\ldots,u_{m-1},v_1,\ldots,v_{n-1})$ and an algebra $\mathcal{B}$ of G, such that

$$\mathrm{Con}(\mathfrak{g},\mathcal{A})=\mathrm{Con}(\mathfrak{h},\mathcal{B}).$$

Suppose that C_i, D_j in $\mathcal{B}$ are such that $C_i \leftrightsquigarrow P_i$ and $D_j \leftrightsquigarrow Q_j$, $i=1,\ldots,m$, $j=1,\ldots,m$. Therefore, Lemma 2.1 leads to $u_iC_i \leftrightsquigarrow x_iP_i$ and $v_jD_j \leftrightsquigarrow y_jQ_j$. So, if $E=\bigcup_{B\in\mathcal{B}}B$, then

$$\{C_1,\ldots,C_{m-1},C_m\cup(G\setminus E),D_1,\ldots,D_n\},$$

along with $\mathfrak{h}$ form a $(m+n)$-paradoxical decomposition. □

Now, using Lemmas 4.1 and 4.3 together with Proposition 4.5, another proof is obtained for the following theorem which is proved in [12].

Theorem 4.6. *The Tarski number of a group is less than the Tarski numbers of its subgroups and quotients.*

Conflicts of interest:
The authors declare that they have no conflicts of interest regarding the publication of this article.

Funding:
This research received no external funding.

References

[1] Can infiniteness of finitely generated groups be read by a paradoxical decomposition? https://mathoverflow.net/questions.298045/ can-infiniteness-of-finitely-generated-groups-be-read-by-a-paradoxical-decompo.
[2] M. Soleimani Malekan, A. Rejali, Configuration equivalence is not equivalent to isomorphism, Internat. J. Algebra Comput. 27 (2017) 1073–1085.
[3] A. Abdollahi, A. Rejali, G. A. Willis, Group properties characterised by configurations, Illinois J. Math. 48 (2004) 861–873.
[4] M. R. Bridson, A. Hafliger, Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften, Springer Berlin Heidelberg, 2013.
[5] P. de la Harpe, Topics in Geometric Group Theory, Chicago Lectures in Mathematics, University of Chicago Press, 2000.
[6] R. C. Lyndon, P. E. Schup, Combinatorial Group Theory, Classics in Mathematics, Springer Berlin Heidelberg, 2015.
[7] M. Soleimani Malekan, A. Rejali, Two-sided conguration equivalence and isomorphism, Journal of Algebra and Its Applications 19 (2020) 2050189.
[8] A. Rejali, M. Soleimani Malekan, Solubility of groups can be characterized by configuration, New York Journal of Mathematics 23 (2017)1427–1445.
[9] A. Rejali, A. Yousofzadeh, Group properties characterized by two-sided configurations, Algebra Colloquium 17 (2010) 583–594.
[10] J. M. Rosenblatt, G. A. Willis, Weak convergence is not strong convergence for amenable groups, Canad. Math. Bull. 44 (2001) 231–241.

[11] V. Runde, Lectures on Amenability, Number 1774 in Lecture Notes in Mathematics, Springer, 2002.
[12] M. V. Sapir, V. S. Guba, M. V. Volkov, Combinatorial Algebra: Syntax and Semantics, Springer Monographs in Mathematics, Springer International Publishing, 2014.
[13] J. Tits, Free subgroups in linear groups, Journal of Algebra 20 (1972) 250–270.

Citation: M. Soleimani Malekan, Configuration sets; a right place for ping-pong arguments, J. Disc. Math. Appl. 9(3) (2024) 191–202.
https://doi.org/10.22061/jdma.2024.11140.1079