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*Research Paper*

# On Pairs of non-abelian finite p-groups

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**Abstract.** Let $(N,G)$ be a pair of non-abelian finite $p$-groups and $K$ be a normal subgroup of $G$ such that $G \cong N \times K$. Moreover, let $|N| = p^n$ and $|N'| = p^k$, where $K$ is a $d$-generator group of order $p^m$. Then $|\mathcal{M}(N,G)| = p^{\frac{1}{2}(n-1)(n-2)+1+(n-1)m-s'}$, where $\mathcal{M}(N,G)$ is the Schur multiplier of the pair $(N,G)$ and $s'$ is a non-negative integer. In this paper, the non-abelian pairs $(N,G)$ for $s' = 0,1,2,3$ are characterized.

**Keywords.** Pair of groups, schur multiplier, finite $p$-groups.

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## 1 Introduction

Schur [21] defined the Schur multiplier of a group $G$. The Schur multiplier of a group $G$ is as the abelian group $\mathcal{M}(G) = R \cap F^{'}/[R,F]$ in which $F/R$ is a free presentation of $G$, (see [8] for more information.) In 1956, Green [5] proved that $|\mathcal{M}(G)| \leq p^{\frac{1}{2}n(n-1)}$ for $p$-groups $G$ of order $p^n$. Thus $|\mathcal{M}(G)| = p^{\frac{1}{2}n(n-1)-t(G)}$ for some $t(G) \geq 0$. In [1,4,6,9,12] all finite $p$-groups are characterized when $t(G) = 0,1,2,...,7$.
Niroomand [14] improved the Green's bound and proved that for non abelian $p$-groups of order $p^n$, $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)+1-s(G)}$, for some $s(G) \geq 0$. The structure of non-abelian $p$-groups for $s(G) = 0,1,2,3$ has been determined in [6,14,16,17].

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A pair of groups $(N,G)$ is a group $G$ with a normal subgroup $N$. In 1998, Eliss [2] introduced the Schur multiplier of the pair $(N,G)$ to be the abelian group $\mathcal{M}(N,G)$ appears in a natural exact sequence

$$H_3(G) \to H_3(\frac{G}{N}) \to \mathcal{M}(N,G) \to \mathcal{M}(G) \to \mathcal{M}(\frac{G}{N}) \\ \to \frac{N}{[N,G]} \to (G)^{ab} \to (\frac{G}{N})^{ab} \to 1,$$

in which $H_3(-)$ is the third homology of a group with integer coefficients. If $N = G$, then $\mathcal{M}(G,G)$ is the usual Schur multiplier of $G$.
Let $(N,G)$ be a pair of groups such that $G \cong N \times K$ with $|N| = p^n$ and $|K| = p^m$. Ellis [4] proved that

$$|\mathcal{M}(N,G)| \leq p^{\frac{1}{2}n(n+2m-1)}. \tag{1}$$

In this paper it is proved that

$$|\mathcal{M}(N,G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1+(n-1)m}. \tag{2}$$

So, $|\mathcal{M}(N,G)| = p^{\frac{1}{2}(n-1)(n-2)+1+(n-1)m-s'}$, where $s'$ is a non-negative integer. The upper bound (2) is better than the upper bound (1). Moreover, all non-abelian finite pairs $(N,G)$ for $s' = 0,1,2,3$ are characterized.

## 2 Preliminaries

In this section, some preliminary results are discussed which will be used in the main theorem. Throughout this paper the following notations are used:
$Q_8$: quaternion group of order 8,
$D_8$: dihedral group of order 8,
$E_1$: extra special p-group of order $p^3$ and exponent $p$,
$E_2$: extra special p-group of order $p^3$ and exponent $p^2$ $(p \neq 2)$,
$C_{p^n}^{(m)}$ : direct product of $m$ copies of the cyclic group of order $p^n$,
$G^{ab}$: the abelianization of group $G$.
$M.N$: the ceneral product of $M$ and $N$.

James [13] classified all $p$-groups of order $p^n$ for $n \leq 6$ up to isoclinism which are denoted by $\Phi_k$. We use his notation in our paper.

**Theorem 2.1.** *( [6,14,16,17]) Let G be a non-abelian p-group of order $p^n$ and $|\mathcal{M}(G)| = p^{\frac{1}{2}(n-1)(n-2)+1-s}$, then*

*(i)* $s = 0$ *if and only if* $G \cong E_1 \times C_p^{(n-3)}$.

