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Research Paper

On the two-sided group digraph with a normal adjacency matrix

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Abstract. In this article, we explore the adjacency matrix of a two-sided group graph and its properties. We introduce the two-sided color group digraph as a generalization of the Cayley color graph and the two-sided group digraph. We also obtain the adjacency matrix of the latter digraph and provide a criterion for determining the normality of the adjacency matrix of a two-sided group graph. Moreover, we prove that if all the two-sided group digraphs of valency two for a certain group G are normal, then G is a Hamiltonian group. We also show that if a strongly connected two-sided group digraph of valency two is normal, then the corresponding group is isomorphic to the product of two groups: a cyclic group with either $T_{m,n}$ or $H_{p,q}$, or an abelian group.

Keywords. Cayley digraph, adjacency matrix, normal matrix.

Mathematics Subject Classification (2010): 05C20.

1 Introduction

A Cayley digraph is a directed graph that has a finite group, G, as its vertex set. An arc (x, y) connects vertex x to vertex y if and only if $x^{-1}y \in S$, where S is a subset of G and the identity element, e, is not an element of S. The digraph is represented as $\overrightarrow{Cay}(G, S)$. If the subset S is symmetric, meaning $S = S^{-1} = \{s^{-1} : s \in S\}$, then the digraph becomes a simple

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undirected graph called a Cayley graph. This graph is represented as $Cay(G,S)$. If the subset S is empty, the related Cayley graph has no edges. The graph is connected if and only if G is generated by S [2]. There have been many generalizations of Cayley digraphs, including the recent introduction of a two-sided group digraph by Iradmusa and Praeger, denoted by $\overrightarrow{2S}(G;L,R)$ [6]. This digraph is defined on a group G as the vertex set, with L and R as two non-empty subsets of G. An arc is formed between vertices x and y if and only if $y = l^{-1}xr$ for some $l \in L$ and $r \in R$. It is important to note that for distinct ordered pairs (l_i, r_i) that occur in the relation $y = l_1^{-1}xr_1 = l_2^{-1}xr_2$, at most one arc from x to y is considered. If the digraph has no loops and the adjacency relation is symmetric, then it is a simple graph called a two-sided group graph, denoted by $2S(G;L,R)$. If a group is abelian, its associated two-sided group digraph will be a Cayley digraph. For the purposes of this paper, we will assume that all groups are non-abelian.

In a two-sided group digraph $\overrightarrow{2S}(G;L,R)$, the arcs that start from a vertex x are referred to as the pairs (x,y), where y is known as the out-neighbor of x. The out-valency of a vertex x is the number of distinct out-neighbors of x. Conversely, the in-valency of a vertex x is the number of distinct in-neighbors of x, represented by the pairs (y,x), where y is called the in-neighbor of x. If all vertices in the digraph have the same out-valency and in-valency c, it is considered as regular of valency c. The following definition provides sufficient and necessary conditions by which a two-sided group digraph is simple with no loops and regular valency $|L||R|$.

We must remark that if $g \in G$ and $L \subseteq G$, then $L^g = g^{-1}Lg$.

Definition 1. [6] Let G be a group with identity element e, L and R be two non-empty subsets of G. A pair (L,R) has the *2S-graph-property* if for all $g \in G$ the following conditions hold:
(i) $L^{-1}gR = LgR^{-1}$,
(ii) $L^g \cap R = \emptyset$,
(iii) $(LL^{-1})^g \cap (RR^{-1}) = \{e\}$.

Note that, unlike Cayley graphs, the condition $G = \langle L \cup R \rangle$ is insufficient $\overrightarrow{2S}(G;L,R)$ graphs to be connected.

A word in a subset L of a group G is a string $w = l_1 l_2 ... l_k$, where each $l_i \in L$. The integer k, known as the length of w and denoted $|w|$, and in G, we always identify w with its evaluation, which is the element obtained by multiplying the l_i in the given order.

