Journal of Discrete Mathematics and Its Applications 9 (3) (2024) 173–179

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Automorphism group of a graph constructed from a lattice

Shahide Malekpour*, **Behnam Bazigaran**

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Kashan, Kashan 87317-53153, I. R. Iran

Academic Editor: Saeid Alikhani

Abstract. Let L be a lattice and S be a $\wedge$-closed subset of L. The graph $\Gamma_S(L)$ is a simple graph with all elements of L as vertex set and two distinct vertex x, y are adjacent if and only if $x \vee y \in S$. In this paper, we verify the automorphism group of $\Gamma_S(L)$ and the relation by automorphism group of the lattice L. Also we study some properties of the graph $\Gamma_S(L)$, where S is a prime filter or an ideal such as the perfect maching.

Keywords: automorphism group of a graph; prime filter; automorphism group of a lattice; perfect maching of a graph.
Mathematics Subject Classification (2010): $05C10$, $06B99$.

1 Introduction

In last decade, lot of mathematicians investigate the graphs associated to various algebraice structures, such as the zero divisor graphs of rings, ordered structures et. al [2,6,7,12].

In [9,10] authores discuss some properties of the graph $\Gamma_S(L)$ which is derived from any finite lattice and determine the realizability of it. In this paper, we determine the group automorphism of $\Gamma_S(L)$ for some especially cases and the relation of it by the automorphism of the lattice L. Furthermore in Section 2, we study some properties of the graph $\Gamma_S(L)$, where S is aprime filter or an ideal.

Now we recall some definitions of Lattice Theory from [4,5].

In this paper L means finite bounded lattice. For two distinct elements x, y of the lattice

*Corresponding author (*Emial address*:malekpoor@grad.kashanu.ac.ir)
Received 11 August 2024 ; Revised 11 August 2024 ; Accepted 25 August 2024
First Publish Date:1 September 2024

L, if $x < y$ and there is no element z in L such that $x < z < y$, we say that y covers x and write $x \prec y$. In bounded lattice L an element $p \in L$ is called an *atom* if $0 \prec p$, also an element $m \in L$ is called a *coatom* of L if $m \prec 1$. The set of all coatoms of L is denoted by $Coatom(L)$ and the set of atoms of L by $Atom(L)$. An element $x \in L$ is called a join irredusible if $x = a \vee b$ implies that $x = a$ or $x = b$. Dually, we have the concept of a meet irreducible element.

A nonempty subset I of a lattice L is called an ideal if $a, b \in I$ implies $a \vee b \in I$ and for any $a \in I$ and $b \in L$, $a \wedge b \in I$. A proper ideal I of a lattice is called prime if $a, b \in L$ and $a \wedge b \in I$ imply $a \in I$ or $b \in I$. Dually, the concept of filter and prime filter are defined.

Let L and L' be two lattices. A mapping $\theta : L \longrightarrow L'$ is called a homomorphism if for any $a, b \in L$, $\theta(a \vee b) = \theta(a) \vee \theta(b)$ and $\theta(a \wedge b) = \theta(a) \wedge \theta(b)$. If the map θ is also bijective, we say that θ is an isomorphism. An isomorphic from L to itself is called automorphism and the set of all automorphisms of the lattice L is denoted by $Aut(L)$.

Let G be an undirected simple graph with vertex set $V(G)$. The notation $\{a, b\} \in E(G)$ means that two vertices a and b are adjacent in G. The notation K_n is used for complete graph with n vertices.

The complement of G is a graph denoted by $\overline{G}$ with the same vertex set of G and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. For $e = \{x, y\} \in E$, $G - e$ is a new graph obtained from graph G by removing the edge e.

An independent edge set in the graph G is a set of edges without common vertices. An independent edge set is also called a matching. An independent edge set M is called perfect matching, if every vertex of G is incident to exactly one edge of M. The size of the largest independent edge set in the graph G is called matching number of G and denoted by $m(G)$.

We recall some graph operations from [8].

Let G and H be two graphs whose the vertex sets are disjoint, the disjoint union $G + H$ is a graph which $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. Also the join graph $G \oplus H$ is isomorphic $\overline{\bar{G} + \bar{H}}$.

