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Research Paper

On the hierarchical product of graphs

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Abstract. The hierarchical product of graphs is a variant of the Cartesian product. It is associative, not commutative, and finite connected graphs have unique first prime factors with respect to it. We present examples of infinite graphs with different first prime factors, and show that homogeneous trees of finite degree have unique prime factorizations with respect to the hierarchical product. On the way, we pose two problems.

Keywords. Hierarchical products of finite and infinite graphs, prime factorizations, trees.

Mathematics Subject Classification (2010): 05C05, 05C25, 05C75, 05C76.

1 Introduction

The hierarchical product of graphs was introduced in 2009 by Barriére, Comellas, Dalfó, and Fiol [3]. It is a special case of a product of graphs, whose spectral properties were studied in 1978 by Godsil and McKay [5]. We are interested in its prime factorization properties, and continue the investigations of [6], where it was shown that each finite connected graph X has a first prime factor G with respect to the hierarchical product of graphs, and that the embedding of G into the product is invariant under automorphisms of X.

For other properties of the hierarchical product, and generalizations such as the rooted

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hierarchical product, the generalized hierarchical product, and the rooted generalized hierarchical product we refer to [1,2,4,6].

Here we present examples of infinite graphs with different first prime factors, and show that homogeneous trees of finite degree have unique prime factorizations with respect to the hierarchical product.

We also pose two problems on hierarchical product of trees, respectively of finite connected graphs.

2 Hierarchical products

Given a graph G, we write $V(G)$ for its vertex set, and $E(G)$ for its edge set. $E(G)$ is a set of unordered pairs ab of distinct vertices of G. If $ab \in E(G)$, then we call a,b *adjacent*, in symbols $a \sim b$ or $a \sim_G b$. For a graph G with a distinguished vertex u, called the *root* of G, we use the notation $G[u]$.

Barriére, Comellas, Dalfó, and Fiol [3] define the hierarchical product as a multiary operation, but we only need it as a binary operation. For two factors the *hierarchical product* $G \sqcap H[v]$ of an unrooted graph G by a rooted graph $H[v]$, is defined as an unrooted graph with vertex set $V(G) \times V(H)$, whose edges are

$$(g,h)(g',h') \in E(G \sqcap H[v]) \text{ if } \begin{cases} gg' \in E(G) \text{ and } h = h' = v, \text{ or} \\ hh' \in E(H) \text{ and } g = g'. \end{cases}$$

Figure 1 depicts $K_2 \sqcap K_2[1]$, where $V(K_2) = \{0,1\}$.

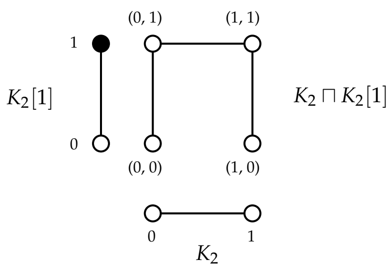

Figure 1. $K_2 \sqcap K_2[1]$.

Hierarchical multiplication is not commutative, because $H[v] \sqcap G$ is not defined. Similarly, it is not associative, because $G_2[v_2] \sqcap G_3[v_3]$ is not defined as a hierarchical product. Hence, $G_1 \sqcap (G_2[v_2] \sqcap G_3[v_3])$ is also not defined, and thus cannot be equal to $(G_1 \sqcap G_2[v_2]) \sqcap G_3[v_3]$.

The unrooted graphs G,H are the *factors* of $G \sqcap H[v]$. For different vertices $v,v' \in V(H)$, the graphs $G \sqcap H[v]$ and $G \sqcap H[v']$ need not be isomorphic. If there is no automorphism of H that maps v into v', then we say that the products $G \sqcap H[v]$ and $G \sqcap H[v']$ are different.

A graph on at least two vertices is *prime with respect to the hierarchical product* if it cannot be represented as the hierarchical product of two graphs different from K_1.

A vertex v of a graph G is a *cutpoint* if $G - v$, that is the graph obtained from G by removal of v and all edges incident with v, is disconnected. It is easy to see that graphs without cutpoints, and graphs with exactly one vertex of degree one, must be prime. Hence all cycles must be prime, and all one-sided infinite paths, which we call *rays*.