*(ii)* $s = 1$ *if and only if* $G \cong D_8 \times C_2^{(n-3)}$ *or* $G \cong C_p^{(4)} \rtimes C_p$ $(p \neq 2)$.

*(iii)* $s = 2$ *if and only if*

(1) $G \cong E(2) \times C_p^{(n-2m-2)} = E.Z(E) \times C_p^{(n-2m-2)}$, *where E is an extra special p-group and* $Z(E)$ *is a cyclic group of order* $p^m$ $(m \geq 2)$,

(2) $G \cong E_2 \times C_p^{(n-3)}$,

(3) $G \cong Q_8 \times C_2^{(n-3)}$,

(4) $G \cong H \times C_p^{(n-2m-1)}$, *where H is an extra special p-group of order* $p^{2m+1}$ $(m \geq 2)$,

(5) $G \cong \langle a,b \mid a^4 = b^4 = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^2b^2\rangle$

(6) $G \cong \langle a,b,c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle$

(7) $G \cong \langle a,b \mid a^{p^2} = 1, b^p = 1, [a,b,a] = [a,b,b] = 1\rangle$

(8) $G \cong C_p \times (C_p^{(4)} \rtimes C_p)$ $(p \neq 2)$

(9) $G \cong \langle a,b \mid a^9 = b^3 = 1, [a,b,a] = 1, [a,b,b] = a^6, [a,b,b,b] = 1\rangle$

(10) $G \cong \langle a,b \mid a^p = 1, b^p = 1, [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1\rangle$ $(p \neq 3)$

(11) $G \cong \phi_2(211)b$ *(This group is omitted in [16], but* $|\mathcal{M}(G)| = p^2$ *and* $n = 4$ *so* $s = 2$*, see [7])*

*(iv)* $s = 3$ *if and only if*

(1) $G \cong \Phi_2(22) = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^p = \alpha_2, \alpha_1^{p^2} = \alpha_2^p = 1\rangle$,

(2) $G \cong \Phi_2(2111)c = \Phi_2(211)c \times C_p$ *where* $\Phi_2(211)c = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1\rangle$ ,

(3) $G \cong \Phi_2(2111)d = E_1 \times C_{p^2}$,

(4) $G \cong \Phi_3(211)a = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1\rangle$,

(5) $G \cong \Phi_3(211)b_r = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha^p = \alpha_3, \alpha_1^{(p)} = \alpha_2^p = \alpha_3^p = 1\rangle$,

(6) $G \cong \Phi_3(1^5) = \Phi_3(1^4) \times C_p$ *where* $\Phi_3(1^4) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3 \mid [\alpha_i, \alpha] = \alpha_{i+1}, \alpha^p = \alpha_i^{(p)} = \alpha_3^p = 1\ (i = 1,2)\rangle$

(7) $G \cong \Phi_7(1^5) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta \mid [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} = \alpha_{i+1}^p = \beta^p = 1\ (i = 1,2)\rangle$,

(8) $G \cong \Phi_{12}(1^6) = E_1 \times E_1$,

(9) $G \cong \Phi_{13}(1^6) = \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_2, \alpha_4] = \beta_2, \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1\ (i = 1,2)\rangle$,

(10) $G \cong \Phi_{15}(1^6) = \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2 \mid [\alpha_i, \alpha_{i+1}] = \beta_i, [\alpha_3, \alpha_4] = \beta_1, [\alpha_2, \alpha_4] = \beta_2^g, \alpha_i^p = \alpha_3^p = \alpha_4^p = \beta_i^p = 1 (i = 1,2)\rangle$ *where g is the smallest positive integer which is a primitive root modulo p*,

(11) $G \cong (C_p^{(4)} \rtimes C_p) \times C_p^{(2)}$,

(12) $G \cong \Phi_{11}(1^6) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3 \mid [\alpha_1, \alpha_2] = \beta_3, [\alpha_2, \alpha_3] = \beta_1, [\alpha_3, \alpha_1] = \beta_2, \alpha_i^{(p)} = \beta_i^p = 1\ (i = 1,2,3)\rangle$,

(13) $G \cong C_2^{(4)} \rtimes C_2$,

(14) $G \cong C_2 \times ((C_4 \times C_2) \rtimes C_2)$,

(15) $G \cong D_{16}$ *dihedral group of order 16*,

*(16)* $G \cong C_4 \rtimes C_4$.