A permutation ϕ on the set of vertices $V(G)$ is called a graph automorphism of G, if it is satisfying the property that $\{a, b\} \in E(G)$ if and only if $\{\phi(a), \phi(b)\} \in E(G)$. All graph automorphisms of the graph G is denoted by $Aut(G)$. The symmetric group consisting of all permutations of n elements is represented by S_n. See [3] for more details.

Theorem 1.1. ([11, Theorem 2.4]) For any graph G, $Aut(G) = Aut(\bar{G})$.

Theorem 1.2. ([11, Theorem 1.6]) The set $Aut(G)$ of all graph automorphisms of a graph G forms a group under function composition.

Theorem 1.3. ([11, Theorem 4.1]) The automorphism group of the complete graph on n vertices, $Aut(K_n)$ is isomorphic to S_n.

Theorem 1.4. ([11, Theorem 4.2]) The automorphism group of the complete graph on n vertices with any single edge removed is isomorphic to $S_2 \times S_{n-2}$.

Theorem 1.5. ([11, Theorem 4.3]) The automorphism group of the complete graph on n vertices with two adjacent edges removed is isomorphic to $S_1 \times S_2 \times S_{n-3}$.

In graph theory, the hamiltonian path is a path that visits each vertex presicely once. A hamiltonian graph is a graph which contains a hamiltonian cycle, otherwise it is called non-hamiltonian. An Eulerian path in the graph G is a path that visits each edge exactely once. The graph G is called Eulerian if it contains an Eulerian path which starts and ends on the same vertex. We recall two following theorem about hamiltonian and Eulerian graph [3].

Theorem 1.6. *The graph G is Eulerian if and only if for every $v \in V(G)$, $deg(v)$ is even.*

Theorem 1.7. *If for every $v \in V(G)$, $deg(v) \geq \frac{n}{2}$, then the graph G is hamiltonian.*

2. The group automorphism of $\Gamma_S(L)$

In this section we investigate the group automorphism of $\Gamma_S(L)$ for some especioally cases.

Proposition 2.1. If $S = L$ or $S = L \setminus \{0\}$, then $Aut(\Gamma_S(L))$ is isomorphic to S_n.

Proof. The proof is straightforward by [1, Proposition 2.4] and Theorem 1.3. □

Proposition 2.2. Let L be a lattice with at most two atoms. Consider $S = L \setminus \{p\}$ or $S = L \setminus \{0, p\}$ for some meet-irreducible atom p of L, then $Aut(\Gamma_S(L))$ is isomorphic to $S_2 \times S_{n-2}$.

Proof. Assume that $S = L \setminus \{p\}$ or $S = L \setminus \{0, p\}$. By [9, Theorem 3.8], $\Gamma_S(L) = K_n - e$ which $e = \{0, p\}$. Then the proof complete by Theorem 1.4. □

Proposition 2.3. Suppose that the lattice L has at most three atoms. If $S = L \setminus \{p_1, p_2\}$ or $S = L \setminus \{0, p_1, p_2\}$ for two meet-irreducible elements $p_1, p_2 \in Atom(L)$, then $Aut(\Gamma_S(L))$ is isomorphic to $S_1 \times S_2 \times S_{n-3}$.

Proof. Suppose that $S = L \setminus \{p_1, p_2\}$ or $S = L \setminus \{0, p_1, p_2\}$. Then S is a $\wedge$-closed subset of L in these cases and the graph $\Gamma_S(L)$ can be defined. One can easily chek that $\Gamma_S(L) = K_n - \{e_1, e_2\}$, which $e_i = \{0, p_i\}$. Hence, the result follows from Theorem 1.5. □

Proposition 2.4. Assume that the lattice L has at most four atoms. If $S = L \setminus \{p_1, p_2, p_3\}$ or $S = L \setminus \{0, p_1, p_2, p_3\}$ for three meet-irreducible elements $p_1, p_2, p_3 \in Atom(L)$, then $Aut(\Gamma_S(L))$ is isomorphic to $S_3 \times S_{n-3}$.