Clearly, each finite graph can be represented as a product of prime graphs. For disconnected graphs this presentation need not be unique, as the example

$$K_2 \sqcap (K_1 + K_2)[0] \cong (K_1 + K_1 + K_1) \sqcap K_2[1]$$

of Anderson, Guo, Tenney, and Wash [1] shows. In particular, note that the first factors are different.

Hence, we restrict attention to prime factorizations of connected graphs.

2.1 First prime factors of hierarchical products

Given a product $X = G \sqcap H[v]$, the subgraph of X induced by the vertices $\{(g,v) \,|\, g \in V(G)\}$ is isomorphic to G. We denote it by $G \times v$. Similarly, for each $g \in V(G)$, the set $\{(g,h) \,|\, h \in V(H)\}$ induces a subgraph isomorphic to H, which we denote by $g \times H$. We also set

$$V(G) \times H = \bigcup_{g \in V(G)} g \times H.$$

Clearly, for all $g \in V(G)$, we have $(G \times v) \cap (g \times H) = \{(g,v)\}$.

With this notation,

$$G \sqcap H[v] = (G \times v) \cup (V(G) \times H),$$

and $X - E(G \times v) = V(G) \times H$ consists of $|G|$ copies of H. In other words, the graphs $g \times H$ are uniquely determined by $G \times v$, and thus also $H[v]$. Moreover, each vertex of $G \times v$ is a cutpoint of $G \sqcap H[v]$.

In Imrich, Kalinowski and Pilsniak [6] the following results are shown:

Theorem 1 ([6, Theorem 1]). *To any finite connected graph $X \neq K_1$, there exists a unique graph G that is prime with respect to the hierarchical product, and a unique rooted graph $H[v]$, possibly trivial, such that*

$$X = G \sqcap H[v].$$

Furthermore, $G \times v$ is invariant under all automorphisms of X.

Corollary 2 (Standard prime factorization with respect to the hierarchical product, [6, Corollary 2]). *Each finite connected graph G has a unique standard prime factorization as a hierarchical product*

$$G = G_1 \sqcap (G_2 \sqcap (G_3 \sqcap (\cdots \sqcap G_k[v_k])[v_{k-1}]) \cdots [v_3])[v_2],$$

of uniquely determined prime factors $G_1, \ldots, G_k$ and uniquely determined roots $v_2, \ldots, v_k$, where each root v_i, for $2 \le i \le k$, is a vertex of the product of the last $k - i + 1$ factors.

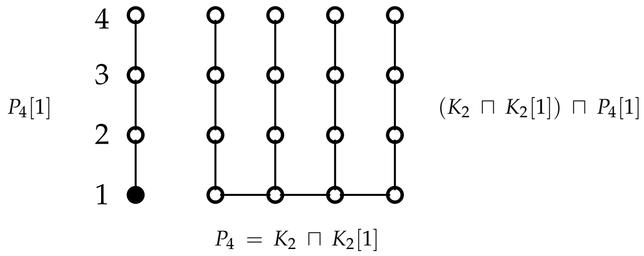

Figure 2. Non-standard prime factorization.

For an example of a prime factorization that is different from the standard prime factorization, let $V(K_2) = \{0,1\}$, and 1 be a leaf of $P_4[1]$. Both K_2 and $P_4[1]$ are indecomposable by the hierarchical product, and $(K_2 \sqcap K_2[1]) \sqcap P_4[1]$ is a prime factorization of the tree with standard prime factorization $K_2 \sqcap (K_2 \sqcap (K_2 \sqcap K_2[1])[0,0])[0,0]$, see Figures 2 and 3.

