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*Review Paper*

# On the conjecture for the sum of the largest signless Laplacian eigenvalues of a graph- a survey

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**Abstract.** Let $G$ be a simple graph with order $n$ and size $m$. Let $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ be its diagonal matrix, where $d_i = \deg(v_i)$, for all $i = 1, 2, \ldots, n$ and $A(G)$ be its adjacency matrix. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of $G$. Let $q_1, q_2, \ldots, q_n$ be the signless Laplacian eigenvalues of $Q(G)$ and let $S_k^+(G) = \sum_{i=1}^{k} q_i$ be the sum of the $k$ largest signless Laplacian eigenvalues. Ashraf et al. [F. Ashraf, G. R. Omidi, B. Tayfeh-Rezaie, On the sum of signless Laplacian eigenvalues of a graph, Linear Algebra Appl. **438** (2013) 4539-4546.] conjectured that $S_k^+(G) \le m + \binom{k+1}{2}$, for all $k = 1, 2, \ldots, n$. We present a survey about the developments of this conjecture.

**Keywords:** Signless Laplacian matrix, signless Laplacian spectrum, clique number, forest
**Mathematics Subject Classification (2010):** 05C50.

## 1 Introduction

Let $G = (V, E)$ be a simple graph with $n$ vertices, $m$ edges having vertex and edge set, respectively, as $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{e_1, e_2, \ldots, e_m\}$. The adjacency matrix $A = (a_{ij})$ of $G$ is a $(0, 1)$-square matrix of order $n$ whose $(i, j)$-entry is equal to 1, if $v_i$ is adjacent to $v_j$ and equal to 0, otherwise. Let $D(G) = \text{diag}(d_1, d_2, \ldots, d_n)$ be the diagonal matrix associated to $G$, where $d_i = \deg(v_i)$, for all $i = 1, 2, \ldots, n$. The matrices $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ are respectively called Laplacian and signless Laplacian matrices and their spectrum are respectively called Laplacian spectrum and signless Laplacian spectrum of the

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graph $G$. Being real symmetric, positive semi-definite matrices, let $0 = \mu_n \leq \mu_{n-1} \leq \cdots \leq \mu_1$ and $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$ be respectively the Laplacian spectrum and signless Laplacian spectrum of $G$. It is well known that $\mu_n$=0 with multiplicity equal to the number of connected components of $G$ and $\mu_{n-1} > 0$ if and only if $G$ is connected. Also, it is well known that $\mu_i = q_i$, for all $i = 1, 2, \ldots, n$, if and only if $G$ is a bipartite graph [8].

Let $S_k(G) = \sum\limits_{i=1}^{k} \mu_i$, $k = 1, 2, \ldots, n$ be the sum of $k$ largest Laplacian eigenvalues of $G$ and let $d_i^*(G) = |\{v \in V(G) : d_v \geq i\}|$, for $i = 1, 2, \ldots, n$. In 1994, Grone and Merris [21] observed that for any graph $G$, we have

$$S_k(G) \leq \sum_{i=1}^{k} d_i^*(G), \quad \text{for all } k = 1, 2, \ldots, n.$$

This observation was proved by Hua Bai [6] and is nowadays called the Grone-Merris theorem. As an analogue to Grone-Merris theorem, Andries Brouwer [7] conjectured the following.

If $G$ is a graph with $n$ vertices and $m$ edges, then for any $k$, $k = 1, 2, \ldots, n$

$$S_k(G) = \sum_{i=1}^{k} \mu_i \leq m + \binom{k+1}{2}.$$

For the progress on this conjecture, we refer to [15, 17–20, 24] and the references therein.

