Journal of Discrete Mathematics and Its Applications 9 (2) (2024) 133–145

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Shahid Rajaee Teacher Training University

Research Paper

A survey on automorphism groups and transmission-based graph invariants

Reza Sharafdini[1, *], **Mehdi Azadimotlagh**[2]

[1] Department of Mathematics, Persian Gulf University, Bushehr 75169, IRAN

[2] Department of Computer Engineering of Jam, Persian Gulf University, Jam, IRAN

Academic Editor: Hamidreza Maimani

Abstract. The distance $d(u,v)$ between vertices u and v of a simple connected graph G is equal to the number of edges in a minimal path connecting them. The transmission of a vertex v is defined by $\sigma(v) = \sum_{u \in V(G)} d(v,u)$. A graph invariant (topological index) is said to be a transmission-based topological index (TT index) if it includes the transmissions $\sigma(u)$ of vertices of G. Because $\sigma(u)$ can be derived from the distance matrix of G, it follows that transmission-based topological indices form a subset of distance-based topological indices. In this article we survey some results on the computation of some transmission-based graph invariants of intersection graph, hypercube graph, Kneser graph, Paley graph and unitary Cayley graph.

Keywords. Transmission, Wiener index, hypercube graph, intersection graph, Kneser graph, unitary Cayley graph, Paley graph.

Mathematics Subject Classification (2010): 05C82, 91D30.

1 Introduction

Let G be a simple connected graph with the finite vertex set $V(G)$ and the edge set $E(G)$, and denote by $n = |V(G)|$ and $m = |E(G)|$ the number of vertices and edges, respectively. Using the standard terminology in graph theory, we refer the reader to [26]. The degree $d(u)$

---

*Corresponding author (Email address: sharafdini@pgu.ac.ir).
Received 2 August 2023; Revised 7 January 2024; Accepted 17 January 2024
First Publish Date: 1 June 2024

of the vertex $u \in V(G)$ is the number of the edges incident to u. A simple connected graph G is called *vertex-degree regular* if $d(u) = d(v)$ for any vertex u and v of G. The edge of the graph G connecting the vertices u and v is denoted by uv.

The *distance* between the vertices u and v in graph G is denoted by $d(u,v)$ and it is defined as the number of edges in a minimal path connecting them. The *eccentricity* $\varepsilon(v)$ of a vertex v is the maximum distance from v to any other vertex. A simple connected graph G is called *self-center* if $\varepsilon(u) = \varepsilon(v)$ for any vertex u and v of G. The *diameter* $\mathrm{diam}(G)$ of G is the maximum eccentricity among the vertices of G. The *transmission* (or *status*, or *(total) distance*) of a vertex v of G is defined as $\sigma(v) = \sigma_G(v) = \sum_{u \in V(G)} d(v,u)$. A graph G is said to be *transmission regular* [1] if $\sigma(u) = \sigma(v)$ for any vertex u and v of G. A transmission regular graph G is called *k-transmission regular* if there exists a positive integer k, for which $\sigma(v) = k$ for any vertex v of G.

The role of molecular descriptors (especially topological descriptors) is remarkable in mathematical chemistry especially in QSPR/QSAR investigations. A topological index is said to be a transmission-based topological index (TT index) if it includes the transmissions $\sigma(u)$ of vertices of G. Because $\sigma(u)$ can be derived from the distance matrix of G, it follows that transmission-based topological indices form a subset of distance-based topological indices. Now let us introduce some of them. The oldest and most famous one is the *Wiener* index, $W(G)$, represented in [16] as

$$W(G) = \frac{1}{2} \sum_{u \in VG)} \sum_{v \in V(G)} d(u,v) = \frac{1}{2} \sum_{u \in VG)} \sigma(u).$$

The *variable transmission Zagreb* index $MS^{\lambda}(G)$, introduced in [19] as

$$MS^{\lambda}(G) = \sum_{u \in V(G)} \sigma(u)^{2\lambda},$$

where λ is a real number. The *transmission variance* $\mathrm{Var}_{\mathrm{Tr}}(G)$, introduced in [19] as

$$\begin{aligned}\mathrm{Var}_{\mathrm{Tr}}(G) &= \frac{1}{n} \sum_{u \in V(G)} \left(\sigma_G(u) - \frac{2W(G)}{n}\right)^2 = \frac{1}{n} \sum_{u \in V(G)} \sigma_G(u)^2 - \frac{4W(G)^2}{n^2} \\ &= \frac{MS^1(G)}{n} - \frac{4W(G)^2}{n^2} \geq 0,\end{aligned} \tag{1}$$

where $\frac{2W(G)}{n}$ is the average vertex transmission of G.

Many transmission-based topological indices can be represented in the form:

$$TI = \sum_{uv \in E(G)} \mathcal{F}(\sigma(u), \sigma(v)), \tag{2}$$

where $\mathcal{F}$ is a real non-negative symmetric function, i.e., $\mathcal{F}(x,y) = \mathcal{F}(y,x)$, defined on a cartesian product $D \times D$, where $D = \{\sigma(v_1), \ldots, \sigma(v_n)\}$. Here we list some particular cases obtained from (2) by appropriate choice of function $\mathcal{F}(x,y)$ that are of interest for the present paper.

