Journal of Discrete Mathematics and Its Applications 9 (1) (2024) 73–79

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Research Paper

On the connectivity of $\Gamma(R_1 \circ R_2)$

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Academic Editor: Herish O. Abdullah

Abstract. The graph $\Gamma(R_1 \circ R_2)$ of the lexicographic product of two commutative rings R_1, R_2 is considered. It was shown that $\Gamma(R_1 \circ R_2)$ is connected and $diam(\Gamma(R_1 \circ R_2)) \leq 2$. We get the several expressions for finding the connectivity $\kappa(\Gamma(R_1 \circ R_2))$ when certain conditions are given.

Keywords. Lexicographic product, connectivity, vertex-cut, zero-divisor graph.

Mathematics Subject Classification (2010): 13C05, 18E40, 13B30, 16D60, 13B25.

1 Introduction

We follow [3] for terminologies and notations of graph theory not defined here.

Let G be a simple undirected graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. For each vertex $v \in V(G)$, the *neighborhood* $N_G(v)$ of v is defined as the set of all vertices adjacent to v and $deg_G(v) = |N_G(v)|$ is the *degree* of v. The number $\delta(G) = \min\{\deg_G(v) \mid v \in V(G)\}$ is the minimum degree of G. Let u, v be vertices in a graph G. The *distance* between u and v is the length of a shortest path between them in G and is denoted by $d(u, v)$. If G is disconnected and u, v are in different components we say $d(u, v) = \infty$. Let v be a vertex of a graph G. The *eccentricity* of v is

$$e(v) = max\{d(u, v) \,|\, u \in V(G)\}.$$

The *diameter* of a graph G is defined as $max\{e(v) \,|\, v \in V(G)\}$ and is denoted by $diam(G)$.

For an arbitrary subset $S \subset V(G)$ we use $G - S$ to denote the graph obtained by removing

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Received 23 October 2023; Revised 02 February 2024; Accepted 15 February 2024
First Publish Date: 01 March 2024

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all vertices in S from G. For any connected graph G, if $G-S$ is disconnected, then S is called a *vertex-cut*. The *connectivity* of a graph G, denoted by $\kappa(G)$, is the minimum cardinality of a set $S \subset V(G)$ such that $G-S$ is either disconnected or the trivial graph K_1. It is known that $\kappa(G) \leq \delta(G)$. If a graph G is disconnected, then we define $\kappa(G)$ as ∞. It is known that when the underlying topology of an interconnection network is modeled by a graph $G=(V,E)$, where V represents the set of processors and E represents the set of communication links in the network, $\kappa(G)$ is an important measurement for the fault tolerance of the network.

The lexicographic product $G_1 \circ G_2$ of two graphs G_1 and G_2 is the graph having

$$V(G_1 \circ G_2) = V(G_1) \times V(G_2), \text{ and}$$
$$E(G_1 \circ G_2) = \{(x_1,y_1)(x_2,y_2) \mid x_1x_2 \in E(G_1) \text{ or } x_1 = x_2, y_1y_2 \in E(G_2)\}.$$

Note that in the sense of isomorphism the lexicographic product does not satisfies the commutative law.

Clearly, $G_1 \circ G_2$ is connected if and only if G_1 is connected.

Theorem 1.1. [12, Theorem 1] *Let G_1 and G_2 be two graphs. If G_1 is non-trivial, non-complete and connected, then $\kappa(G_1 \circ G_2) = \kappa(G_1).|V(G_2)|$.*

The lexicographic product has generated a lot of interest mainly due to its various applications. According to [5], the lexicographic product of two graphs first was defined in [4]. Connectivity and super connectivity of lexicographic product of graphs have been studied in [12] and [7], respectively. For more information about lexicographic product, see [6,9] and [11,12].

In section 2, we deal with the lexicographic product of two commutative rings R_1, R_2 and give their examples. We show that $\Gamma(R_1 \circ R_2)$ is connected and $diam(\Gamma(R_1 \circ R_2)) \leq 2$, and then we find the expressions for finding $\kappa(\Gamma(R_1 \circ R_2))$ when certain conditions are given.

In section 3, we investigate the connectivity of special subgraphs of $\Gamma(R_1 \circ R_2)$.