## 3 Main Results

In this section, some upper bounds for the Schur multiplier of pairs of groups are obtained that are better than the upper bound of (1). Then they are used for characterizing the pair of non-abelian finite $p$-groups. The following results are used in our proofs.

**Lemma 3.1.** *( [14], Main Theorem) Let $G$ be a non-abelian finite $p$-group of order $p^n$. If $|G'| = p^k$, then we have*

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1}.$$

*In particular, $|\mathcal{M}(G)| \leq p^{\frac{1}{2}(n-1)(n-2)+1}$, and the equality holds in this last bound if and only if $G \cong E_1 \times Z$, where $Z$ is an elementary abelian $p$-group.*

**Lemma 3.2.** *( [18], Corollary 1.2) Let $(N, G)$ be a pair of groups and $K$ be the complement of $N$ in $G$. Then*

$$|\mathcal{M}(N, G)| = |\mathcal{M}(N)||N^{ab} \otimes K^{ab}|.$$

**Lemma 3.3.** *( [15], Theorem 2.2) Let $G$ be a $p$-group of order $p^n$ $(n \geq 4)$ such that $|G'| = p^{(n-2)}$, then*

$$|\mathcal{M}(G)| \leq \begin{cases} \frac{|G'|}{2} & p = 2 \\ |G'| & otherwise. \end{cases}$$

**Proposition 3.4.** *Let $(N, G)$ be a pair of groups and $K$ be the complement of $N$ in $G$. Also, let $|N| = p^n$, $|N'| = p^{n-2}$ We have*

$$|\mathcal{M}(N, G)| \leq \begin{cases} \frac{|N'|}{2}|N^{ab} \otimes K^{ab}| & p = 2 \\ |N'||N^{ab} \otimes K^{ab}| & otherwise. \end{cases}$$

*Proof.* We can obtain the results using Lemmas 3.2 and 3.3. □

**Lemma 3.5.** *( [15], Corollary 2.3) Let $G$ be a $p$-group of order $p^n$ with derived subgroup of order $p^k$. Then*

$$|\mathcal{M}(G)| \leq p^{\frac{1}{2}n(n-1)-\frac{1}{2}k(k+1)},$$

*equality holds if and only if $G$ is elementary abelian or $G \cong E_1$.*

**Proposition 3.6.** *Let $(N, G)$ be a pair of non-abelian finite $p$-groups and $K$ be a normal subgroup of $G$ such that $G \cong N \times K$. Also, let $|N| = p^n$, $|N'| = p^k$, where $K$ is a group of order $p^m$. Then*

$$|\mathcal{M}(N, G)| \leq p^{\frac{1}{2}n(n+2m-1)-\frac{1}{2}k(k+1+2m)}$$

*and the equality holds if and only if $G \cong E_1 \times C_p^{(m)}$.*

*Proof.* We obtain the result from Lemma 3.5 and Lemma 3.2.

$$\begin{aligned}|\mathcal{M}(N,G)| &= |\mathcal{M}(N)||N^{ab} \otimes K^{ab}| \leq p^{\frac{1}{2}n(n-1)-\frac{1}{2}k(k+1)}.p^{(n-k)m} \\ &= p^{\frac{1}{2}n(n+2m-1)-\frac{1}{2}k(k+1+2m)}.\end{aligned}$$

□

**Proposition 3.7.** *Let $(N,G)$ be a pair of non-abelian finite p-groups and K be a normal subgroup of G such that $G \cong N \times K$. Also, let $|N| = p^n$, $|N'| = p^k$, where K is a d-generator group of order $p^m$. Then*

$$|\mathcal{M}(N,G)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1+(n-k)d}.$$

*Proof.* We have $|N^{ab} \otimes K^{ab}| \leq p^{(n-k)d}$ and by Lemma 3.1 $|\mathcal{M}(N)| \leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1}$. Thus, by Lemma 3.2, we obtain

$$\begin{aligned}|\mathcal{M}(N,G)| &= |\mathcal{M}(N)||N^{ab} \otimes K^{ab}| \\ &\leq p^{\frac{1}{2}(n+k-2)(n-k-1)+1+(n-k)d}.\end{aligned}$$