Lemma 1.1. *[6] Let L,R be two non-empty subsets of a group G. If $\overrightarrow{2S}(G;L,R)$ is connected, then $G = \langle L \rangle \langle R \rangle$ and there exist words w in $L \cup L^{-1}$ and w' in $R \cup R^{-1}$, with lengths having opposite parity, such that ww' evaluates to the identity element e in G.*

In this discussion, we have focused on color graphs that involve labeling the arcs of digraphs with complex numbers. A color graph is defined as a pair consisting of a vertex set and a mapping that associates a color to each arc. Non-edges are represented by the color 0. The adjacency matrix of a color graph is obtained by evaluating the color mapping on each pair of vertices. Cayley color graphs, a specific type of color graph, are defined using a group and a function that maps group elements to complex numbers.

If the matrix N satisfies the relation $N^T N = NN^T$ [10], then it is considered normal. The relationship between the eigenvalues of these matrices and their entries is significant, as stated in Theorem 2.5.3 [5]. Basically, an $n \times n$ matrix $A = [a_{ij}]$ is considered normal if and only if $\sum_{i,j} |a_{ij}|^2 = \sum_{i} |\lambda_i|^2$, where λ_i's are the eigenvalues of the matrix and $1 \leq i \leq n$. This relation shows a connection between the normality of the adjacency matrix of a digraph and its spectrum. Several authors have recently studied the computation of eigenvalues of digraphs [3,7,11]; moreover, this issue justifies how this topic is essential.

We emphasize that a digraph is normal only if its adjacency matrix is normal. It is clear that whenever $N = N^T$, N is a normal matrix; therefore, if all arcs of a digraph are double and both (x,y) and (y,x) are arcs, for all vertices x,y, the adjacency matrix of the digraph is symmetric, and so the digraph is normal. A proper normal digraph (PND) is a normal digraph where all arcs are not double and without loops [10] .

In this paper, we establish certain conditions that ensure the adjacency matrix of $\Gamma = \overrightarrow{2S}(G;L,R)$ is normal. A Hamiltonian group is a non-abelian group where all subgroups are normal. We prove that if all two-sided group digraphs of valency two for a certain group G are normal, then G is a Hamiltonian group. This outcome is similar to the previously researched Cayley digraphs of valency two [8]. Furthermore, we indicate that if a strongly connected two-sided group digraph of valency two is normal, then $G = K.\langle z \rangle$ in which K is an abelian subgroup of G and $\langle z \rangle$ is a cyclic group either $G \cong T_{m,n}.\langle z \rangle$ or $G \cong H_{p,q}.\langle z \rangle$ in which $T_{m,n}, H_{p,q}$ are as follows:

$T_{m,n} = \langle x,y | \ x^{2m} = y^n = e, \ x^{-1}yx = y^{-1} \rangle$, where $m \geq 1$ *and* $n \geq 3$; and

$H_{p,q} = \langle x,y | \ x^{4p} = e, \ y^{2q} = x^{2p}, \ x^{-1}yx = y^{-1} \rangle$, where $p \geq 1$ *and* $q \geq 1$.

2 Two-sided Color Group Digraphs and their adjacency matrices

Now, we generalize a Cayley color graph, a digraph defined in [1], and we name it a two-sided color group digraph. Let G be a finite group, $\alpha : G \to \mathbb{C}$ and $\beta : G \to \mathbb{C}$ be two functions, and for $g,h \in G$, we define the function $c_\Gamma : G \times G \to \mathbb{C}$ as follows:

$$c_\Gamma(g,h) = \frac{1}{t_{g,h}} \sum_{x \in G} \alpha(g^{-1}xh)\beta(x), \tag{1}$$

where $T_{g,h} = \{x \in G : \alpha(g^{-1}xh) = \beta(x) \neq 0\}$ and

$$t_{g,h} = \begin{cases} |T_{g,h}| & if \ T_{g,h} \neq \emptyset \\ 1 & otherwise \end{cases} \tag{2}$$

We introduce a two-sided color group digraph $\Gamma = \Gamma(G;\alpha,\beta)$ with the vertex set $V(\Gamma) = G$ and $c_\Gamma(g,h)$, where $c_\Gamma : G \times G \to \mathbb{C}$ defined by relation (1).