2 Connectivity of $\Gamma(R_1 \circ R_2)$

Let R be a commutative ring. An element a of R is called a *zero-divisor* of R if there exists a non-zero element b in R such that $ab = 0_R$. Let $Z(R)$ denote the set of all zero-divisors of R. For a subset S of R, let $S - \{0_R\}$ be denoted S^*. By the *zero-divisor graph* $\Gamma(R)$ of R we mean the graph whose vertices are elements of $Z(R)$, such that two distinct vertices x and y are adjacent if and only if $xy = 0_R$. Furthermore, $\Gamma_0(R)$ is a subgraph of $\Gamma(R)$ with $V(\Gamma_0(R)) = Z(R)^*$.

By definition, $\Gamma(R)$ is connected. It was shown that $\Gamma_0(R)$ is connected with diameter less than or equal three. For more results and the history of this topic the reader is refereed to [1,2] and [10].

Also, $\tilde{\Gamma}(R)$ is a graph with vertices all elements of R and two distinct elements x,y of R are adjacent if and only if $xy = 0_R$. Clearly, $\Gamma_0(R)$ is a subgraph of $\Gamma(R)$ which is a subgraph of $\tilde{\Gamma}(R)$.

We define $\Gamma(R_1 \circ R_2)$ as a simple graph with $V(\Gamma(R_1 \circ R_2)) = Z(R_1 \times R_2)$ and two distinct vertices (x_1,y_1) and (x_2,y_2) are adjacent if and only if $x_1x_2 = 0_{R_1}$ or $x_1 = x_2$ and $y_1y_2 = 0_{R_2}$.

When divided by a positive integer m, the set of all integers with remainders forms a commutative ring. This ring is called the *ring of integers modulo m*, and is denoted by $\mathbb{Z}_m$.

Example 2.1. *We take $R_1 = \mathbb{Z}_2$ and $R_2 = \mathbb{Z}_2{}^2 (= \mathbb{Z}_2 \times \mathbb{Z}_2)$. For convenience, let $0_{R_1} = \bar{0}$, $x_1 = \bar{1}$ in R_1, and let*

$$0_{R_2} = (0,0), y_1 = (1,0), y_2 = (0,1), y_3 = (1,1)$$

in R_2. Then $\Gamma(R_1) \cong K_1$ and $\Gamma(R_2) \cong K_3$. So $\Gamma(R_1) \circ \Gamma(R_2) \cong K_3$ and $\kappa(\Gamma(R_1) \circ \Gamma(R_2)) = 2$.

To draw the graph $\Gamma(R_1 \circ R_2)$, first complete the following table:

	Elements of $R_1 \times R_2$	*Vertices of $\Gamma(R_1 \circ R_2)$*	*Degree*
	$(0_{R_1}, 0_{R_2})$	$(0_{R_1}, 0_{R_2})$	6
	$(x_1, 0_{R_2})$	$(x_1, 0_{R_2})$	6
	$(0_{R_1}, y_1)$	$(0_{R_1}, y_1)$	6
	(x_1, y_1)	(x_1, y_1)	6
	$(0_{R_1}, y_2)$	$(0_{R_1}, y_2)$	6
	(x_1, y_2)	(x_1, y_2)	6
	$(0_{R_1}, y_3)$	$(0_{R_1}, y_3)$	6
	(x_1, y_3)	*No*	-
Total	8	7	42

This table lists the vertices of $\Gamma(R_1 \circ R_2)$ and its degrees. For example,

$$N_{\Gamma(R_1 \circ R_2)}(x_1, y_2) = \{(x_1, 0_{R_2}), (x_1, y_1), (0_{R_1}, 0_{R_2}), (0_{R_1}, y_1), (0_{R_1}, y_2), (0_{R_1}, y_3)\},$$

So $\Gamma(R_1 \circ R_2) \cong K_7$ and $\kappa(\Gamma(R_1 \circ R_2)) = 6$.

The following lemma holds by definition.