Let $S_k^+(G) = \sum_{i=1}^{k} q_i$, be the sum of $k$ largest signless Laplacian eigenvalues of a graph $G$. Motivated by the definition of $S_k(G)$ and Brouwer's conjecture, Ashraf et al. in [5] proposed the following conjecture about $S_k^+(G)$. For any graph with $n$ vertices and $m$ edges, we have

$$S_k^+(G) = \sum_{i=1}^{k} q_i \leq m + \binom{k+1}{2}, \quad \text{for all } k = 1, 2, \ldots, n.$$

The signless Laplacian energy $QE(G)$ of a graph $G$ is defined as

$$QE(G) = \sum_{i=1}^{n} |q_i - \frac{2m}{n}|.$$

Let $\sigma^+$ be the number of signless Laplacian eigenvalues greater than or equal to average degree $\overline{d} = \frac{2m}{n}$. Then by a simple observation, it follows that

$$QE(G) = \sum_{i=1}^{n} |q_i - \frac{2m}{n}| = 2S_\sigma^+(G) - \frac{4m\sigma}{n} = \max_{1 \leq i \leq n-1} \left\{ 2S_i^+(G) - \frac{4mi}{n} \right\},$$

where $S_\sigma^+(G) = \sum\limits_{i=1}^{\sigma} q_i$, is the sum of $\sigma$-largest signless Laplacian eigenvalues of $G$.

Various papers can be found in the literature containing some unsolved problems and conjectures on the topic of signless Laplacian energy of a graph. For some work on signless

Laplacian eigenvalues, we refer to [1, 2, 4, 25]. It is clear from the above definition that any upper or lower bound for $S_k(G)$ can be utilized to obtain a corresponding bound for signless Laplacian energy $QE(G)$. Therefore, any development on the Conjecture 1 can lead us to possible solution of these unsolved problems.

The conjecture is still open and seems difficult even for a particular class of graphs. For the progress on this conjecture and related results, we refer to [16, 28, 30] and the references therein.

A clique of a graph $G$ is the maximal complete subgraph of the graph $G$. The order of the maximal clique is called clique number of the graph $G$ and is denoted by $\omega(G)$ or $\omega$. If $H$ is a subgraph of the graph $G$, then we denote the graph obtained by removing the edges in $H$ from $G$ by $G \setminus H$. As usual, $K_n$ and $K_{s,t}$ denote, respectively, the complete graph on $n$ vertices and the complete bipartite graph on $s+t$ vertices. For other undefined notations and terminology(if any), the readers are referred to [8, 26]. Recently, Brouwer-type conjecture for the eigenvalues of distance signless Laplacian matrix of a graph was given in [3].

## 2 Progress on Conjecture 1

Using computations on a computer, Ashraf et al. [5] verified the truth of this conjecture for all graphs with at most 10 vertices. For $k=1$, the conjecture follows from the well-known inequality $q_1(G) \leq \frac{2m}{n-1} + n - 2$ and $m \geq n-1$ (see [10]). Also the cases $k=n$ and $k=n-1$ are straightforward. The conjecture is true for trees and follows from the fact that, Brouwer's conjecture holds for trees and for trees Laplacian and signless Laplacian eigenvalues are same. Ashraf et al. [5] showed that the conjecture is true for all graphs when $k=2$ and is also true for regular graphs.

Yang et al. [32] obtained various upper bounds for $S_k^+(G)$ and proved that the conjecture is also true for unicyclic graphs, bicyclic graphs and tricyclic graphs (except for $k=3$). One of the results obtained by Yang et al. [32] is the following upper bound for $S_k^+(G)$, in terms of the clique number $\omega$ and the number of edges $m$:

$$S_k^+(G) \leq k(\omega-2) + 2m - \omega(\omega-2). \tag{1}$$

Das et al. [9] obtained an upper for $S_k(G)$ of a graph with $n$ vertices, in terms of the vertex covering number $\tau$ and the number of edges $m$. Using similar analysis, the following upper bound can be obtained for $S_k^+(G)$, in terms of the vertex covering number $\tau$ and the number of edges $m$:

$$S_k^+(G) \leq m + k\tau, \tag{2}$$

with equality if and only if $G \cong K_{1,n-1}$.

Pirzada et al. [28] obtained upper bounds for $S_k^+(G)$, in terms of various graph parameters, which improve some previously known upper bounds and showed that the conjecture is true for some new classes of graphs.