In the following theorem, we show that $\Gamma = \Gamma(G;\alpha,\beta)$ is a two-sided group digraph for two specific functions α, and β.

Proposition 2.1. *Let G be a group. If a two-sided color group digraph $\Gamma = \Gamma(G;\alpha,\beta)$, in which two functions α,β are from G to $\mathbb{C}$, have the values in $\{0,1\}$, then Γ is a two-sided group digraph.*

Proof. It suffices to have $\alpha,\beta : G \to \mathbb{C}$ such that $\alpha(g),\beta(g) \in \{0,1\}$ for all $g \in G$, $R = \{r \in G : \alpha(r) = 1\}$, and $L = \{l \in G : \beta(l) = 1\}$, then Γ is a two-sided group digraph denoted by $\Gamma = \overrightarrow{2S}(G;L,R)$.

Let $g,h \in G$, for all $x \in G$, if $\alpha(g^{-1}xh) = 0$ or $\beta(x) = 0$, then $g^{-1}xh \notin R$ or $x \notin L$, therefore (g, h) is not an arc, and according to relation (1), we have $c_\Gamma(g,h) = 0$. Now, if there is at least one $x \in G$ such that $\alpha(g^{-1}xh) = 1$ and $\beta(x) = 1$, then $g^{-1}xh = r$ and $x = l$ for some $r \in R$, $l \in L$, so $g^{-1}lh = r$, hence (g, h) forms an arc and, in this case, $c_\Gamma(g,h) = 1$. □

According to the proposition 1, the adjacency matrix of $\Gamma = (G;\alpha,\beta)$ is as follows:

$$A = \frac{1}{|G|} \sum_{g,h\in G} \sum_{x\in G} \frac{1}{t_{g,h}} \alpha(g^{-1}xh)\beta(x)L_g.R_h,$$

where $L_g = [l_{xy}(g)]_{x,y\in G}$, $R_g = [r_{xy}(g)]_{x,y\in G}$, $l_{xy}(g) = \begin{cases} 1 & if\ g^{-1}x = y \\ 0 & otherwise \end{cases}$, and $r_{xy}(g) = \begin{cases} 1 & if\ xg = y \\ 0 & otherwise \end{cases}$.

Now, we have the matrix $\Lambda_{g,h} = [\lambda_{xy}(g,h)]_{x,y\in G}$, where

$$\lambda_{xy}(g,h) = \begin{cases} 1 & if\ \ y = g^{-1}xh \\ 0 & otherwise. \end{cases}$$

Further, it is clear that $L_g R_h = \Lambda_{g,h}$, so the adjacency matrix of a two-sided color group digraph is represented by the following relation.

$$A = \frac{1}{|G|} \sum_{g,h\in G} \sum_{x\in G} \frac{1}{t_{g,h}} \alpha(g^{-1}xh)\beta(x)\Lambda_{g,h}.$$