Lemma 2.2. *Let R_1 and R_2 be commutative rings and $(x,y) \in Z(R_1 \times R_2)$.*

1. *If $(x,y) \in R_1^* \times R_2^*$, then*

$$N_{\Gamma(R_1 \circ R_2)}(x,y) = (\{0_{R_1}\} \times R_2) \dot\cup (N_{\Gamma_0(R_1)}(x) \times R_2) \dot\cup (\{x\} \times \{0_{R_1}\}) \dot\cup (\{x\} \times N_{\Gamma_0(R_2)}(y)).$$

2. *If $(x,y) \in \{0_{R_1}\} \times R_2$, then*

$$N_{\Gamma(R_1 \times R_2)}(x,y) = Z(R_1 \times R_2).$$

3. *If $(x,y) \in R_1^* \times \{0_{R_2}\}$, then*

$$N_{\Gamma(R_1 \circ R_2)}(x,y) = (\{0_{R_1}\} \times R_2) \dot\cup (N_{\Gamma_0(R_1)}(x) \times R_2) \dot\cup (\{x\} \times R_2).$$

Corollary 2.3. *Let R_1 and R_2 be commutative rings and $(x,y) \in Z(R_1 \times R_2)$.*

1. *If $(x,y) \in R_1^* \times R_2^*$, then*

$$deg_{\Gamma(R_1 \circ R_2)}(x,y) = |R_2| + deg_{\Gamma_0(R_1)}(x) \cdot |R_2| + 1 + deg_{\Gamma_0(R_2)}(y).$$

2. *If $(x,y) \in \{0_{R_1}\} \times R_2$, then*

$$deg_{\Gamma(R_1 \times R_2)}(x,y) = |Z(R_1 \times R_2)| - 1.$$

3. *If $(x,y) \in R_1^* \times \{0_{R_2}\}$, then*

$$deg_{\Gamma(R_1 \times R_2)}(x,y) = 2|R_2| + |R_2| \cdot deg_{\Gamma(R_1)}(x).$$

Proof. This is established by Lemma 2.2. □

For every $x_i \in R_1^*$, let $V(\Gamma(R_{x_i2}))$ be the set of all vertices (x_i,y_j) of $V(\Gamma(R_1 \circ R_2))$ for all $y_j \in R_2$. It is easy to check that if $x_i^2 = 0_{R_1}$, then $\Gamma(R_{x_i2}) \cong K_r$ where $r = |R_2|$. Let $x_i^2 \neq 0_{R_1}$. If $x_i \notin Z(R_1)$, then $\Gamma(R_{x_i2}) \cong \Gamma(R_2)$. If $x_i \in Z(R_1)$, then $\Gamma(R_{x_i2}) \cong \tilde{\Gamma}(R_2)$.

Also, for every $y_j \in R_2$, let $V(\Gamma(R_{1y_j}))$ be the set of all vertices $(0_{R_1},y_j)$ of $V(\Gamma(R_1 \circ R_2))$. Therefore, $V(\Gamma(R_1 \circ R_2)) = \bigcup_{x_i \in R_1^*}(V(\Gamma(R_{x_i2}))) \cup V(\Gamma(R_{1y_j}))$.

Theorem 2.4. *Let R_1 and R_2 be two commutative rings. Then $\Gamma(R_1 \circ R_2)$ is connected and $diam(\Gamma(R_1 \circ R_2)) \leq 2$.*

Proof. All vertices of $\Gamma(R_{1y_j})$ are adjacent to all vertices of $\Gamma(R_{x_i2})$. Hence $diam(\Gamma(R_1 \circ R_2)) \leq 2$. □

In the rest, we consider $R_1 = \{0_{R_1}, x_1, \cdots, x_n\}$ and $R_2 = \{0_{R_2}, y_1, \cdots, y_m\}$.

Theorem 2.5. *Let R_1 and R_2 be two commutative rings with $Z(R_1) = \{0_{R_1}\}$, $|R_1| \geq 3$, and $Z(R_2) = \{0_{R_2}\}$. Then $\kappa(\Gamma(R_1 \circ R_2)) = |R_2|$.*

Proof. There is no path between $(x_i,0_{R_2})$ and $(x_t,0_{R_2})$ for $i \neq t$ in $\Gamma(R_1 \circ R_2) - \Gamma(R_{1y_j})$. Hence $\kappa(\Gamma(R_1 \circ R_2)) \leq |V(\Gamma(R_{1y_j}))| = |R_2|$.