Proof. Assume that $S = L \setminus \{p_1, p_2, p_3\}$ or $S = L \setminus \{0, p_1, p_2, p_3\}$. Thus S is a $\wedge$-closed subset of L in these cases, so the graph $\Gamma_S(L)$ can be defined. Also the graph $\Gamma_S(L) = K_n - \{e_1, e_2, e_3\}$, which $e_i = \{0, p_i\}$. By Theorem 1.1, $Aut(\Gamma_S(L)) = Aut(\overline{\Gamma_S(L)}) = Aut(\overline{K_{n-3}} \cup S_3) = S_{n-3} \times S_3$. □

In view of Proposition 2.4, we have the following theorem.

Theorem 2.5. Let L be a lattice and $|Atom(L)| \geq k$. If $S = L \setminus \{p_1, p_2, \ldots, p_k\}$ for meet-irreducible elements $p_1, p_2, \ldots, p_k \in Atom(L)$, then $Aut(\Gamma_S(L))$ is isomorphic to $S_k \times S_{n-k}$.

Proof. Let $S = L \setminus \{p_1, p_2, \ldots, p_k\}$ for meet-irreducible elements $p_1, p_2, \ldots, p_k \in Atom(L)$. Thus S is a $\wedge$-closed subset of L and the graph $\Gamma_S(L)$ can be defined. Also the graph $\Gamma_S(L) = K_n - \{e_1, e_2, \ldots, e_k\}$, which $e_i = \{0, p_i\}$, for $1 \leq i \leq k$. Hence, by Theorem 1.1, $Aut(\Gamma_S(L)) = Aut(\overline{\Gamma_S(L)}) = Aut(\overline{K_{n-k}} \cup S_k) = Aut(\overline{K_{n-k}}) \times Aut(S_k) = S_{n-k} \times S_k$. □

Theorem 2.6. If S is an ideal of the lattice L, then $Aut(\Gamma_S(L)) = S_{|S|} \times S_{|S^c|}$.

Proof. Let S be an ideal of the lattice L. By [10, Theorem 2.7], $\Gamma_S(L) = K_{|S|} + \overline{K}_{|S^c|}$, so by Theorem 1.3, $Aut(\Gamma_S(L)) = S_{|S|} \times S_{|S^c|}$. □

Theorem 2.7. If S is a prime filter of the lattice L, then $Aut(\Gamma_S(L)) = S_{|S|} \times S_{|S^c|}$.

Proof. Let S be a prime filter of the lattice L. By [10, Theorem 2.9], $\Gamma_S(L) = K_{|S|} \oplus \overline{K}_{|S^c|}$, and by Theorem 1.1, $Aut(\Gamma_S(L)) = Aut(\overline{\overline{K_{|S|}} + K_{|S^c|}}) = Aut(\overline{K_{|S|}} + K_{|S^c|}) = S_{|S|} \times S_{|S^c|}$. □

In view of [10, Proposition 2.11] and Theorem 1.1, we have the following lemma.

Lemma 2.8. Assume that $\alpha : L \longrightarrow L'$ is a lattice isomorphism and S is a prime ideal or a filter of L, then

$$Aut(\Gamma_S(L)) \cong Aut(\Gamma_{\alpha(S)^c}(L')).$$

Corollary 2.9. If S is a filter or prime ideal of the lattice L, then $Aut(\Gamma_S(L)) = Aut(\Gamma_{S^c}(L))$.

Proof. Let S be a filter or prime ideal of the lattice L. By [10, Corollary 2.12], $\overline{\Gamma_S(L)} = \Gamma_{S^c}(L)$. Hence, by Theorem 1.1, $Aut(\Gamma_S(L)) = Aut(\Gamma_{S^c}(L))$. □

It is clear that $Aut(L)$ and $Aut(\Gamma_S(L))$ need not be isomorphic. We illustrate this concept with two examples.

Example 2.10. Let the lattice L which is shown in Fig. 1. Then $Aut(L) = \{id,\theta\}$.

$$\theta = \begin{pmatrix} 1\ a\ b\ c\ d\ e\ 0 \\ 1\ b\ a\ c\ d\ e\ 0 \end{pmatrix}.$$

Now consider $S = \{1,a,b,c\}$. Since S is a prime filter of the lattice L, by [10, Theorem 2.9], $\Gamma_S(L) = K_4 \oplus \overline{K_3}$. So by Theorem 2.7, $Aut(\Gamma_S(L)) = S_4 \times S_3$.