1. For $\mathcal{F}(x,y) = x + y$, we get the *first transmission Zagreb* index $MS_1(G)$, introduced in [21] as
$$MS_1(G) = \sum_{uv\in E(G)} \sigma(u) + \sigma(v) = \sum_{u\in V(G)} d(u)\sigma(u).$$

2. For $\mathcal{F}(x,y) = xy$, we get the *second transmission Zagreb* index $MS_2(G)$, introduced in [21] as
$$MS_2(G) = \sum_{uv\in E(G)} \sigma(u)\sigma(v).$$

3. For $\mathcal{F}(x,y) = \frac{1}{\sqrt{xy}}$, we get the *transmission Randić* index $RS(G)$, introduced in [19] as
$$RS(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{\sigma(u)\sigma(v)}}.$$

4. For $\mathcal{F}(x,y) = \frac{1}{\sqrt{x+y}}$, we get the *transmission ordinary sum-connectivity* index $XS(G)$, introduced in [19] as
$$XS(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{\sigma(u) + \sigma(v)}}.$$

5. For $\mathcal{F}(x,y) = \frac{2}{x+y}$, we get the *transmission ordinary sum-connectivity* index $XS(G)$, introduced in [19] as
$$HS(G) = \sum_{uv\in E(G)} \frac{2}{\sigma(u) + \sigma(v)}.$$

6. For $\mathcal{F}(x,y) = \frac{2\sqrt{xy}}{x+y}$, we get the *transmission geometric-arithmetic* index $GAS(G)$, introduced in [19] as
$$GAS(G) = \frac{n}{2m} \sum_{uv\in E(G)} \frac{2\sqrt{\sigma(u)\sigma(v)}}{\sigma(u) + \sigma(v)}.$$

7. For $\mathcal{F}(x,y) = x^{2\lambda-1} + y^{2\lambda-1}$, where λ is an arbitrary real number, the *variable degree transmission Zagreb* index $MSD^{\lambda}(G)$, introduced in [19] as
$$MSD^{\lambda}(G) = \sum_{uv\in E(G)} \sigma(u)^{2\lambda-1} + \sigma(v)^{2\lambda-1} = \sum_{u\in V(G)} d(u)\sigma(u)^{2\lambda-1}.$$

8. For $\mathcal{F}(x,y) = |x - y|$, the *transmission irregularity* $\mathrm{irr}_{\mathrm{Tr}}(G)$, introduced in [19] as
$$\mathrm{irr}_{\mathrm{Tr}}(G) = \sum_{uv\in E(G)} |\sigma_G(u) - \sigma_G(v)|. \tag{3}$$

The *Balaban* index $J(G)$ and the *sum-Balaban* index $SJ(G)$ represent a particular class of transmission-based topological indices. They are defined as [3–5,7],

$$J(G) = \frac{m}{m-n+2} \sum_{uv \in E(G)} \frac{1}{\sqrt{\sigma(u)\sigma(v)}} = \frac{m}{m-n+2} RS(G),$$

$$SJ(G) = \frac{m}{m-n+2} \sum_{uv \in E(G)} \frac{1}{\sqrt{\sigma(u)+\sigma(v)}} = \frac{m}{m-n+2} XS(G).$$

If ω is a vertex weight of graph G, then one can see that

$$\sum_{\{u,v\} \subseteq V(G)} (\omega(u) + \omega(v))\, d(u,v) = \sum_{v \in V(G)} \omega(v)\sigma(v). \tag{4}$$

The *eccentric distance sum* of a graph G, denoted by $\xi^d(G)$, defined as [13]

$$\xi^d(G) = \sum_{u \in V(G)} \varepsilon(u)\sigma(u).$$

It follows from Eq. (4) that

$$\xi^d(G) = \sum_{\{u,v\} \subseteq V(G)} (\varepsilon(u) + \varepsilon(v)) d(u,v) = \sum_{v \in V(G)} \varepsilon(v)\sigma(v). \tag{5}$$

Note that $MS_1(G)$ coincides with the *degree distance* $DD(G)$ that was introduced in [8, 14], and [24]. In fact by Eq. (4),

$$DD(G) = \sum_{\{u,v\} \subseteq V(G)} (d(u) + d(v)) d(u,v) = \sum_{v \in V(G)} d(v)\sigma(v) = MS_1(G). \tag{6}$$

Consequently, if G is a k-transmission regular graph with m vertices, then

$$DD(G) = MS_1(G) = 2mk.$$

In this article we aim to survey the method which applies group theory to graph theory. This method is applied to calculate transmissions-based graph invariants of intersection graph, hypercube graph, Kneser graph, unitary Cayley graph and Paley graph.