Now, let S be a vertex-cut of $\Gamma(R_1 \circ R_2)$. Then $\Gamma(R_1 \circ R_2) - S$ has at least two distinct components, say Γ_1 and Γ_2. Let $(x_a,y_b) \in \Gamma_1$ and $(x_c,y_d) \in \Gamma_2$. Therefore, $x_a x_c \neq 0_{R_1}$, that is, $x_a \neq 0_{R_1}$ and $x_c \neq 0_{R_1}$. If $x_a = x_c$, then $y_b = y_d = 0_{R_2}$, a contradiction. So $x_a \neq x_c$ and $S = \Gamma(R_{1y_j})$. Therefore $\kappa(\Gamma(R_1 \circ R_2)) = |R_2|$. □

Let in the Theorem 2.5, $Z(R_2) \neq \{0_{R_2}\}$. By using notations of the proof of theorem, if $x_a = x_c$, then $y_b y_d \neq 0_{R_2}$. By [12], there are $\kappa_2 = \kappa(\Gamma(R_2))$ internally disjoint paths $P_1, \cdots, P_{\kappa_2}$ between y_b and y_d in $\Gamma(R_2)$. By choosing one vertex y_t of each path P_t for $t \in \mathbb{Z}_{\kappa_2}$ we get $S = \{(x_a,y_t)\} \cup \Gamma(R_{1y_j})$ where $|S| = \kappa_2 + |R_2|$. Also, for the case that $x_a \neq x_c$ we get $S = \Gamma(R_{1y_j})$.

Hence, we have the following result.

Corollary 2.6. *Let R_1 be a commutative ring with $Z(R_1) = \{0_{R_1}\}$ and $|R_1| \geq 3$. Then*

$$\kappa(\Gamma(R_1 \circ R_2)) = |R_2|,$$

for every commutative ring R_2.

Theorem 2.7. *Let R_1 be a commutative ring with $Z(R_1) = \{0_{R_1}\}$. The followings hold for every commutative ring R_2.*

1. *If $|R_1| = 1$, then $\kappa(\Gamma(R_1 \circ R_2)) = |Z(R_2)| - 1$.*
2. *If $|R_1| = 2$ and $Z(R_2) = \{0_{R_2}\}$ then $\kappa(\Gamma(R_1 \circ R_2)) = |R_2|$.*
3. *If $|R_1| = 2$ and $|Z(R_2)| \geq 2$ then $\kappa(\Gamma(R_1 \circ R_2)) = |R_2| + \kappa_2$.*

Proof.
1. If $R_1 = \{0_{R_1}\}$, then $\Gamma(R_1 \circ R_2) \cong K_r$ where $r = |Z(R_2)|$. Hence $\kappa(\Gamma(R_1 \circ R_2)) = r - 1$.
2. Let $R_1 = \{0_{R_1}, x_1\}$. Then, $V(\Gamma(R_1 \circ R_2)) = \{(0_{R_1}, y_j), (0_{R_1}, 0_{R_2}), (x_1, 0_{R_2}) | y_j \in R_2\}$ and $\Gamma(R_1 \circ R_2) \cong K_{m+2}$, as needed.
3. By similar argument just pirior to Corollary 2.6, we get $\kappa(\Gamma(R_1 \circ R_2)) = |R_2| + \kappa_2$.

□

Note that, if $Z(R) = R$ then in general R is not the null ring. Take $R = \{0,2,4,6\}$ where addition is addition mod 8 and multiplication is multiplication mod 8. Then R is a ring with $Z(R) = R$.

Theorem 2.8. *Let R_1 be a commutative rings with $Z(R_1) = R_1$. Then*

$$\kappa(\Gamma(R_1 \circ R_2)) = |R_2|(\kappa(\Gamma(R_1)),$$

for every commutative ring R_2.

Furthermore, if $\Gamma(R_1)$ is complete, then $\kappa(\Gamma(R_1 \circ R_2)) = n|R_2| + \kappa(\tilde{\Gamma}(R_2))$.