□

**Theorem 3.8.** *Under assumption of Proposition 3.7*

$$|\mathcal{M}(N,G)| = p^{\frac{1}{2}(n-1)(n-2)+1+(n-1)m-s'}.$$

*Moreover, we have*

*(i) $s' = 0$ if and only if $(N,G)$ is isomorphic to the following pair*

$$(E_1 \times C_p^{(n-3)}, E_1 \times C_p^{(n-3)} \times C_p^{(m)}).$$

*(ii) $s' = 1$ if and only if $(N,G)$ is isomorphic the following pair*

$$(D_8 \times C_2^{(n-3)}, D_8 \times C_2^{(n-3)} \times C_2^{(m)}).$$

*(iii) $s' = 2$ if and only if $(N,G)$ is isomorphic to one of the following pairs:*
*(1) $(E_1, E_1 \times K)$ where K is a group with $m = d+1$,*
*(2) $(E(2) \times C_p^{(n-6)}, E(2) \times C_p^{(n-6)} \times C_p^{(m)})$,*
*(3) $(E_2 \times C_p^{(n-3)}, E_2 \times C_p^{(n-3)} \times C_p^{(m)})$,*
*(4) $(Q_8 \times C_2^{(n-3)}, Q_8 \times C_2^{(n-3)} \times C_2^{(m)})$,*
*(5) $(H \times C_p^{(n-2l-2)}, H \times C_p^{(n-2l-2)} \times C_p^{(m)})$, where H is an extra special p-group of order $p^{2l-1}$, $(l \geq 2)$,*
*(6) $(\langle a,b \mid a^4 = b^4 = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^2b^2\rangle, \langle a,b \mid a^4 = b^4 = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^2b^2\rangle \times C_p^{(m)})$,*

*(7)* $(\langle a,b,c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle, \langle a,b,c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab\rangle \times C_p^{(m)})$,
*(8)* $(\langle a,b \mid a^{p^2} = 1, b^p = 1, [a,b,a] = [a,b,b] = 1\rangle, \langle a,b \mid a^{p^2} = 1, b^p = 1, [a,b,a] = [a,b,b] = 1\rangle \times C_p^{(m)})$,
*(9)* $(\phi_2(211)b, \phi_2(211)b \times C_p)$.

*(iv)* $s' = 3$ *if and only if* $(N,G)$ *is isomorphic to one of the following pairs:*
*(1)* $(D_8, D_8 \times K)$*, where K is a group with* $m = d+1$*,*
*(2)* $((C_p^{(4)} \rtimes C_p) \times C_p^{(2)}, (C_p^{(4)} \rtimes C_p) \times C_p^{(2)})$
*(3)* $(\langle a,b \mid a^9 = b^3 = 1, [a,b,a] = 1, [a,b,b] = a^6, [a,b,b,b] = 1\rangle, \langle a,b \mid a^9 = b^3 = 1, [a,b,a] = 1, [a,b,b] = a^6, [a,b,b,b] = 1\rangle)$,
*(4)* $(\langle a,b \mid a^9 = b^3 = 1, [a,b,a] = 1, [a,b,b] = a^6, [a,b,b,b] = 1\rangle, \langle a,b \mid a^9 = b^3 = 1, [a,b,a] = 1, [a,b,b] = a^6, [a,b,b,b] = 1\rangle \times C_p)$,
*(5)* $(\langle a,b \mid a^p = 1, b^p = 1, [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1\rangle, \langle a,b \mid a^p = 1, b^p = 1, [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1\rangle)$
*(6)* $(\langle a,b \mid a^p = 1, b^p = 1, [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1\rangle, \langle a,b \mid a^p = 1, b^p = 1, [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1\rangle \times C_p)$,
*(7)* $(\phi_2(22), \phi_2(22) \times C_p^{(m)})$,
*(8)* $(\phi_3(211)a, \phi_3(211)a)$,
*(9)*$(\phi_3(211)b_r, \phi_3(211)b_r)$,
*(10)* $(\phi_2(2111)c, \phi_2(2111)c)$,
*(11)* $(\phi_2(2111)d, \phi_2(2111)d)$,
*(12)* $(\phi_3(1^5), \phi_3(1^5))$,
*(13)* $(\phi_7(1^5), \phi_7(1^5))$,
*(14)* $(\phi_{12}(1^6), \phi_{12}(1^6))$,
*(15)* $(\phi_{13}(1^6), \phi_{13}(1^6))$,
*(16)* $(\phi_{15}(1^6), \phi_{15}(1^6))$,
*(17)* $(\phi_{11}(1^6), \phi_{11}(1^6))$,
*(18)* $(D_{16}, D_{16})$,
*(19)* $(C_4 \rtimes C_4, C_4 \rtimes C_4)$,
*(20)* $(C_2^{(4)} \rtimes C_2, C_2^{(4)} \rtimes C_2)$,
*(21)* $(C_2 \times ((C_4 \times C_2) \rtimes C_2), C_2 \times ((C_4 \times C_2) \rtimes C_2))$.