2 The method

Let Γ be a group acting on a set X. We shall denote the action of $\alpha \in \Gamma$ on $x \in X$ by x^α. Then $U \subseteq X$ is call an *orbit* of Γ on X if for every $x, y \in U$ there exists $\alpha \in \Gamma$ such that $x^\alpha = y$. The action of group Γ on X is called *transitive* if X is itself an orbit of Γ on X.

Let G be a graph. A bijection α on $V(G)$ is called an *automorphism* of G if it preserves $E(G)$. In other words, α is an automorphism if for each $u, v \in V(G)$, $e = uv \in E(G)$ if and only if $u^\alpha v^\alpha \in E(G)$. Let us denote by $\mathrm{Aut}(G)$ the set of all automorphisms of G.

It is known that $\mathrm{Aut}(G)$ forms a group under the composition of mappings. This is a subgroup of the symmetric group on $V(G)$. Note that $\mathrm{Aut}(G)$ acts on $V(G)$ naturally, i.e., for each $\alpha \in \mathrm{Aut}(G)$ and $v \in V(G)$ the action of α on v, v^α, is defined as $\alpha(v)$. The action of

$\mathrm{Aut}(G)$ on $V(G)$ induces an action on $E(G)$. In fact, for $\alpha \in \mathrm{Aut}(G)$ and $e = uv \in E(G)$, the action of α on $e = uv$, e^α, is defined as $u^\alpha v^\alpha$.

A graph G is called *vertex-transitive* (*edge-transitive*) if the action of $\mathrm{Aut}(G)$ on $V(G)$ ($E(G)$) is transitive. Let G be a graph, $V_1, V_2, \ldots, V_t$ be the orbits of $\mathrm{Aut}(G)$ under its natural action on $V(G)$. Then for each $1 \leq i \leq t$ and for $u, v \in V_i$,

$$\sigma(u) = \sigma(v), \quad \varepsilon(u) = \varepsilon(v), \quad d(u) = d(v).$$

In particular, if G is vertex-transitive (i.e., $t = 1$), then for each $u, v \in V(G)$,

$$\sigma(u) = \sigma(v), \quad \varepsilon(u) = \varepsilon(v), \quad d(u) = d(v).$$

In fact, vertex-transitive graphs are transmission regular, self-center and vertex-degree regular [9,12], but note vice versa, see Figure 1.

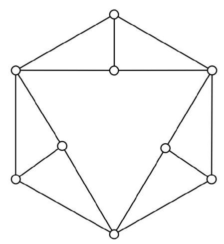

Figure 1. A transmission regular graph but not vertex-degree regular graph with the smallest order

Example 2.1. *Consider the graph* Γ *depicted in Figure 2. The orbits of* $\mathrm{Aut}(G)$ *on* $V(G)$ *are*

$$V_1 = \{x\},\ V_2 = \{y, z\},\ V_3 = \{u, v, w, h\}.$$

The orbits of $\mathrm{Aut}(G)$ *on* $E(G)$ *are*

$$E_1 = \{xy, xz\},\ E_2 = \{yu, yv, zw, zh\}.$$

By a simple calculation, we have

$$\sigma(x) = 10, \quad \sigma(y) = \sigma(z) = 11, \quad \sigma(u) = \sigma(v) = \sigma(w) = \sigma(h) = 16.$$

$$\varepsilon(x) = 2, \quad \varepsilon(y) = \varepsilon(z) = 3, \quad \varepsilon(u) = \varepsilon(v) = \varepsilon(w) = \varepsilon(h) = 4.$$

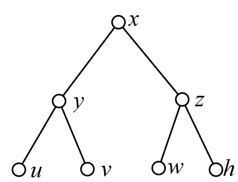

Figure 2. The graph Γ in Example 2.1

Let G be a group with identity 1. For $S \subset G$, $1 \notin S$ and $S^{-1} := \{s^{-1} \mid s \in S\} = S$ the Cayley graph $\Gamma = \mathrm{Cay}(G,S)$ is the undirected graph having vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{\{a,b\}|ab^{-1} \in S\}$. The Cayley graph Γ is vertex-degree regular of degree $|S|$. Its connected components are the right cosets of the subgroup generated by S. So Γ is connected, if S generates G. Every Cayley graph is vertex-transitive but the converse is not true. The Petersen graph on 10 vertices is the smallest example of a vertex-transitive graph which is not a Cayley graph. More information about Cayley graphs can be found in the books on algebraic graph theory by Biggs [6].

Theorem 2.2. *[18] Let Γ be a graph. Then $\Gamma \cong \mathrm{Cay}(G,S)$ for some group G and $S \subseteq G$ if and only if $\mathrm{Aut}(\Gamma)$ has a regular subgroup isomorphic to G.*

In this article the following results are frequently used:

Lemma 2.3. *Let G be a connected k-transmission regular graph with m edges. Then $MS_1(G) = 2mk$ and $MS_2(G) = mk^2$.*

Theorem 2.4. *[2] Let G be a connected graph on n vertices with the automorphism group $\mathrm{Aut}(G)$ and the vertex set $V(G)$. Let $V_1, V_2, \cdots, V_t$ be all orbits of the action $\mathrm{Aut}(G)$ on $V(G)$. Suppose that for each $1 \leq i \leq t$, k_i are the transmission of vertices in the orbit V_i, respectively. Then*

$$W(G) = \frac{1}{2}\sum_{i=1}^{t}|V_i|k_i.$$

Specially if G is vertex-transitive (i.e., $t = 1$), then $W(G) = \frac{1}{2}nk$, where k is the transmission of each vertex of G respectively.