Proof. Since $Z(R_1) = R_1$, we get $\Gamma(R_1 \circ R_2) \cong \Gamma(R_1) \circ \tilde{\Gamma}(R_2)$. If $\Gamma(R_1)$ is non-complete, then the result holds by using Theorem 1.1.
Assume that $\Gamma(R_1) \cong K_{n+1}$. We can consider $\Gamma(R_1 \circ R_2)$ as a complete graph with vertices $\Gamma(R_{1y_j}) \cup \Gamma(R_{x_i2})$ for every $x_i \in R_1$. By definition, $S = \bigcup_{i=2}^{n} \Gamma(R_{x_i2}) \cup \Gamma(R_{1y_j}) \cup \{(x_1, y_t)\}$ for $t \in \mathbb{Z}_{\kappa(\tilde{\Gamma}(R_2))}$ is a minimum vertex-cut in $\Gamma(R_1 \circ R_2)$. Thus, $\kappa(\Gamma(R_1 \circ R_2)) = n|R_2| + \kappa(\tilde{\Gamma}(R_2))$. □

Theorem 2.9. *Let R_1 and R_2 be two commutative rings with $Z(R_1) \neq R_1$ and $Z(R_1) \neq \{0_{R_1}\}$. Then $\kappa(\Gamma(R_1 \circ R_2)) = |R_2|$.*

Proof. By hypothesis there exists $x_i \in R_1 - Z(R_1)$. There is no path between (x_i, y_j) and (x_t, y_j) in $\Gamma(R_1 \circ R_2) - \Gamma(R_{1y_j})$ for $x_t \in Z(R_1)^*$ and $y_j \in Z(R_2)$. Hence, $\kappa(\Gamma(R_1 \circ R_2)) \leq |R_2|$.

Now, let S be a vertex-cut of $\Gamma(R_1 \circ R_2)$. Assume that Γ_1 and Γ_2 are distinct components of $\Gamma(R_1 \circ R_2) - S$. Let $(x_a, y_b) \in \Gamma_1$ and $(x_c, y_d) \in \Gamma_2$. Therefore $x_a x_c \neq \{0_{R_1}\}$. Let $x_a, x_c \in Z(R_1)^*$ and $A = \{x'_1, \cdots, x'_{\kappa_1}\}$ be a minimum vertex-cut in $\Gamma(R_1)$ where $\kappa(\Gamma(R_1)) = \kappa_1$. Clearly, $0_{R_1} \in A$. There are two cases:

Case 1. Let $x_a \neq x_c$. By [12] there are κ_1 internally disjoint paths $P_1, \cdots, P_{\kappa_1}$ between x_a and x_c in $\Gamma(R_1)$. Choose one vertex x_t of each path P_t for $t \in \mathbb{Z}_{\kappa_1}$. Then $S = \{(x_t, y_j) | y_j \in R_2\}$ where $|S| = \kappa_1.|R_2|$.

Case 2. Let $x_a = x_c$. Then $y_b y_d \neq 0_{R_2}$. There are two subcases:

Subcase 1. Let $Z(R_2) = \{0_{R_1}\}$. Then

$$S = \{(x_i, y_j), (x_a, 0_{R_2}) | x_i \in N_{\Gamma(R_1)}(x_a), y_j \in R_2\}$$

with $|S| = deg_{\Gamma(R_1)}(x_a).|R_2| + 1 \geq \delta(\Gamma(R_1)).|R_2| + 1 \geq \kappa_1.|R_2| + 1$.

Subcase 2. Let $Z(R_2) \neq \{0_{R_2}\}$. If $y_b \notin Z(R_2)^*$ or $y_d \notin Z(R_2)^*$, then S is same as Subcase 1.

Let $y_b, y_d \in Z(R_2)^*$. By [12] there are $\kappa_2 = \kappa(\Gamma(R_2))$ internally disjoint paths $Q_1, \cdots, Q_{\kappa_2}$ between y_b and y_d in $\Gamma(R_2)$. Choose one vertex y_u of each path Q_u for $u \in \mathbb{Z}_{\kappa_2}$. Then, $S = \{(x_i, y_j), (x_a, y_u) | x_i \in N_{\Gamma(R_1)}(x_a), y_j \in R_2\}$ with $|S| = deg_{\Gamma(R_1)}(x_a).|R_2| + \kappa_2 \geq \delta(\Gamma(R_1)).|R_2| + \kappa_2 \geq \kappa_1.|R_2| + \kappa_2$.

Furthermore, in the case that $x_a \notin Z(R_1)^*$ or $x_b \notin Z(R_1)^*$ we get $S = \Gamma(R_{1y_j})$.