Example Let $G = D_6 = \langle a,b | a^3 = b^2 = 1, b^{-1}ab = a^{-1}\rangle$ be the dihedral group of order 6 and $L = \{a,a^2\}$, $R = \{ab,a^2b\}$. Here, corresponding permutation matrices are $L_1 = R_1 = I_6$, the identity matrix of order 6, and $L_a = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$, $L_{a^2} = \begin{pmatrix} B & O \\ O & B \end{pmatrix}$, $L_b = \begin{pmatrix} O & C \\ C & O \end{pmatrix}$, $L_{ab} = \begin{pmatrix} O & D \\ D & O \end{pmatrix}$, $L_{a^2b} = \begin{pmatrix} O & E \\ E & O \end{pmatrix}$, $R_a = \begin{pmatrix} B & O \\ O & B \end{pmatrix}$, $R_{a^2} = \begin{pmatrix} A & O \\ O & A \end{pmatrix}$, $R_b = \begin{pmatrix} O & I \\ I & O \end{pmatrix}$, $R_{ab} = \begin{pmatrix} O & B \\ B & O \end{pmatrix}$, $R_{a^2b} = \begin{pmatrix} O & E \\ E & O \end{pmatrix}$ where O, A, B, C, D, E are matrices of order 3 as follows:
$A = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$, $B = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}$, $C = \begin{pmatrix} 1&0&0 \\ 0&0&1 \\ 0&1&0 \end{pmatrix}$, $D = \begin{pmatrix} 0&1&0 \\ 1&0&0 \\ 0&0&1 \end{pmatrix}$, $E = \begin{pmatrix} 0&0&1 \\ 0&1&0 \\ 1&0&0 \end{pmatrix}$, and O represents zero matrix.

With simple computations and mentioning $\alpha(g),\beta(g) \in \{0,1\}$, we have

$$\sum_{g,h\in G} \sum_{x\in G} \frac{1}{t_{g,h}} \alpha(g^{-1}xh)\beta(x)L_g.R_h = 6\begin{pmatrix} O & J_3 \\ J_3 & O \end{pmatrix} = 6T$$

, where $J_3 = \begin{pmatrix} 1&1&1 \\ 1&1&1 \\ 1&1&1 \end{pmatrix}$ and T is the adjacency matrix of $\Gamma = 2S(D_6;L,R)$.

3 Two-sided Group Digraph with Normal Adjacency Matrix

According to definition, all arcs of a two-sided digraph $\overrightarrow{2S}(G;L,R)$ are double if and only if $L^{-1}gR = LgR^{-1}$, for each $g \in G$, making the related digraph an undirected graph and a normal digraph (graph). Further, the relation $L^g \cap R = \emptyset$, for all $g \in G$, indicates that $\overrightarrow{2S}(G;L,R)$ is a digraph without a loop.

Corollary 3.1. *Let G be a group with two non-empty subsets L and R. Assume that the two-sided group digraph $\Gamma = \overrightarrow{2S}(G;L,R)$ is normal. Then it is a PND if and only if $L^{-1}gR \neq LgR^{-1}$, for some $g \in G$, and $L^g \cap R = \emptyset$, for all $g \in G$.*

It is evident that the following result is true by citing Proposition 4 from reference [10].

Corollary 3.2. *There is no non-abelian group G with two non-empty subsets L, and R, such that the two-sided group digraph $\Gamma = \overrightarrow{2S}(G;L,R)$ be a proper normal connected tree or a proper normal connected unicyclic digraph with at least one vertex of valency one.*

For the adjacency matrix of a digraph to be considered normal, it must satisfy the condition that the number of common in-neighbors between any two vertices is equal to the number of common out-neighbors. This is a well-known fact that can be found in Proposition 1 of [10]. To apply this concept to a two-sided group digraph $\Gamma = \overrightarrow{2S}(G;L,R)$, where L and R be two non-empty subsets of a group G, we can use the fact that $L^{-1}gR$, LgR^{-1} represent the sets of out-neighbors and in-neighbors of g ($g \in G$), respectively. Therefore, the adjacency matrix of Γ will be normal if and only if the intersection of $L^{-1}gR$ and $L^{-1}hR$ equals to the intersection of LgR^{-1} and LhR^{-1}, for all $g,h \in G$.

Proposition 3.3. *Let $G = D_{2n} = \langle a,b | a^n = b^2 = e,\ b^{-1}ab = a^{-1}\rangle$, L and R be two non-empty subsets of G. If one of the following three cases holds, then $\Gamma = \overrightarrow{2S}(G;L,R)$ is normal :*
(i) $L,R \subseteq \{a^ib | 0 \leq i \leq n-1\}$;
(ii) $L,R \subseteq \langle a\rangle$;
(iii) One of the sets L or R is a subset of $\langle a\rangle$, and another is a subset of $\{a^ib | 0 \leq i \leq n-1\}$.