The following result holds by the same idea of that of Theorem 2.4:

Theorem 2.5. *[25] Let G be a connected graph on n vertices with the automorphism group $\mathrm{Aut}(G)$ and the vertex set $V(G)$. Let $V_1, V_2, \cdots, V_t$ be all orbits of the action of $\mathrm{Aut}(G)$ on $V(G)$. Suppose that for each $1 \leq i \leq t$, d_i and k_i are, respectively, the vertex-degree and the transmission of vertices in the orbit V_i. Then*

$$MS_1(G) = \sum_{i=1}^{t}|V_i|d_ik_i.$$

Specially if G is vertex-transitive (i.e., $t = 1$), then

$$MS_1(G) = ndk, \quad MS_2(G) = \frac{1}{2}ndk^2,$$

where d and k are the vertex-degree and the transmission of each vertex of G, respectively.

Lemma 2.6. *[19] Let G be a connected vertex-transitive graph with n vertices and m edges and the with the vertex-degree r. Then*

$$SJ(G) = \frac{m^2\sqrt{n}}{2(m-n+2)\sqrt{W(G)}}, \quad GAS(G) = \frac{2W(G)}{n},$$

$$HS(G) = \frac{nm}{2W(G)} = \frac{n^2 r}{4W(G)},$$

$$J(G) = \frac{m^2 n}{2(m-n+2)W(G)} = \frac{mn^2 r}{4(m-n+2)W(G)}.$$

Theorem 2.7. *[19] Let G be a connected graph with n vertices and m edges. Let us denote the orbits of the action of* $\mathrm{Aut}(G)$ *on* $E(G)$ *by* $E_1, E_2, \ldots, E_l$*. Suppose that for each* $1 \le i \le l$, $e_i = u_i v_i$ *is a fixed edge in the orbit* E_i*. Then*

$$HS(G) = \sum_{i=1}^{l} \frac{2|E_i|}{\sigma(u_i) + \sigma(v_i)}, \quad SJ(G) = \frac{m}{m-n+2} \sum_{i=1}^{l} \frac{|E_i|}{\sqrt{\sigma(u_i) + \sigma(v_i)}},$$

$$GAS(G) = \frac{n}{2m} \sum_{i=1}^{l} \frac{|E_i|\sqrt{\sigma(u_i)\sigma(v_i)}}{\sigma(u_i) + \sigma(v_i)}, \quad \mathrm{irr}_{\mathrm{Tr}}(G) = \sum_{i=1}^{l} |E_i|\, |\sigma(u_i) - \sigma(v_i)|,$$

$$MS_1(G) = \sum_{i=1}^{l} |E_i|(\sigma(u_i) + \sigma(v_i)), \quad MS_2(G) = \sum_{i=1}^{l} |E_i|\sigma(u_i)\sigma(v_i).$$

Example 2.8. *Let us apply Theorem 2.7 to calculate some transmission-based topological indices of the graph* Γ *considered in Example 2.1.*

$$HS(G) = \frac{4}{21} + \frac{8}{27}, \quad SJ(G) = 6\left(\frac{2}{\sqrt{21}} + \frac{4}{\sqrt{27}}\right),$$

$$GAS(G) = \frac{7}{12}\left(\frac{2\sqrt{10 \times 11}}{21} + \frac{4\sqrt{11 \times 16}}{27}\right), \quad \mathrm{irr}_{\mathrm{Tr}}(G) = 2 \times |10 - 11| + 4 \times |11 - 16|,$$

$$MS_1(G) = 2(10 + 11) + 4(11 + 16), \quad MS_2(G) = 2(10 \times 11) + 4(11 \times 16).$$

3 Main results

In this section we will consider a few families of graphs and find their transmission-based indices using the results mentioned in Section 2.

Following [15] we recall intersection graphs as follows. Let S be a set and $\mathcal{F} = \{S_1, \ldots, S_q\}$ be a non-empty family of distinct non-empty subsets of S such that $S = \bigcup_{i=1}^{q} S_i$. The intersection graph of S which is denoted by $\Omega(\mathcal{F})$ has $\mathcal{F}$ as its set of vertices and two distinct vertices S_i, S_j, $i \neq j$, are adjacent if and only if $S_i \bigcap S_j \neq \emptyset$. Here we will consider a set S of cardinality p and let $\mathcal{F}$ be the set of all subsets of S of cardinality t, $1 < t < p$, which is denoted by $S^{\{t\}}$. Upon convenience we may set $S = \{1, 2, \ldots, p\}$. Let us denote the intersection graph $\Omega(S^{\{t\}})$ by $\Gamma_p^{\{t\}} = (V, E)$. The number of vertices of this graph is $\binom{p}{t}$, the degree d of each vertex is as follows:

$$d = \begin{cases} \binom{p}{t} - \binom{p-t}{t} - 1 & p \ge 2t; \\ \binom{p}{t} - 1 & p < 2t. \end{cases}$$