*Proof.* By Lemmas 3.1 and 3.2, we obtain

$$p^{\frac{1}{2}(n-1)(n-2)+1+(n-1)m-s'} = p^{\frac{1}{2}(n-1)(n-2)+1-s}.|N^{ab} \otimes K^{ab}|$$

Thus,

$$p^{(n-1)m-s'} = p^{-s}.|N^{ab} \otimes K^{ab}|$$

and so,

$$p^{(n-1)m} = p^{s'-s}.|N^{ab} \otimes K^{ab}| \leq p^{s'-s}.p^{(n-k)d}$$
$$\leq p^{s'-s}.p^{(n-1)m}$$

Therefore, $s' \geq s$.

**Case** $s' = 0$**.** Then $s = 0$, and by Theorem 2.1 $N \cong E_1 \times C_p^{(n-3)}$. We have $|N'| = p$ and so, $|N^{ab} \otimes K^{ab}| \leq p^{(n-1).d}$. Now, we have

$$p^{(n-1)m} \leq p^{(n-1).d} \leq p^{(n-1)m}$$

Thus, $d = m$ and $G \cong E_1 \times C_p^{(n-3)} \times C_p^{(m)}$.

**Case** $s' = 1$**.** If $s = 0$, then $N \cong E_1 \times C_p^{(n-3)}$. Let $N \cong E_1 \times C_p^{(n-3)}$, then

$$p^{(n-1)m} = p|N^{ab} \otimes K^{ab}| \leq p.p^{(n-1).d} = p^{(n-1).d+1}.$$

Thus, $(n-1)m \leq (n-1).d + 1$ and so, $(n-1)(m-d) \leq 1$. This implies that $n = 1$ or 2 and $m = d$, thus $|\mathcal{M}(N)| = p^{1/2(n-1)(n-2)}$ which is impossible.

If $s = 1$, then $N \cong D_8 \times C_p^{(n-3)}$ or $N \cong C_p^{(4)} \rtimes C_p$ $(p \neq 2)$. Suppose that $N \cong D_8 \times C_2^{(n-3)}$, then $|N'| = p = 2$. Now, we have

$$p^{(n-1)m} = p^{s'-s}.|N^{ab} \otimes K^{ab}| = |N^{ab} \otimes K^{ab}|.$$

Hence, $p^{(n-1)m} \leq p^{(n-1).d} \leq p^{(n-1)m}$, this implies that $d = m$ and so $G \cong D_8 \times C_2^{(n-3)} \times C_2^{(m)}$.

If $N \cong C_p^{(4)} \rtimes C_p$ $(p \neq 2)$, then $|N'| = p^2$ and so, $p^{(n-1)m} \leq p^{(n-2).d} \leq p^{(n-2)m}$, which is impossible.

**Case** $s' = 2$**.** Let $s = 0$ ,then $N \cong E_1 \times C_p^{(n-3)}$. Hence

$$p^{(n-1)m} = p^2.|N^{ab} \otimes K^{ab}| \leq p^{(n-1).d+2}.$$

Hence, $(n-1)m \leq (n-1).d + 2$ and so, $(n-1)(m-d) \leq 2$. Thus, $n = 3$ and $d = m$ or $m = d + 1$. If $m = d$, then $|\mathcal{M}(N)| = p^{1/2(n-1)(n-2)-1}$, which is contradiction by Theorem 2.1, else $G \cong E_1 \times K$, where $m = d + 1$.