Proof. (i) In this case, since L and R are inverse-closed subsets ($L = L^{-1}$ and $R = R^{-1}$), then Γ is a simple undirected graph; so, the adjacency matrix of Γ is symmetric, which is normal. (ii) Let $L,R \subseteq \langle a\rangle$, then $L = \{a^{i_1},\dots,a^{i_s}\}$ and $R = \{a^{j_1},\dots,a^{j_m}\}$. We must show that $|L^{-1}a^kb^wR \cap L^{-1}a^{k'}b^{w'}R| = |La^kb^wR^{-1} \cap La^{k'}b^{w'}R^{-1}|$, for all k,k',w,w', where $0 \leq k,k' \leq n-1$, $0 \leq w,w' \leq 1$. It is convenient to call these two recent subsets A and B, respectively. We define the function $\phi : A \longrightarrow B$ as follows. If $x \in A$, then $x = a^{-i_p+k}b^wa^{j_q} = a^{-i_r+k'}b^{w'}a^{j_t}$, for some $i_p,i_r \in \{i_1,\dots,i_s\}$ and $j_q,j_t \in \{j_1,\dots,j_m\}$. Therefore, $a^{i_r+k}b^wa^{-j_t} = a^{i_p+k'}b^{w'}a^{-j_q}$. It implies that $y = a^{i_r+k}b^wa^{-j_t} = a^{i_p+k'}b^{w'}a^{-j_q}$ belongs to set B. Hence, it is sufficient to define $\phi(x) = y$. One can prove that ϕ is bijective; therefore, the adjacency matrix of Γ is normal. (iii) It is similar to the case (ii). □

By Proposition 2, the Dihedral group does not have a PND in case (i), and if cases (ii) or (iii) occur, it likely has some. For example, if we consider the Dihedral group of order 12 with $L = \{a,a^2\}$, $R = \{a^3\}$, then $\Gamma = \overrightarrow{2S}(G;L,R)$ is a PND.

Now, we determine the necessary condition under which the adjacency matrix of the two-sided group digraph is normal.

Lemma 3.4. *Let G be a finite group and $L, R \subseteq G$ and $|L| \geq 2$. If the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal, then $(LL^{-1})^y RR^{-1} = (L^{-1}L)^y R^{-1}R$, for all $y \in G$.*

Proof. Suppose that $y \in G$ and $e \neq g \in (LL^{-1})^y RR^{-1}$, then $g = (l_1 l_2^{-1})^y r_2 r_1^{-1}$, for some $l_1, l_2 \in L$, $r_1, r_2 \in R$. Assume that $x = yg$, then $g = y^{-1}x$ and $x = (l_1 l_2^{-1}) y r_2 r_1^{-1}$. Hence, $l_1^{-1} x r_1 = l_2^{-1} y r_2 = z$ is a common out-neighbor of x, y, i.e., the set of common out-neighbors of x and y is non-empty. Since the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal, thus the number of common in-neighbors of x, y equals the number of common out-neighbors of x, y. So, the set common in-neighbors of x and y is not empty. Therefore, $l_3 x r_3^{-1} = l_4 y r_4^{-1}$, for some $l_3, l_4 \in L$ and $r_3, r_4 \in R$. Thus $x = l_3^{-1} l_4 y r_4^{-1} r_3$, so $g = y^{-1}x = (l_3^{-1} l_4)^y r_4^{-1} r_3 \in (L^{-1}L)^y R^{-1}R$, and it implies $(LL^{-1})^y RR^{-1} \subseteq (L^{-1}L)^y R^{-1}R$. Similarly, the reverse is correct, so $(LL^{-1})^y RR^{-1} = (L^{-1}L)^y R^{-1}R$, for all $y \in G$. □

Here, a sufficient condition of normality of $\Gamma = \overrightarrow{2S}(G; L, R)$ is presented.