Example 2.11. Let $X = \{a,b,c\}$ and $L = (P(X),\subseteq)$ (Fig. 2).
Then $Aut(L) = \{id,\theta_1,\theta_2,\theta_3\}$.

$$\theta_1 = \begin{pmatrix} \{\} & \{a\} & \{b\} & \{c\} & \{a,b\} & \{a,c\} & \{b,c\} & \{a,b,c\} \\ \{\} & \{a\} & \{c\} & \{b\} & \{a,c\} & \{a,b\} & \{b,c\} & \{a,b,c\} \end{pmatrix},$$

$$\theta_2 = \begin{pmatrix} \{\} & \{a\} & \{b\} & \{c\} & \{a,b\} & \{a,c\} & \{b,c\} & \{a,b,c\} \\ \{\} & \{b\} & \{a\} & \{c\} & \{a,b\} & \{b,c\} & \{a,c\} & \{a,b,c\} \end{pmatrix},$$

$$\theta_3 = \begin{pmatrix} \{\} & \{a\} & \{b\} & \{c\} & \{a,b\} & \{a,c\} & \{b,c\} & \{a,b,c\} \\ \{\} & \{c\} & \{b\} & \{a\} & \{b,c\} & \{a,c\} & \{a,b\} & \{a,b,c\} \end{pmatrix}.$$

Now consider $S = \{\{c\},\{b,c\}\}$. The associated graph $\Gamma_S(L)$ is shown in Fig. 3. So the automorphism gruop of $\Gamma_S(L)$ is isomorphic by $S_1 \times S_2 \times S_1 \times S_4$.

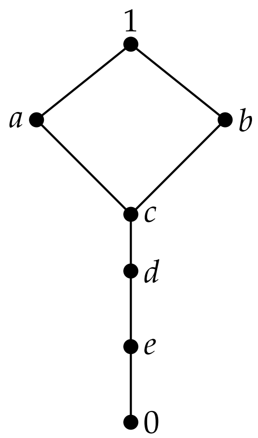

Fig. 1: The lattice L.

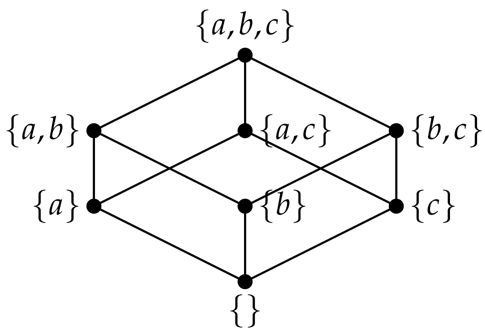

Fig. 2: The lattice of power set $X = \{a,b,c\}$.

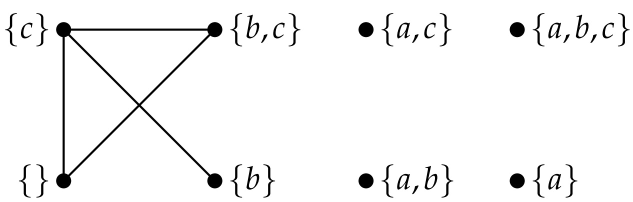

Fig. 3: The graph $\Gamma_S(L)$.

Theorem 2.12. Let $Aut(L,S) = \{\alpha \in Aut(L) : \ \alpha(S) = S\}$.

1. $Aut(L,S)$ is a subgroup of $Aut(L)$,
2. $Aut(L,S)$ is a subgroup of $Aut(\Gamma_S(L))$,
3. $Aut(L) \cap Aut(\Gamma_S(L) = Aut(L,S)$.

Proof. 1. Assume that $\alpha, \beta \in Aut(L,S)$. Since $\alpha \in Aut(L)$, by Theorem 1.2 $\alpha^{-1} \in Aut(L)$ and $\alpha^{-1}(S) = S$, so $\alpha^{-1} \in Aut(L,S)$. Also $\alpha\beta \in Aut(L)$ and $\alpha\beta(S) = \alpha(S) = S$, thus $\alpha\beta \in Aut(L,S)$. Hence $Aut(L,S)$ is a subgroup of $Aut(L)$.