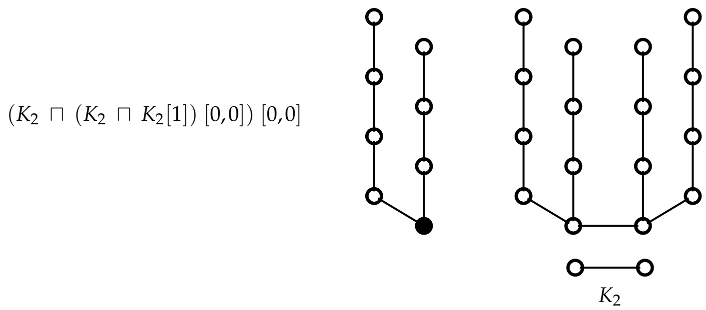

Figure 3. Standard prime factorization.

Given a standard representation, one cannot always regroup parentheses by choosing appropriate roots, as the following example shows:

$$\begin{aligned} K_2 \sqcap P_4[1] &= K_2 \sqcap (K_2 \sqcap K_2[1])[0,0] \\ &\not\cong (K_2 \sqcap K_2[1]) \sqcap K_2[1] \\ &= P_4 \sqcap K_2[1], \end{aligned}$$

because $K_2 \sqcap P_4[1]$ has two vertices of degree 1, but $P_4 \sqcap K_2[1]$ has four.

This leads to the following problems.

Problem 3. *Given a tree T with the representation*

$$T = T_1 \sqcap (T_2 \sqcap [T_3 \sqcap (\cdots \sqcap T_n[v_n])[v_{n-1}]) \cdots [v_3])[v_2],$$

and an arbitrary binary bracketing of $T_1, T_2, \dots T_n$, is there a criterion to decide whether there exist $n-1$ roots, such that the hierarchical product of the factors with the given bracketing and these roots is isomorphic to T?

For example, let $n = 4$ and $(T_1, T_2)(T_3, T_4)$ be a given binary bracketing of $T_1, \dots, T_4$. When are there roots $v_2 \in V(T_2), v_4 \in V(T_4)$, and $v = (u_3, u_4) \in V(T_3) \times V(T_4)$, such that

$$T \cong (T_1 \sqcap T_2[v_2])(T_3 \sqcap T_4[v_4])[v]?$$

Note that the number of binary bracketings of n letters are the Catalan numbers C_{n-1}, where $C_n = \frac{1}{n+1}\binom{2}{n}$.

We formulated the problem for trees, where it seems more tractable. But, of course the analogous problem holds if one replaces the T_i by connected graphs G_i.

Problem 4. *Given a prime factoritzation of a connected graph G in standard form, and a prime factorization of G in non-standard form. Are the prime factors isomorphic as unrooted graphs?*

3 First prime factors in infinite graphs

Easy examples of infinite graphs without unique first prime factor are free products. Consider the free product $C_m * C_n$, where m, n are integers ≥ 2. It consists of cycles C_m, C_n, where each vertex is contained in exactly one C_m, and exactly one C_n. Clearly its number of vertices is countably infinite, and both C_m and C_n are prime with respect to the hierarchical product.

To present $X = C_m * C_n$ as a hierarchical product

$$C_m \sqcap C'_m[v],$$

let C'_m be a component of the graph obtained from X by removal of the edges of a single C_n, and v be the vertex of degree 2 in C'_m.

Note that C'_m is prime, because it has only one vertex of degree 2, say w, and because this vertex is not a cutpoint. If C'_m were a non-trivial product $G \sqcap H[h]$, then w must be contained in some $g \times H$, where $g \in V(G)$. If $w = (g, u) \neq (g, h)$, then $(g', u) \in V(C'_m)$ for all g' $V(G)$. All these vertices have degree 2 in C'_m, and because G is non-trivial, this implies that w is not the only vertex of degree 2. Hence $w = (g, h) \in G \times h$, and thus a cutpoint of C'_m, but this is not possible, because w is in an m-cycle.

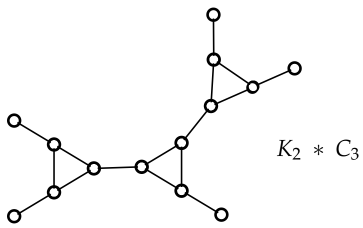

Figure 4. Free product of K_2 by C_3.