Lemma 3.5. *Let G be a finite group and $L, R \subseteq G$. If one of the following conditions holds, then the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal.*
(i) $l_1 l_2 = l_2 l_1$, $r_1 r_2 = r_2 r_1$, for all $l_1, l_2 \in L$ and $r_1, r_2 \in R$,
(ii) $l_1^2 = l_2^2$, $r_1^2 = r_2^2$, for all $l_1, l_2 \in L$, and $r_1, r_2 \in R$.

Proof. (i) Let $x, y \in G$ and if z_1, z_2 are distinct common out-neighbors of x, y, then $z_1 = l_1^{-1} x r_1 = m_1^{-1} y n_1$ and $z_2 = l_2^{-1} x r_2 = m_2^{-1} y n_2$ for some $l_1, l_2, m_1, m_2 \in L$ and $r_1, r_2, n_1, n_2 \in R$. Since $l_1 l_2^{-1} = l_2^{-1} l_1$, $r_1 r_2^{-1} = r_2^{-1} r_1$ for all $l_1, l_2 \in L$ and $r_1, r_2 \in R$, we obtain distinct common in-neighbors $z_1' = l_1 y r_1^{-1} = m_1 x n_1^{-1}$, $z_2' = l_2 y r_2^{-1} = m_2 x n_2^{-1}$. Thus, in this case, we have a one-to-one mapping from $L^{-1}xR \cap L^{-1}yR$ to $LxR^{-1} \cap LyR^{-1}$. Therefore, the number of common in-neighbors x, y equals the number of common out-neighbors of x, y. (ii) The proof is similar to (i). □

If L and R are two single-member subsets of G, then the sets of common out-neighbors and common in-neighbors of two distinct elements x, y of G are empty; then the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal. In additon, it can be the unique PND, up to isomorphism, the directed cycle C_n [10].

Corollary 3.6. *Let G be a finite group with two subsets $L = \{x, y\}$ and R, in which R is an arbitrary singleton. Then the adjacency matrix of $\Gamma = \overrightarrow{2S}(G; L, R)$ is normal, if and only if $x^2 = y^2$ or $xy = yx$.*

Proof. If we use Lemmas 2 and 3, the proof easily follows. □

Since $\overrightarrow{2S}(G; L, R) \cong \overrightarrow{2S}(G; R, L)$, then Corollary 3 explains that if a two-sided group digraph of valency two is normal, then both elements of L (or R if $|R| = 2$) are commuted or have the same square.

The following theorem shows that if $\Gamma = \overrightarrow{2S}(G; L, R)$ of valency two is normal, then G is a Hamiltonian group. This consequence is similar to Cayley digraphs of valency two with a normal adjacency matrix (see [8]). The next proposition presents a relationship between normality in matrices and group theory.

Proposition 3.7. *For a finite group G; if the adjacency matrix of any two-sided group digraph $\Gamma = \overrightarrow{2S}(G; L, R)$ of valency two is normal, then any subgroup of G is normal.*

Proof. Let H be an arbitrary subgroup of G, $a \in H$, and $b \in G$. Let L_1, L_2 be $\{a,b\}$, $\{ba,a\}$ respectively, and R be an arbitrary singleton of G. Since both $\Gamma = \overrightarrow{2S}(G;L_1,R)$ and $\Gamma = \overrightarrow{2S}(G;L_2,R)$ are normal if $ab \neq ba$, Corollary 3 concludes that $a^2 = b^2$ and $(ba)^2 = a^2$ and this implies $b^{-1}ab = a^{-1}$; therefore H is a normal subgroup. □

As a result, if both $\Gamma = \overrightarrow{2S}(G;L_1,R)$ and $\Gamma = \overrightarrow{2S}(G;L_2,R)$ are normal where L_1, L_2 be $\{a,b\}$, $\{ba,a\}$ respectively, with an arbitrary singleton R, then $b^{-1}ab = a$ or $b^{-1}ab = a^{-1}$.