Let $\Gamma_1$ be the family of all connected graphs except for the graphs $G$, where the vertices in the vertex covering set $S = \{v_1, v_2, \ldots, v_{\omega-1}\}$ of the subgraph $K_\omega$ have the property that there are pendent vertices incident to some $v_i \in S$ or any two vertices of $S$ forms a triangle with a vertex $v \in V(G) \setminus C$, where $C$ is the vertex covering set of $G$.

The following theorem gives an upper bound for $S_k^+(G)$ in terms of the clique number $\omega$, the vertex covering number $\tau$ and the number of edges $m$ of the graph $G$. The number of vertices in a graph $G$ is denoted by $n(G)$ and the number of vertices adjacent to a vertex $v$ is denoted by $N(v)$.

**Theorem 2.1.** [28] Let $G \in \Gamma_1$ be a connected graph of order $n \geq 2$ with $m$ edges having clique number $\omega$ and vertex covering number $\tau$. Then, for $1 \leq k \leq n$,

$$S_k^+(G) \leq k(\tau - 1) + m - \frac{\omega(\omega - 3)}{2}, \tag{3}$$

with equality if and only if $G \cong K_n$.

*Remark* 1. *For a graph $G \in \Gamma_1$, it is easy to see that the upper bound given by* (3) *is better than the upper bound given by* (1) *for all $m \geq k(\tau - \omega + 1) + \frac{\omega(\omega-1)}{2}$. In particular, for the graph with $\tau = \omega$ and $k \leq n - \omega$, the upper bound* (3) *is better than the upper bound* (1).

*Remark* 2. *Clearly for the graph $G \in \Gamma_1$ the upper bound given by* (3) *is always better than the upper bound given by* (2).

Let $\Gamma_2$ be the family of all connected graphs except for the graphs $G$, where the vertices in the vertex covering set $S = \{v_1, v_2, \ldots, v_{\lfloor \frac{d}{2} \rfloor}\}$ of the subgraph $P_d$ has the property that there are pendent vertices incident at some $v_i \in S$ or any two vertices of $S$ forms a triangle with a vertex $v \in V(G) \setminus C$, where $C$ is the vertex covering set of $G$. Rocha et al. [31] obtained an upper bound for $S_k(G)$ in terms of diameter of the graph $G$. Using similar analysis, the following upper bound is obtained for $S_k^+(G)$, in terms of the diameter $d - 1$ of the graph $G$ [28].

$$S_k^+(G) \leq 2(m - d) + 1 - n + 4k + p + \cos\left(\frac{k\pi}{d}\right) + \frac{\cos(\frac{\pi}{d})\sin(\frac{k\pi}{d}) + \sin(\frac{k\pi}{d})}{\sin(\frac{\pi}{d})}, \tag{4}$$

where $p$ is the number of isolated vertices in the graph obtained by removing the edges of $P_d$ from $G$.

The following theorem gives an upper bound for $S_k^+(G)$, in terms of the diameter, the number of edges $m$ and the vertex covering number $\tau$ of the graph $G$.

**Theorem 2.2.** [28] Let $G \in \Gamma_2$ be a connected graph of order $n \geq 3$ with $m$ edges having diameter $d - 1$ and vertex covering number $\tau$. Then for $1 \leq k \leq n$,

$$S_k^+(G) \leq (\tau - \lfloor\frac{d}{2}\rfloor + 2)k + m - d + \cos\left(\frac{k\pi}{d}\right) + \frac{\cos(\frac{\pi}{d})\sin(\frac{k\pi}{d}) + \sin(\frac{k\pi}{d})}{\sin(\frac{\pi}{d})}, \tag{5}$$

with equality if and only if $G \cong P_n$.

*Remark* 3. *For the connected graphs $G \in \Gamma_2$, it is easy to see that the upper bound given by* (5) *is better than the upper bound given by* (4) *for all $k \leq \frac{m-n-d+1+p}{\tau-\lfloor \frac{d}{2} \rfloor -2}$. In particular, if $G \in \Gamma_2$ is such that $\tau \leq \lfloor \frac{d}{2} \rfloor + 2$ and $m \geq n+d-1-p$, the upper bound* (5) *is always better than the upper bound* (4).