The number of its edges is as follows:

$$|E| = \begin{cases} \frac{1}{2}\binom{p}{t}\left(\binom{p}{t} - \binom{p-t}{t} - 1\right) & p \geq 2t; \\ \frac{1}{2}\binom{p}{t}\left(\binom{p}{t} - 1\right) & p < 2t. \end{cases}$$

Lemma 3.1. *[9] The following hold for the intersection graph $\Gamma_p^{\{k\}}$:*

1. *The automorphism group of $\Gamma_p^{\{k\}}$ has a subgroup isomorphic to the symmetric group on n letters.*
2. *For $p \geq 2k$, the automorphism group of $\Gamma_p^{\{k\}}$ has $k-1$ orbits on the set E of edges of $\Gamma_p^{\{k\}}$. If $p < 2k$, then $\mathrm{Aut}\left(\Gamma_p^{\{k\}}\right)$ has $n-k$ orbits on $E\left(\Gamma_p^{\{k\}}\right)$.*

Lemma 3.2. *[9] The intersection graph $\Gamma_p^{\{t\}}$ is vertex-transitive and for any t-element subset A of S we have*

$$\sigma_{\Gamma_p^{\{t\}}}(A) = \begin{cases} \binom{p}{t} + \binom{p-t}{t} - 1 & p \geq 2t; \\ \binom{p}{t} - 1 & p < 2t. \end{cases}$$

Moreover,

$$W(\Gamma_p^{\{t\}}) = \begin{cases} \frac{1}{2}\binom{p}{t}\left(\binom{p}{t} + \binom{p-t}{t} - 1\right) & p \geq 2t; \\ \frac{1}{2}\binom{p}{t}\left(\binom{p}{t} - 1\right) & p < 2t. \end{cases}$$

As a direct consequence of Theorem 2.5 and Lemma 3.2, we have the following:

Theorem 3.3. *[25]*

$$MS_1(\Gamma_p^{\{t\}}) = \begin{cases} \binom{p}{t}\left(\binom{p}{t} - \binom{p-t}{t} - 1\right)\left(\binom{p}{t} + \binom{p-t}{t} - 1\right) & p \geq 2t; \\ \binom{p}{t}\left(\binom{p}{t} - 1\right)^2 & p < 2t. \end{cases}$$

$$MS_2(\Gamma_p^{\{t\}}) = \begin{cases} \frac{1}{2}\binom{p}{t}\left(\binom{p}{t} - \binom{p-t}{t} - 1\right)\left(\binom{p}{t} + \binom{p-t}{t} - 1\right)^2 & p \geq 2t; \\ \frac{1}{2}\binom{p}{t}\left(\binom{p}{t} - 1\right)^3 & p < 2t. \end{cases}$$

The vertex set of the hypercube H_n consists of all n-tuples $(b_1, b_2, \cdots, b_n)$ with $b_i \in \{0,1\}$. Two vertices are adjacent if the corresponding tuples differ in precisely one place. Moreover, H_n has exactly $2n$ vertices and $n2^{n-1}$ edges. Darafsheh [9] proved that H_n is vertex-transitive and for every vertex u, $\sigma_{H_n}(u) = n2^{n-1}$.

Lemma 3.4. *[9] The automorphism group of H_n is isomorphic to a group of the shape $2^n : \mathbb{S}_n$. In particular H_n is both vertex-transitive and edge-transitive.*

Theorem 3.5. *[9] The Wiener index of H_n is equal to $W(H_n) = 2^{2(n-1)}n$.*

It follows from Lemma 2.3 and Lemma 3.4 that

Theorem 3.6. *[25] For hypercube H_n,*

$$MS_1(H_n) = n^2 2^{2n-1} \text{ and } MS_2(H_n) = n^3 2^{3n-3}. \tag{7}$$

Corollary 3.7. *[19]*

$$SJ(H_n) = \frac{n^2 2^{2(n-1)}}{(n2^{n-1} - 2n + 2)\sqrt{n2^n}}, \quad GAS(H_n) = n, \quad HS(H_n) = 2n^2 2^{2(n-1)},$$

$$J(H_n) = \frac{n^2 2^{2(n-1)}}{(n2^{n-1} - 2n + 2)n2^{n-1}}.$$

The *Kneser graph* $KG_{p,k}$ is the graph whose vertices correspond to the k-element subsets of a set of p elements, and where two vertices are adjacent if and only if the two corresponding sets are disjoint. Clearly we must impose the restriction $p \geq 2k$. The Kneser graph $KG_{p,k}$ has $\binom{p}{k}$ vertices and it is regular of degree $\binom{p-k}{k}$. Therefore the number of edges of $KG_{p,k}$ is $\frac{1}{2}\binom{p}{k}\binom{p-k}{k}$ (see [10]). The Kneser graph $KG_{n,1}$ is the complete graph on n vertices. The Kneser graph $KG_{2p-1,p-1}$ is known as the *odd graph* O_p. The odd graph $O_3 = KG_{5,2}$ is isomorphic to the Petersen graph, see Figure 3.