If $s = 1$, then $N \cong D_8 \times C_2^{(n-3)}$ or $N \cong C_p^{(4)} \rtimes C_p$ $(p \neq 2)$. Let $N \cong D_8 \times C_2^{(n-3)}$, then $|N'| = 2$. Also, $2(m-d) \leq 1$. Thus, $m = d$, $K \cong C_2^{(m)}$, hence, $|\mathcal{M}(N)| = p^{1/2(n-1)(n-2)-1}$, which is contradiction using Theorem 2.1

Now, assume that $N \cong C_p^{(4)} \rtimes C_p$ $(p \neq 2)$, then $|N'| = p^2$ and we have

$$p^{(n-1)m} \leq p.p^{(n-2).d} \leq p^{1+(n-2)m}$$

, so, $m \leq 1$ and $\mathcal{M}(N) = p^{1/2(n-1)(n-2)-1}$, which is contradiction by Theorem 2.1.

Let $s = 2$, then by Theorem 2.1, we have $N \cong E(2) \times C_p^{(n-2m-2)} = E.Z(E) \times C_p^{(n-2m-2)}$, where $E$ is an extra special $p$-group and $Z(E)$ is a cyclic group of order $p^m$ $(m \geq 2)$

$N \cong E_2 \times C_p^{(n-3)}$, $N \cong Q_8 \times C_2^{(n-3)}$,
$N \cong H \times C_p^{(n-2m-1)}$, where $H$ is an extra special $p$-group of order $p^{2m+1}$ $(m \geq 2)$,
$N \cong \langle a,b \mid a^4 = b^4 = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^2b^2 \rangle$,
$N \cong \langle a,b,c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle$,
$N \cong \langle a,b \mid a^{p^2} = 1, b^P = 1, [a,b,a] = [a,b,b] = 1 \rangle$,
$N \cong C_p \times (C_p^{(4)} \rtimes C_p)$ $(p \neq 2)$,
$N \cong \langle a,b \mid a^9 = b^3 = 1, [a,b,a] = 1, [a,b,b] = a^6, [a,b,b,b] = 1 \rangle$
$N \cong \langle a,b \mid a^p = 1, b^p = 1, [a,b,a] = [a,b,b,a] = [a,b,b,b] = 1 \rangle$
$N \cong \phi_2(211)b$
Similar to the previous cases we obtain $G \cong E(2) \times C_p^{(n-6)} \times C_p^{(m)}$, $G \cong E_2 \times C_p^{(n-3)} \times C_p^{(m)}$, $G \cong Q_8 \times C_2^{(n-3)} \times C_2^{(m)}$, $G \cong H \times C_p^{(n-2l-2)} \times C_p^{(m)}$ $(l \geq 2)$, $G \cong \langle a,b \mid a^4 = b^4 = 1, [a,b,a] = [a,b,b] = 1, [a,b] = a^2b^2 \rangle \times C_p^{(m)}$, $G \cong \langle a,b,c \mid a^2 = b^2 = c^2 = 1, abc = bca = cab \rangle \times C_p^{(m)}$ or $G \cong \langle a,b \mid a^{p^2} = 1, b^p = 1, [a,b,a] = [a,b,b] = 1 \rangle \times C_p^{(m)}$.
**Case** $s' = 3$. If $s = 0$, then $N \cong E_1 \times C_p^{(n-3)}$ and we have

$$\begin{aligned} p^{(n-1)m} &= p^3.|N^{ab} \otimes K^{ab}| = p^3.|E_1 \times C_p^{(n-3)} \otimes K^{ab}| \\ &\leq p^3.|E_1 \otimes K^{ab}||C_p^{(n-3)} \otimes K^{ab}| \\ &\leq p^3.p^{2d}.p^{(n-3).d} = p^{3+(n-1)d}. \end{aligned}$$

Hence $(n-1)(m-d) \leq 3$, in this case we obtain $n = 4$ and $m = d + 1$.