Let $\Gamma_3$ be the family of all connected graphs except for the graphs $G$, where the vertices in the vertex set $S = \{v_1, v_2, \dots, v_{s_1}, u_1, u_2, \dots u_{s_2}\}$ of the subgraph $K_{s_1,s_2}$, $s_1 \leq s_2$, has the property that there are pendent vertices incident at some $v_i$ or $u_j \in S$ or any two vertices of $S$ forms a triangle with a vertex $v \in V(G) \setminus C$, where $C$ is the vertex covering set of $G$.

Let $K_{s_1,s_2}$ $s_1 \leq s_2$, be the maximal complete bipartite subgraph of a graph $G$. Using the fact that the vertex covering number of $K_{s_1,s_2}$ $s_1 \leq s_2$, is $s_1$ and its signless Laplacian spectrum is $\{s_1+s_2, s_1^{[s_2-1]}, s_2^{[s_1-1]}, 0\}$, and proceeding similarly as in Theorem 2.1, we obtain the following upper bound for $S_k^+(G)$.

**Theorem 2.3.** [28] Let $G \in \Gamma_3$ be a connected graph of order $n \geq 2$ with $m$ edges having vertex covering number $\tau$. If $K_{s_1,s_2}$ $s_1 \leq s_2$, is the maximal complete bipartite subgraph of the graph $G$, then

$$S_k^+(G) \leq k(\tau + s_2 - s_1) + m - s_1(s_2 - 1), \tag{6}$$

with equality if and only if $G \cong K_{s_1,s_2}$ and $s_1 + s_2 = n$.

If $s_1 = s_2$, for the graphs $G \in \Gamma_3$, it is easy to see that the upper bound (6) is always better than the upper bound (2).

**Theorem 2.4.** [28] If $G \in \Gamma_1$ is a connected graph of order $n \geq 12$ with $m$ edges having clique number $\omega$, then for $\omega \geq \frac{3+\sqrt{3n^2-14n+9}}{2}$,

$$S_k^+(G) \leq m + \frac{k(k+1)}{2},$$

for all $k \in \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$.

Let $\Omega_n$ be a family of those connected graphs $G \in \Gamma_1$ for which the vertex covering number $\tau \in \{\omega - 1, \omega, \omega + 1\}$, that is,

$$\Omega_n = \{G \in \Gamma_1 : \tau = \omega - 1 \text{ or } \omega \text{ or } \omega + 1\}.$$

For the family of graphs $\Omega_n$, we have the following observation.

**Theorem 2.5.** [28] If $G \in \Omega_n$, then

$$S_k(G) \leq m + \frac{k(k+1)}{2},$$

holds for all $k$, if $\tau = \omega - 1$; holds for all $k$ except for $k = \omega - 2, \omega - 1$, if $\tau = \omega$; holds for all $k$, $k \leq \frac{2\omega - 1 - \sqrt{8\omega+1}}{2}$ and $k \geq \frac{2\omega-1+\sqrt{8\omega+1}}{2}$, if $\tau = \omega + 1$.

**Theorem 2.6.** [28] Let $G \in \Gamma_2$ be a connected graph of order $n \geq 2$ with $m$ edges having vertex covering number $\tau$. Let $K_{s_1,s_1}$ be the maximal complete bipartite subgraph of $G$. Then Conjecture 1 holds for all $k$, if $\tau \leq \frac{1+\sqrt{8s_1(s_1-1)}}{2}$;
holds for all $k \leq \frac{2\tau-1-\sqrt{(2\tau-1)^2-8s_1(s_1-1)}}{2}$ and $k \geq \frac{2\tau-1+\sqrt{(2\tau-1)^2-8s_1(s_1-1)}}{2}$, if $\tau \geq \frac{1+\sqrt{8s_1(s_1-1)}}{2}$.