In the following, we explain a theorem that is similar to one pertaining to Cayley graphs of valency two as referenced in [8]. The methods employed in proving this theorem closely resemble those used in Theorem 1 of [8]. We should mention that Q_8 is the quaternion group of order 8, that is, $Q_8 = \langle x,y|\ x^4 = e,\ y^2 = x^2,\ y^{-1}xy = x^{-1}\rangle$. Further, Corollary 3 implies that two non-commuting elements of L, when $|L| = 2$, are the same square; therefore $\Gamma = \overrightarrow{2S}(G;L,R)$ is normal, and obviously, any two elements of $Q_8 \times \mathbb{Z}_2^n$ has this condition while $\mathbb{Z}_2^n = \mathbb{Z}_2 \times ... \times \mathbb{Z}_2$ for some $n \in \mathbb{Z}_+$. Theorem 1 shows that $Q_8 \times \mathbb{Z}_2^n$, up to isomorphism, is the only example that satisfies the assumptions.

Theorem 3.8. *Let G be a finite non-abelian group. If the adjacency matrix of every two-sided group digraph $\Gamma = \overrightarrow{2S}(G;L,R)$ of valency two is normal, then $G \cong Q_8 \times \mathbb{Z}_2^n$ for some $n \in \mathbb{Z}^+$.*

Proof. Let G be a non-abelian group with elements $x,y \in G$ such that $xy \neq yx$. Using Corollary 3, we can show that $x^2 = y^2$. Furthermore, the adjacency matrix of the group digraph $\Gamma = \overrightarrow{2S}(G;L,R)$ is normal when $L = \{x,y\}$ or $L = \{yx,x\}$. This implies that $y^{-1}xy = x^{-1}$, and so $x^{-2} = x^2$. Thus, we have $x^4 = e$, and the subgroup $\langle x,y\rangle$ is isomorphic to Q_8 and that is a normal subgroup of G by using Proposition 3.

Suppose that $\alpha \in G$ does not commute with x and y. Using the same method as in the proof of Proposition 3, we can show that α commutes xy, so α commutes with at least one of x, y, xy. If $\alpha x = x\alpha$ and $y\alpha = \alpha y$, then we can use the adjacency matrix of the two-sided group digraph with $L = \{x\alpha, y\}$ to show that $\alpha^2 = e$ and thus α lies in the center of G. If $y\alpha \neq \alpha y$, then we have $\alpha^2 = y^2 = x^2$ and $(x\alpha)^2 = x^4 = e$, implying that $x\alpha$ lies in the center of G. Therefore, G is generated by Q and $Z(G)$, and since $Z(G)$ has exponent 2, we can choose a subgroup X of index 2 of $Z(G)$ such that $X \cap \langle x,y\rangle = \{e\}$. This implies that $G = \langle x,y\rangle \times X$, and since X is a subgroup of $Z(G)$, it is a group of exponent 2 and is isomorphic to $\mathbb{Z}_2^n$ (the other cases are similar to the considered one). □

The next proposition and paragraph present more details about groups $T_{m,n}$, and $H_{p,q}$.

Proposition 3.9. *[8] For the groups $T_{m,n}$ and $H_{p,q}$, the following three statements hold:*
(1) $T_{m,n} \cong T_{m',n'}$ iff $m = m'$ and $n = n'$;
(2) $H_{p,q} \cong H_{p',q'}$ iff $p = p'$ and $q = q'$;
(3) $H_{p,q} \ncong T_{m,n}$ for any p,q,m,n.