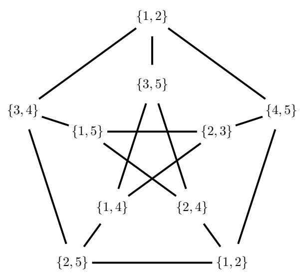

Figure 3. The odd graph $O_3 = KG_{5,2}$ is isomorphic to the *Petersen graph*

Theorem 3.8. *[10] The automorphism group of the Kneser graph $KG_{p,k}$ contains a subgroup isomorphic to the symmetric group on p letters.*

Vertex and edge transitivity of the Kneser graph are obtained from Theorem 3.8 and Theorem 2.2.

Lemma 3.9. *[10] The Kneser graph is both vertex-transitive and edge-transitive.*

It is proved that $KG_{p,k}$ is connected if $p \geq 2k+1$, where $k \geq 2$, (see [10, Corollary 3.5.]).

Lemma 3.10. *[10] Let $k \geq 2$ and $n \geq 2k+2$. Then for each k-subset A,*

1. *If $p \geq 3k-1$, then we have*

$$\sigma(A) = \left(\binom{p-k}{k} + 2\left(\binom{p}{k} - 1 - \binom{p-k}{k}\right)\right).$$

2. *If $p < 3k-1$, then we have*

$$\sigma(A) = \left(\binom{p-k}{k} + 2\alpha + 3\beta\right),$$

where

$$\alpha = \sum_{j=1}^{p-2k} \binom{k}{k-j}\binom{p-2k}{j},$$
$$\beta = \left(\binom{p}{k} - 1 - \binom{p-k}{k} - \sum_{j=1}^{p-2k} \binom{k}{k-j}\binom{p-2k}{j}\right).$$

It follows from Lemma 3.9 and Theorem 2.4 that

$$W(KG_{p,k}) = \frac{1}{2}\binom{p}{k}\sigma(A),$$

where A is an arbitrary k-subset. Hence by Lemma 3.10 we have:

Theorem 3.11. *[10] Let $k \geq 2$ and $n \geq 2k+2$. Then*

1. *If $p \geq 3k-1$, then we have*

$$W\left(KG_{p,k}\right) = \frac{1}{2}\binom{p}{k}\left(\binom{p-k}{k} + 2\left(\binom{p}{k} - 1 - \binom{p-k}{k}\right)\right).$$

2. *If $p < 3k-1$, then we have*

$$W(KG_{p,k}) = \frac{1}{2}\binom{p}{k}\left(\binom{p-k}{k} + 2\alpha + 3\beta\right),$$

where

$$\alpha = \sum_{j=1}^{p-2k} \binom{k}{k-j}\binom{p-2k}{j},$$
$$\beta = \left(\binom{p}{k} - 1 - \binom{p-k}{k} - \sum_{j=1}^{p-2k} \binom{k}{k-j}\binom{p-2k}{j}\right).$$

The next proposition follows from Theorem 2.5 and Lemma 3.9.

Proposition 3.12. *[25] For the Kneser graph $KG_{p,k}$, we have*

$$MS_1(KG_{p,k}) = 2W(KG_{p,k})\binom{p-k}{k},$$

and

$$MS_2(KG_{p,k}) = \binom{p-k}{k}\left[\frac{2(W(KG_{p,k}))^2}{\binom{p}{k}}\right].$$

Let $\mathbb{F}_q$ denote the field with q elements, where $q \equiv 1(\text{mod}4)$. The Paley graph $P(q)$ has $\mathbb{F}_q$ as the set of its vertices and two vertices x and y are joined by an edge if and only if $x-y$ is a non-zero square in $\mathbb{F}_q$. If we put $S = \{a^2 \mid 0 \neq a \in \mathbb{F}_q\}$, then clearly $P(q)$ is the Cayley graph of the additive group of $\mathbb{F}_q$ with S as the connecting set. The condition $q \equiv 1(\text{mod}4)$ implies that $-1 \in \mathbb{F}_q$, hence $S = -S$ and $P(q)$ is an undirected graph. Because S generates $\mathbb{F}_q$ we deduce that $P(q)$ is a connected graph. $P(q)$ is a regular graph of degree $\frac{q-1}{2}$, hence the number of edges in $P(q)$ is $\frac{q(q-1)}{4}$.