Let $G$ be a connected bipartite graph of order $n$ having the vertex covering number $\tau$. For bipartite graphs, it is well known that $\tau \leq \frac{n}{2}$. With this in mind, we have the following observation for bipartite graphs.

**Theorem 2.7.** [28] Let $G \in \Gamma_3$ be a connected bipartite graph of order $n \geq 4$ with $m$ edges having the vertex covering number $\tau$. If $K_{s_1,s_1}$, with $s_1 \geq \frac{n}{4}$, is the maximal complete bipartite subgraph of the graph $G$, then

$$S_k(G) \leq m + \frac{k(k+1)}{2},$$

for all $k \leq \frac{n}{7} - 1$ and $k \geq \frac{6n}{7}$.

For graphs with girth $g \geq 5$, Rocha et al. [31] showed that Brouwer's conjecture holds for all $k \leq \lfloor \frac{g}{5} \rfloor$. Using similar analysis, we have the following observation.

**Theorem 2.8.** [28] For connected graphs with girth $g \geq 5$, Conjecture 1 holds for all $k$, $1 \leq k \leq \lfloor \frac{g}{5} \rfloor$.

Using Theorem 2.12, the fact that

$$\cos\left(\frac{k\pi}{d}\right) + \frac{\cos(\frac{\pi}{d})\sin(\frac{k\pi}{d}) + \sin(\frac{k\pi}{d})}{\sin(\frac{\pi}{d})} \leq 2k+1$$

and proceeding similarly as in above theorems, we arrive at the following observation.

**Theorem 2.9.** [28] Let $G \in \Gamma_2$ be a connected graph of order $n \geq 3$ with $m$ edges having diameter $d-1$ and vertex covering number $\tau$. Then for $1 \leq k \leq n$, Conjecture 1 holds for all $k$, if $\tau \leq \frac{2\lfloor \frac{d}{2} \rfloor - 7 + \sqrt{8(d-1)}}{2}$; holds for all $k$, $k \leq \frac{2\tau - 2\lfloor \frac{d}{2} \rfloor + 7 - \sqrt{2\tau - 2\lfloor \frac{d}{2} \rfloor + 7 - 8(d-1)}}{2}$ and $k \geq \frac{2\tau - 2\lfloor \frac{d}{2} \rfloor + 7 + \sqrt{2\tau - 2\lfloor \frac{d}{2} \rfloor + 7 - 8(d-1)}}{2}$, if $\tau \geq \frac{2\lfloor \frac{d}{2} \rfloor - 7 + \sqrt{8(d-1)}}{2}$.

Recently, Pirzada et al [29] verified Conjecture 1 for several more families of graphs. The following upper bound for $S_k^+(G)$ is in terms of the clique number $\omega$ and the number of vertices $n$ of the graph $G$.

**Theorem 2.10.** [29] Let $G$ be a connected graph of order $n \geq 4$ with $m$ edges having clique number $\omega \geq 2$. Let $H = G \setminus K_\omega$ be a forest having $r \geq 1$ non-trivial components and $p \geq 0$ trivial components. Then, for $\alpha = \sum_{i=1}^{r} \frac{2k_i - 2}{n_i}$, we have

$$S_k^+(G) \leq \begin{cases} \omega(\omega-1) + n - p - 2r + 2k - \alpha, & \text{if } k \geq \omega, \\ k\omega + \omega + n - p - 2r - \alpha, & \text{if } k \leq \omega - 1. \end{cases} \quad (7)$$

The next upper bound for $S_k^+(G)$ is in terms of the positive integers $s_1,\ s_2$ and the number of vertices $n$ of the graph $G$.