2. It is sufficient to show that $Aut(L,S)$ is a subset of $Aut(\Gamma_S(L))$.

   Let $\alpha \in Aut(L,S)$, hence

$$\{x,y\} \in E(\Gamma_S(L)) \Longleftrightarrow x \vee y \in S \Longleftrightarrow \alpha(x \vee y) \in \alpha(S) = S \\ \Longleftrightarrow \alpha(x) \vee \alpha(y) \in S \Longleftrightarrow \{\alpha(x), \alpha(y)\} \in E(\Gamma_S(L)),$$

   which implies that $\alpha \in Aut(\Gamma_S(L))$.

3. By Part (1) and (2), the set $Aut(L,S) \subset Aut(L) \cap Aut(\Gamma_S(L)$. Now suppose that $\beta \in Aut(L) \cap Aut(\Gamma_S(L)$, we show that $\beta(S) = S$. Let $s \in S$, clearly $\{0,s\} \in E(\Gamma_S(L))$. Since $\beta \in Aut(L)$, $\{\beta(0), \beta(s)\} \in E(\Gamma_S(L))$. We know that $\beta(0) = 0$, so $\{0, \beta(s)\} \in E(\Gamma_S(L))$. By definition of the graph $\Gamma_S(L)$, $\beta(s) \in S$. Thus $\beta(S) \subseteq S$ and this completes the proof. □

Remark 1. *In the Example 2.10, $Aut(L,S) = Aut(L)$. Also one can easily check that for $\theta \in Aut(L,S)$ and $\phi \in Aut(\Gamma_S(L))$, $\phi o \theta o \phi^{-1} \notin Aut(L,S)$.*

$$\phi = \begin{pmatrix} 1\ a\ b\ c\ d\ e\ 0 \\ 1\ c\ b\ a\ d\ e\ 0 \end{pmatrix}.$$

Hence, it is not nessesary that $Aut(L,S)$ is a normal subgroup of $Aut(\Gamma_S(L)$.

References

[1] M. Afkhami, Z. Barati, K. Khashayarmanesh, A graph associated to a lattice, Ric. Mat. 63 (2014) 67–78.

[2] M. Alizadeh, A. K. Das, H. R. Maimani, M. R. Pouranki, S. Yassemi, On the diameter and girth of zero-divisor graphs of posets, Discrete Appl. Math. 160 (2012) 1319–1324.

[3] J. A. Bondy, U. S. R. Murty, Graph Theory with Applications, American Elsevier Publishing Co. Inc. New York, 1976.

[4] T. Donnellan, Lattice Theory, Pergamon Press, Oxford-New York-Toronto, Ont. 1968

[5] G. Grätzer, Lattice Theory: Foundation, Birkhäuser/Springer Basel AG, Basel, 2011.

[6] R. Halas, H. Langer, The zero divisor graph of a qoset, Order 27 (2010) 343–351.

[7] V. V. Joshi, A. U. Khiste, On the zero divisor graphs of pm-lattices, Discrete Math 312 (2012) 2076–2082.
[8] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graph, Second edition. with a foreword by Peter Winkler. Discrete Mathematics and its Applications, Boca Raton, CRC Press, Boca Raton, FL, 2011.
[9] S. Malekpour, B. Bazigaran, On Realizability of Graphs as $\Gamma_S(L)$ for some Lattice L, Southeast Asian Bull. Math. 43 (2019) 537–551.
[10] S. Malekpour, B. Bazigaran, Some properties of a graph associated to a lattice, Quasigroups and Related Systems 24 (2016) 93–102.
[11] L. Rodriguez, Automorphism groups of simple graphs, Mathematics (2014) 1–25.
[12] K. Samei, The zero divisor graph of a reduced ring, J. Pure and Appl. Algebra 209 (2007) 813–821.

Citation: S. Malekpour, B. Bazigaran, Automorphism group of a graph constructed from a lattice, J. Disc. Math. Appl. 9(3) (2024) 173–179.
https://doi.org/10.22061/jdma.2024.11157.1082