Figure 4 shows $Y = K_2 * C_3$, that is, the graph obtained from $C_2 * C_3$ by replacing multiple edges by single edges. By the same arguments as before, both K_2 and C_3 are first prime factors of Y.

One can easily obtain a tree of maximal degree 3 with nonunique first prime factors from it. One just chooses a root v in Y and removes all triangle edges whose endpoints have the same distance from v. Clearly one obtains a tree, say Z, depicted in Figure 5.

It is not hard see that both the path P_3 and K_2 are first factors of Z with respect to the hierarchical product:

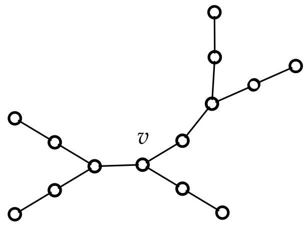

Figure 5. The tree Z.

For K_2 as a first factor, construct a second factor $B'[u]$ as follows: Let u be the vertex of degree 2 in the binary tree B, and subdivide each edge of B by a single point. See Figure 6.

For P_3 as a first factor, construct a second factor $B''[v]$ from B' by adding a single edge to u and let its other endpoint be the root v of B''. See Figure 7.

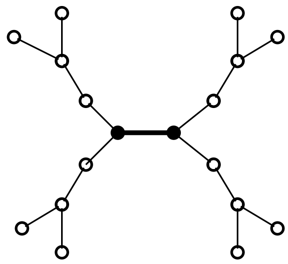

Figure 6. $K_2 \sqcap B'[1]$.

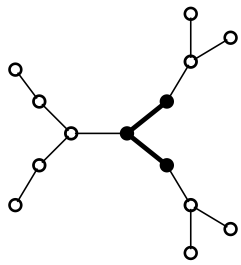

Figure 7. $P_3 \sqcap B''[1]$.

Figure 8. Two different first prime factors for the tree in Figure 5.

Finally, let us point out that for any given finite tree T,

$$T \sqcap T_{\aleph_0}[v] \cong T_{\aleph_0},$$

where $T_{\aleph_0}$ is the homogeneous tree of degree $\aleph_0$, and v an arbitrary vertex of $T_{\aleph_0}$. See Figure 9. It means that any finite tree is a first factor of $T_{\aleph_0}$, and thus any finite tree that is prime with respect to the hierarchical product is a first factor of $T_{\aleph_0}$.

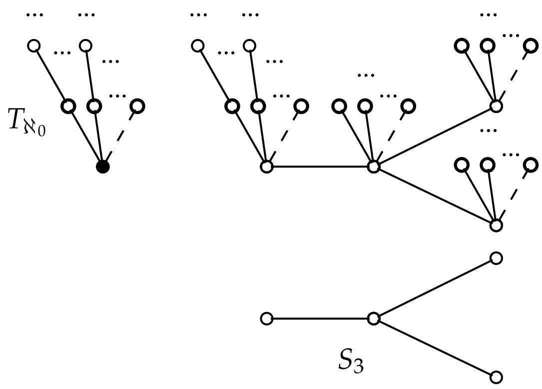

Figure 9. $S_3 \sqcap T_{\aleph_0} \cong T_{\aleph_0}$.

4 Factorization of homogeneous trees of finite degree

We begin with the observation that the two-sided infinite path, which can also be considered as a homogeneous tree of degree 2, denoted T_2, can be factored as

$$T_2 = K_2 \sqcap R[0],$$

where R is a ray, and 0 its only vertex of degree 1. It is easily seen that this is the only factorization, because if $T_2 = G \sqcap H[h]$, then each vertex of G has to be of degree 1, otherwise the product would contain vertices of degree ≥ 3. Hence $G = K_2$, and H is a ray.

Therefore, we have a unique first prime factor K_2. Contrary to the finite case, $K_2 \times 0$ is not invariant under automorphisms of T_2.

Recalling that R is prime, this means that T_2 has unique prime factorization with respect to the hierarchical product. The question arises whether this is true for all homogeneous trees of finite degree. We show that this is indeed the case.