Hence, in any case $|S| \geq |R_2|$, as needed. □

3 Connectivitiy of the subgraphs of $\Gamma(R_1 \circ R_2)$

In this section, we investigate the connectivity of subgraphs of $\Gamma(R_1 \circ R_2)$ for two commutative rings R_1 and R_2. It is easy to check that the subgraph of $\Gamma(R_1 \circ R_2)$ whose vertices are elements of $Z(R_1)^* \times Z(R_2)^*$ is equivalent to $\Gamma_0(R_1) \circ \Gamma_0(R_2)$.

Definition 1. Let R_1 and R_2 be two commutative rings.

The subgraph of $\Gamma(R_1 \circ R_2)$ whose vertices are elements of $Z(R_1)^* \times R_2$ is denoted by $\Gamma_{01}(R_1 \circ R_2)$.

Also, the subgraph of $\Gamma(R_1 \circ R_2)$ whose vertices are elements of $R_1 \times Z(R_2)^*$ is denoted by $\Gamma_{02}(R_1 \circ R_2)$.

Example 3.1. *Let $R_1 = \mathbb{Z}_4$ and $R_2 = \mathbb{Z}_8$. Clearly, $\Gamma_0(R_1) \cong K_1$, $\Gamma_0(R_2) \cong K_{1,2}$. Imagin $x = \bar{2}$. Thus*

$$V(\Gamma_{01}(R_1 \circ R_2)) = \{(x, y) | y \in R_2\}.$$

Now, $x^2 = 0_{R_1}$ implies that $\Gamma_0(R_1) \circ \Gamma_0(R_2) \cong K_3$, $\Gamma_{01}(R_1 \circ R_2) \cong K_8$ and $\kappa(\Gamma_{01}(R_1 \circ R_2)) = 7$.

By definition, if $Z(R_1) = \{0_{R_1}\}$ then $\Gamma_{01}(R_1 \circ R_2)$ is null. Consider Example 2.1.

Theorem 3.2. *Let R_1 and R_2 be two commutative rings with $Z(R_1) \neq \{0_{R_1}\}$. Then $\kappa(\Gamma_{01}(R_1 \circ R_2)) = \kappa(\Gamma_0(R_1))|R_2|$.*

Proof. Let $\Gamma_0(R_1)$ be complete. Then $\Gamma_{01}(R_1 \circ R_2)$ is a complete graph with vertices $\Gamma(R_{x_i 2})$ where each $\Gamma(R_{x_i 2})$ is isomorphic to $\tilde{\Gamma}(R_2)$. Hence $\kappa(\Gamma_{01}(R_1 \circ R_2)) = \kappa(\Gamma_0(R_1))|R_2|$.

Now, let $\Gamma_0(R_1)$ be non-complete. By [1], $\Gamma_0(R_1)$ is connected with $diam(\Gamma_0(R_1)) \leq 3$. The result holds by using Theorem 1.1. □

Example 3.3. *Let $R_1 = \mathbb{Z}_2$ and $R_2 = \mathbb{Z}_6$. Let $y_1 = \bar{2}, y_2 = \bar{3}, y_3 = \bar{4}$ in $\mathbb{Z}_6$. Thus*

$$V(\Gamma_{02}(R_1 \circ R_2)) = \{(x, y_j) | x \in R_1, j \in \mathbb{Z}_3\}.$$

So, $\Gamma_{02}(R_1 \circ R_2) \cong K_6$ and $\kappa(\Gamma_{02}(R_1 \circ R_2)) = 5$.

By definition, if $Z(R_2) = \{0_{R_2}\}$ then $\Gamma_{02}(R_1 \circ R_2)$ is null.

Let $Z(R_2) \neq \{0_{R_2}\}$. Now, by $\Gamma_{02}(R_1 \circ R_2) \cong \tilde{\Gamma}(R_1) \circ \Gamma_0(R_2)$ and using Theorem 1.1, we have the following result.

Theorem 3.4. *Let R_1 and R_2 be two commutative rings with $Z(R_2) \neq \{0_{R_2}\}$. Then $\kappa(\Gamma_{02}(R_1 \circ R_2)) = \kappa(\tilde{\Gamma}(R_1))|Z(R_2)^*|$.*

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Citation: Z. Ghasemi Khangahi, R. Varmazyar, On the connectivity of $\Gamma(R_1 \circ R_2)$, J. Disc. Math. Appl. 9(1) (2024) 73–79.
https://doi.org/10.22061/jdma.2024.10816.1071

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