If $s = 1$, then $N \cong D_8 \times C_2^{(n-3)}$ or $C_p^{(4)} \rtimes C_p$, $p \neq 2$. Let $N \cong D_8 \times C_2^{(n-3)}$, then $p^{(n-1)m} \leq p^2.p^{(n-1).d}$, so $p^{(n-1)m} \leq p^{2+(n-1)d}$. Therefore $(m-d) \leq 1$, hence $m = d$ or $m = d+1$. If $m = d$ then $|\mathcal{M}(N)| = p^{1/2(n-1)(n-2)-2}$, which is contradiction and there isn't any group in this case. Therefore $G \cong D_8 \times K$, where $K$ is a group with $m = d + 1$.

Let $N \cong C_p^{(4)} \rtimes C_p$, $p \neq 2$, then $|N'| = p^2$ and we have $p^{(n-1)m} \leq p^2.p^{(n-2).d}$, so $p^{4m} \leq p^{2+3d}$, hence $m \leq 2$. Also $|\mathcal{M}(N,G)| = p^6 \leq p^{3d}$, hence $d \geq 2$ and $G \cong C_p^{(4)} \rtimes C_p \times C_{p^2}$ or $G \cong C_p^{(4)} \rtimes C_p \times C_p^{(2)}$, $p \neq 2$.

Now, suppose that $s = 2$, then $N$ is isomorphism with one of 10 groups by Theorem 2.1. In 1 to 7 groups, $|N'| = p$, we have $p^{(n-1)m} = p.|N^{ab} \otimes K^{ab}| \leq p.p^{(n-1)d}$, hence $(n-1)(m-d) \leq 1$, so $n = 1$ or $n = 2$, $m = d$ or $m = d+1$. By the definition of these groups we can not have $n = 1$ or $n = 2$, since $n \neq 2$, $m \neq d+1$, therefore $m = d$ and $K \cong C_p^{(m)}$.

In the first group $d(N) = n - 2l + 1$, so $p^{(n-1)m} \leq p.p^{(n-2l+1)m}$, thus $-2m - 2lm \leq 1$. By the definition of $N$, $l \geq 2$, therefore $m = 0$ and $|\mathcal{M}(N)| = p^{1/2}(n-1)(n-2) - 2$, which is contradiction using Theorem 2.1. So there isn't any pair of groups in this case.

Now suppose that $s = 3$, then $N$ is an isomorphism with one of the 16 groups in the Theorem 2.1. In this case we argue by the $|N|$ and $|N'|$.

If $N \cong \Phi_2(22)$, then $|N| = p^4$ and $|N'| = p$, so $p^{3m} \leq p^{3d} \leq p^{3m}$. Hence $d = m$ and $G \cong \Phi_2(22) \times C_p^{(m)}$.

If $N \cong \Phi_3(211)a$ or $N| \cong \Phi_3(211)b_r$, $|N| = p^4$, $|N'| = p^2$ and $d(N) = 2$, hence $p^{3m} \leq p^{2d} \leq$

$p^{2m}$, therefore $m = 0$ and $G \cong N$.

If $N \cong \Phi_2(2111)c$ or $N \cong \Phi_2(2111)d$, $|N| = p^5$, $|N'| = p$ and $d(N) = 3$, hence $p^{4m} \leq p^{4d} \leq p^{4m}$. therefore $d = m$, Also on the other hand $p^{4m} \leq p^{3m}$, so $m = 0$ and $G \cong N$ in both cases.

If $N \cong \Phi_3(1^5)$ or $N \cong \Phi_3(1^7)$, $|N| = p^5$, $|N'| = p^2$, and $p^{4m} \leq p^{3d} \leq p^{3m}$, therefore $m = 0$ and $G \cong N$ in both cases.

In other cases when $p \neq 2$ we have $|N| = p^6$, $|N'| = p^2$ or $|N'| = p^3$. We can show similar to the previous cases $m = 0$ and $G \cong N$.

If $p = 2$ and $N \cong D_{16}$ or $C_4 \rtimes C_4$, then $|N| = 2^4$ and $|N'| \leq 2^2$. Therefore $p^{3m} \leq p^{2d} \leq p^{2m}$, So $m = 0$ and $G \cong N$. Also if $N \cong C_2^{(4)} \rtimes C_2$ or $N \cong C_2 \times (C_4 \times C_2) \rtimes C_2$, then $|N| = 2^5$ and $|N'| = 2^2$, hence $p^{4m} \leq p^{3d} \leq p^{3m}$, hence $m = 0$ and $G \cong N$. □

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