For $n \in \mathbb{N}, k \in \{1, \ldots, n-1\}$ we define T_n^{n-k} as an infinite tree with a unique vertex u such that

$$deg(u) = n - k \text{ and } \forall v \in V(T) \setminus \{u\} : deg(v) = n.$$

If we consider a rooted tree $T_n^{n-k}[0]$, then the root 0 will always be the unique vertex of degree $n - k$. We also set $T_n^n = T_n$.

We now prove that each tree T_n has a unique presentation as a hierarchical product in the standard form. We begin with the following observation

Lemma 5. *For $n \geq 2$ K_2 is a possible first prime factor of T_n.*

Proof. Let $n \geq 2$ and consider $T_n^{n-1}[0]$ where 0 denotes the unique vertex of degree $n-1$ in T_n^{n-1}. Then $K_2 \sqcap T_n^{n-1}[0]$ is a regular tree of degree n, and thus T_n. □

The Figure 10 illustrates the case for T_4.

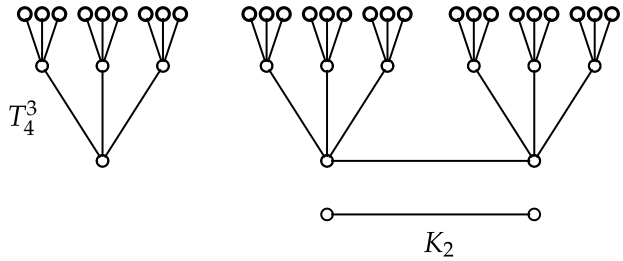

Figure 10. T_4

Now we might ask, what other graph could be the possible first prime factor of T_n. Because T_n is a tree, its factors also have to be trees. Moreover, T_n is regular, so the first factor also must be regular. Therefore, the only possibilities for a first factor of T_n (different from K_2) are T_k for $k \in \{2, \dots, n-1\}$, which are not prime by Remark 5.

Corollary 6. *For each $n \in \mathbb{N}$, $G = K_2$ is the only possible first prime factor of T_n. Furthermore,*

$$T_n = K_2 \sqcap T_n^{n-1}[0].$$

In order to find the standard form of T_n we have to factor T_n^{n-1}.

Lemma 7. *For $n \geq 2, k \in \{0, \dots, n-2\}$ the following equation holds*

$$T_n^{n-k} = T_{k+1}^1 \sqcap T_n^{n-k-1}[0].$$

Proof. Let $G = T_{k+1}^1$. This graph has a unique vertex of degree 1, all other vertices have degree $k+1$. The unique vertex of degree $n-k$ in the product must be in $G \times \{0\}$, and the root of the second factor has to be of degree $n-k-1$. Because all other vertices in the product must have degree n, we conclude that the second factor is T_n^{n-k-1}. □

This helps us to find a factorization of T_n^{n-k} in the general case. Clearly, T_{k+1}^1 is prime, because it has a unique vertex of degree one. It is thus a first prime factor of T_n^{n-k}. Considering another candidate for the first prime factor, it is clear that it needs to be a tree with one unique vertex u of a certain degree, and that all other vertices must be of the same degree, which different from $deg(u)$. Hence, it will be of the form T_{k+l}^l, where $l > 1$. But then, it is not prime by Lemma 7.

Theorem 8. *For each $n \in \mathbb{N}$, the regular infinite tree T_n has a unique presentation as a hierarchical product in the standard form. It is given by the formula*

$$T_n = T_1^1 \sqcap (T_2^1 \sqcap (\cdots \sqcap T_n^1[v_n])[v_{n-1}]) \cdots)[v_2],$$

where each root v_i, $2 \leq i \leq n$, is the unique vertex of degree $n-i+1$ in the product of the last $n-i+1$ factors.

Proof. By Corollary 6 we know that $T_n = K_2 \sqcap T_n^{n-1}[v_2] = T_1^1 \sqcap T_n^{n-1}[v_2]$.

Now an application of Lemma 7 shows that

$$T_n = T_1^1 \sqcap (T_2^1 \sqcap T_n^{n-2}[v_3])[v_2],$$

and the proof is completed by induction. □

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