According to research, both $T_{m,n}$ and $H_{p,q}$ are metacyclic groups. A metacyclic group G is a group that has a cyclic normal subgroup N, and the quotient group G/N is also cyclic [9]. In a specific case, $T_{1,n}$ is proven to be the dihedral group D_{2n} of order $2n$, while $H_{1,q}$ is a dicyclic group of order $8q$ which is an extension of the cyclic group of order two by a cyclic group of order $4q$. Further, $T_{m,n}$ contains subgroup $\langle x^m,y\rangle \cong D_{2n}$ if m is odd, and $\langle x^p,y\rangle$ is a subgroup of $H_{p,q}$ which is isomorphic to a dicyclic group of order $8q$ if p is odd. There is a full classification of such groups in [4].

Using the definition, a digraph Γ is strongly connected if, for any two vertices v and w, there is a path from v to w in Γ. In the final theorem, we show that if $\Gamma = \overrightarrow{2S}(G;L,R)$ is a strongly connected two-sided group digraph of valency two with a normal adjacency matrix, G has very particular conditions.

Theorem 3.10. *Let G be a finite group. Suppose that $\Gamma = \overrightarrow{2S}(G;L,R)$ is a strongly connected two-sided group digraph of valency two with $L = \{x,y\}$, $R = \{z\}$ whose adjacency matrix is normal. Then, G is isomorphic to one of the following groups:*
(1) $K.\langle z\rangle$, where K is an abelian group;
(2) $T_{m,n}.\langle z\rangle$, where $m \geqslant 1$ and $n \geqslant 3$;
(3) $H_{p,q}.\langle z\rangle$, where $p \geqslant 1$ and $q \geqslant 1$.

Proof. Since Γ is strongly connected, then it is connected, and using Lemma 1 implies $G = \langle L\rangle\langle R\rangle = \langle x,y\rangle\langle z\rangle$ and there exist words w in $L \cup L^{-1}$ and w' in $R \cup R^{-1}$, with lengths having opposite parity, such that ww' evaluates to the identity element e in G. Applying Corollary 3, we have $x^2 = y^2$ or $xy = yx$. If $xy = yx$, then the case (1) happens, otherwise $x^2 = y^2$ and suppose that $t = x^{-1}y$. It is clear that $\langle L\rangle = \langle x,t\rangle$ and we have $x^2 = y^2 = xtxt$ or, equivalently, $x^{-1}tx = t^{-1}$. If the order of t is two, then $t^{-1} = t$. It implies $x^{-1}tx = t$, and thus $\langle L\rangle$ must be abelian and this is a contradiction. Therefore, the order of t is at least three. Assume that r is the order of x; since $x^2 \in Z(\langle L\rangle)$ and $x \notin Z(\langle L\rangle)$, we know that $r = 2m$ for some m. Let n be the smallest natural number such that $t^n = x^s$ for some $s \in \{0,...,2m-1\}$. We get $t^n = x^{-1}t^nx = (x^{-1}tx)\ldots(x^{-1}tx) = t^{-n}$ and therefore $x^{2s} = t^{2n} = e$. Hence, either $s = 0$ or $s = m$. If $s = 0$, then $\langle L\rangle = T_{m,n}$. In case $s = m$, then $\langle L\rangle$ has the form $\langle x,t|x^{2m} = e, t^n = x^m, x^{-1}tx = t^{-1}\rangle$. Since $x^2 \in Z(\langle L\rangle)$, $x^m \in Z(\langle L\rangle)$, and $x \notin Z(\langle L\rangle)$, we have $m = 2p$ for some p. If n is odd, then $(tx^m)^n = t^nx^m = t^{2n} = e$. Furthermore, $x^{-1}(tx^m)x = t^{-1}x^m = x^mt^{-1} = (tx^m)^{-1}$. So, we can conclude that $\langle L\rangle$ is isomorphic to $T_{m,n}$. Finally, if $n = 2q$ for some q, then $\langle L\rangle = H_{p,q}$. Therefore, $G \cong T_{m,n}.\langle z\rangle$ or $G \cong H_{p,q}.\langle z\rangle$. □

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