We refer the reader to [17] for the automorphism group of $P(q)$. By construction of $P(q)$, the graph $P(q)$ has the following automorphisms: translation by an element of $\mathbb{F}_q$, multiplication by an element of S, and by applying any field automorphism of $\mathbb{F}_q$. For q odd these operations generate the group

$$A\Delta L_1(q) = \{v \mapsto av^\gamma + b \mid a \in S, b \in \mathbb{F}_q, \gamma \in \text{Aut}(\mathbb{F}_q)\}.$$

Therefore $\text{Aut}(P(q))$ has a subgroup isomorphic to $A\Delta L_1(q)$. In fact from [17], it follows that:

Lemma 3.13. *[17] If $q \equiv 1(\text{mod}4)$ then $\text{Aut}(P(q)) \cong A\Delta L_1(q)$.*

From above it is easily verified that the Paley graph is both vertex and edge transitive graph. It is know that $P(q)$ is of diameter 2, and hence it is $(2(q-1) - \frac{q-1}{2})$-transmission regular graph. Hence by Theorem 2.4 the following hold:

Theorem 3.14. *[23] For Paley graph $P(q)$, $\sigma(0) = \frac{3}{2}(q-1)$. Consequently,*

$$W(P(q)) = \frac{3q}{4}(q-1).$$

By Theorem 2.5 we obtain that

Theorem 3.15. *For Paley graph $P(q)$,*

$$MS_1(P(q)) = \frac{3}{4}q(q-1)^2, \quad MS_2(P(q)) = \frac{9}{16}q(q-1)^3.$$

The Dihedral group D_{2n} is the symmetry group of an n-sided regular polygonal which it has the following presentation $D_{2n} = \langle a, b \mid a^2 = b^n = 1, (ab)^2 = 1\rangle$. Considering subset $S_1 = \{a, ab, b^{n-1}, b\}$ of D_{2n}, the Cayley graph $\text{Cay}(D_{2n}, S_1)$ is defined.

For some integer $n \geq 3$, the generalized Quaternion Q_{2^n} is a non-abelian group of order 2^n with the presentation $Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = 1, a^{2^{n-2}} = b^2, b^{-1}ab = a^{-1}\rangle$. The ordinary Quaternion group corresponds to the case $n = 3$. Now, we regard the subset $S_2 = \left\{a, a^{2^{n-1}-1}, b, a^{2^{n-2}}\right\}$ of Q_{2^n} and define the Cayley graph $\text{Cay}(Q_{2^n}, S_2)$.

It is obvious that $S_1 = S_1^{-1}$ and $S_2 = S_2^{-1}$, and also S_1 and S_2 are generating sets of groups D_{2n} and Q_{2^n}, respectively. So the Cayley graphs $\text{Cay}(D_{2n}, S_1)$ and $\text{Cay}(Q_{2^n}, S_2)$ are both undirected connected graphs.

Proposition 3.16. *[20] The following hold:*

1. *For the Cayley graph* $\Gamma_1 = \mathrm{Cay}\,(D_{2n}, S_1)$*,* $\sigma_{\Gamma_1}(1) = \frac{n}{2}(n+1)$*. Consequently,*

$$W(\Gamma_1) = \frac{1}{2}n^2(n+1).$$

2. *For the Cayley graph* $\Gamma = \mathrm{Cay}\,(Q_{2n}, S_2)$*,* $\sigma_{\Gamma_2}(1) = 2^n\left(2^{n-4}+1\right) - 2$*. Consequently,*

$$W(\Gamma_2) = 2^{n-1}\left(2^n\left(2^{n-4}+1\right) - 2\right).$$

Let R be a finite commutative ring. Let us denoted by $G_R = \mathrm{Cay}(R, R^\times)$ the unitary Cayley graph of R which is a graph with vertex set R and edge set

$$\left\{\{a,b\} \mid a,b \in R, a-b \in R^\times\right\},$$

where $R^\times$ is the set of units of R. For a positive integer $n > 1$ the unitary Cayley graph $\mathbb{X}_n = \mathrm{Cay}(\mathbb{Z}_n, U_n)$ is defined by the additive group of the ring $\mathbb{Z}_n$ of integers modulo n and the multiplicative group U_n of its units. If we represent the elements of $\mathbb{Z}_n$ by the integers $0, 1, \cdots, n-1$, then it is well known that

$$U_n = \{a \in Z_n \mid \gcd(a,n) = 1\}.$$

So $\mathbb{X}_n$, has the vertex set $V(\mathbb{X}_n) = \mathbb{Z}_n = \{0, 1, \cdots, n-1\}$ and $ab \in E(\mathbb{X}_n)$ if and only if $\gcd(a-b, n) = 1$. The graph $\mathbb{X}_n$ is regular of degree $|U_n| = \phi(n)$, where $\phi(n)$ denotes the Euler function. If $n = p$ is a prime number, then $\mathbb{X}_n = K_p$ is the complete graph on p vertices. If $n = p^t$ is a prime power then $\mathbb{X}_n$ is a complete p-partite graph.

Lemma 3.17. *[22] The Wiener index of unitary Cayley graph* $\mathbb{X}_n$ *is as follows:*

$$W(\mathbb{X}_n) = \begin{cases} \frac{1}{2}n(n-1) & \text{if } n \text{ is a prim number,} \\ \frac{3}{4}n^2 - n & \text{if } n = 2^\alpha, \alpha > 1, \\ n^2 - \frac{1}{2}n\phi(n) - n & \text{if } n \text{ is odd but not a prime number,} \\ \frac{5}{4}n^2 - n\phi(n) - n & \text{if } n \text{ is even and has and odd prime divisor.} \end{cases}$$

References

[1] M. Aouchiche, P. Hansen, On a conjecture about Szeged index, European J. Combin. 31 (2010), 1662–1666.

[2] Ashrafi, A.R. Wiener Index of Nanotubes, Toroidal Fullerenes and Nanostars. In *The Mathematics and Topology of Fullerenes* Cataldo, F., Graovac, A. and Ori, O.; Eds.; Springer Netherlands: Dordrecht, 2011; pp. 21–38.

[3] A. T. Balaban, P. V. Khadikar, S. Aziz, Comparison of topological indices based on iterated sum versus product operations, Iranian J. Math. Chem. 1 (2010) 43–67.

[4] A. T. Balaban, Topological indices based on topological distances in molecular graphs, Pure Appl. Chem. 55 (1983) 199–206.

[5] A. T. Balaban, Highly discriminating distance based numerical descriptor, Chem. Phys. Lett. 89 (1982) 399–404.

[6] Biggs, N. Algebraic graph theory. Second Edition. Cambridge Mathematical Library. Cambridge University Press, 1993.
[7] H. Deng, On the sum-Balaban index, MATCH Commun. Math. Comput. Chem. 66 (2011) 273–284.
[8] A. Dobrynin, A. A. Kochetova, Degree distance of a graph: A degree analogue of the Wiener index, J. Chem. Inf. Comput. Sci. 34 (1994) 1082–1086.
[9] Darafsheh M. R. Computation of topological indices of some graphs. Acta. Appl. Math. 110, 1225–1235 (2010).
[10] R. Mohammadyari, M. R. Darafsheh, Topological indices of the Kneser graph $KG_{n;k}$, Filomat 26 (2012) 665–672.
[11] R. C. Entringer, D. E. Jackson, D. A. Snyder, Distance in graphs, Czechoslovak Math. J. 26 (1976) 283–296.
[12] C. Godsil, G. Royle, Algebraic graph theory, Springer-Verlag New York, Inc. 2001.
[13] S. Gupta, M. Singh, A. K. Madan, Eccentric distance sum: A novel graph invariant for predicting biological and physical properties, J. Math. Anal. Appl. 275 (2002) 386–401.
[14] I. Gutman, Selected properties of the Schultz molecular topological index, J. Chem. Inf. Comput. Sci. 34 (1994) 1087–1089.
[15] Harary, F.: Graph Theory. Addison Wesley, Reading (1968).
[16] H. Hosoya, Topological index. A newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons, Bull. Chem. Soc. Jpn. 44 (1971) 2332–2339.
[17] G. A. Jones, Paley and the Paley graphs, Isomorphisms, symmetry and computations in algebraic graph theory, Springer Proc. Math. Stat. 305, Springer, Cham, (2020) 155–183.
[18] G. Sabidussi, On a class of fixed-point-free graphs, Proc. Amer. Math. Soc. 9 (1958) 800–804.
[19] R. Sharafdini, T. Réti, On the Transmission-Based Graph Topological Indices, *Kragujevac J. Math.* 44(1) (2020) 41–63.
[20] S. Shokrolahi Yancheshmeh, R. Modabbernia and M. Jahandideh, The Topological Indices Of The Cayley Graphs of Dihedral Group D_{2n} and the Generalized Quaternion Group Q_{2^n}, Italian Journal Of Pure And Applied Mathematics, 40 (2018) 424–433.
[21] H. S. Ramane, A. S. Yalnaik, Status connectivity indices of graphs and its applications to the boiling point of benzenoid hydrocarbons, J. Appl. Math. Comput. (2016) 1–19.
[22] A. Loghman, Computing Wiener and hyper-Wiener indices ofunitary Cayley graphs, Iranian J. Math. Chem. (2012) 121–125.
[23] R. Modabernia, Some Topological Indices Related to Paley Graphs, Iranian J. Math. Chem. 11 (2) (2020) 107–112.
[24] A. I. Tomescu, Unicyclic and bicyclic graphs having minimum degree distance, Discrete Appl. Math. 156 (2008) 125–130.
[25] H.S. Ramane, A.S. Yalnaik and R. Sharafdini, Status connectivity indices and co-indices of graphs and its computation to some distance-balanced graphs. AKCE Int. J. Graphs Comb. 17(20) (2020) 98–108.
[26] D. B. West, Introduction to graph theory, Prentic-Hall, Upper Saddle River, NJ. 1996.

Citation: R. Sharafdini, M. Azadimotlagh, Asurveyonautomorphismgroupsandtransmission-basedgraph invariants, J. Disc. Math. Appl. 9(2) (2024) 133–145.

https://doi.org/10.22061/jdma.2024.10625.1076