**Theorem 2.11.** [29] Let $G$ be a connected graph of order $n \geq 4$ having $m$ edges. Let $K_{s_1,s_2}$, $s_1 \geq s_2 \geq 2$, be the maximal complete bipartite subgraph of graph $G$. If $H = G \setminus K_{s_1,s_2}$ is a forest having $r \geq 1$ non-trivial components and $p \geq 0$ trivial components, then for $\alpha = \sum_{i=1}^{r} \frac{2k_i-2}{n_i}$, we have

$$S_k^+(G) \leq \begin{cases} 2s_1s_2 + n - p - 2r + 2k - \alpha, & \text{if } k \geq s_1 + s_2 - 1, \\ (k+1)s_2 + s_1s_2 - s_2^2 + n - p - 2r + 2k - \alpha, & \text{if } s_2 + 1 \leq k \leq s_1 + s_2 - 2, \\ ks_1 + s_2 + n - p - 2r + 2k - \alpha, & \text{if } k \leq s_2. \end{cases} \quad (8)$$

If in particular $s_1 = s_2$, we have the following consequence of Theorem 2.11.

**Corollary 2.12.** [29] Let $G$ be a connected graph of order $n \geq 4$ having $m$ edges. Let $K_{s,s},\ s \geq 2$, be the maximal complete bipartite subgraph of the graph $G$. If $H = G \setminus K_{s,s}$ is a forest, having $r \geq 1$ non-trivial components and $p \geq 0$ trivial components, then for $\alpha = \sum_{i=1}^{r} \frac{2k_i-2}{n_i}$, we have

$$S_k^+(G) \leq \begin{cases} 2s^2 + n - p - 2r + 2k - \alpha, & \text{if } k \geq 2s - 1, \\ s + k(s+2) + n - p - 2r - \alpha, & \text{if } k \leq 2s - 2. \end{cases}$$

Now, we see that Conjecture 1 almost holds for all graphs which belong to some special families of graphs.

**Theorem 2.13.** [29] Let $G$ be a connected graph of order $n \geq 4$ with $m$ edges having clique number $\omega \geq 2$. If $H = G \setminus K_\omega$ is a forest having $r \geq 1$ non-trivial components, then for $\alpha = \sum_{i=1}^{r} \frac{2k_i-2}{n_i}$, Conjecture 1 holds for all $k \in [\omega+1, n]$ and for all $k \in [1, \frac{(2\omega-1)-\sqrt{8\omega+1-8r-8\alpha}}{2}]$.

If a graph $G$ has the property that the graph $H = G \setminus K_\omega$ is a forest with large number of non-trivial components $r$, $1 \leq r \leq \omega$, then we have the following observation, the proof of which follows from the proof of Theorem 2.13.

**Corollary 2.14.** [29] For a connected graph $G$ of order $n \geq 4$ with $m$ edges having clique number $\omega \geq 2$. If $H = G \setminus K_\omega$ is a forest, then Conjecture 1 holds for all $k$, if $H = G \setminus K_\omega$ has $\omega$ non-trivial components; holds for all $k,\ k \neq \omega - 1, \omega$, if $H = G \setminus K_\omega$ has $\omega - 1$ non-trivial components; holds for all $k,\ k \neq \omega - 2, \omega - 1, \omega$, if $H = G \setminus K_\omega$ has $\omega - 2$ or $\omega - 3$ non-trivial components; holds for all $k,\ k \neq \omega - 3, \omega - 2, \omega - 1, \omega$, if $H = G \setminus K_\omega$ has $\omega - 4$ or $\omega - 5$ or $\omega - 6$ non-trivial components, and so on.

*Remark* 4. *From Theorem 2.13, it is clear that if a connected graph $G$ has the property that after removing the edges of the maximal complete subgraph $K_\omega$, the resulting graph is a forest, then Conjecture 1 holds for all $k$, if $\omega = 2$; holds for all $k,\ k \neq 3$, if $\omega = 3$; holds for all $k,\ k \in [1, 85]$ and $k \in [101, n]$, if $\omega = 100$; holds for all $k,\ k \in [1, 954]$ and $k \in [1001, n]$, if $\omega = 1000$ and so on.*

When a graph has a special kind of symmetry so that we can write its associated matrix in the form

$$M = \begin{pmatrix} X & \beta & \cdots & \beta & \beta \\ \beta^t & B & \cdots & C & C \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \beta^t & C & \cdots & B & C \\ \beta^t & C & \cdots & C & B \end{pmatrix}, \tag{9}$$

where $X \in R^{t\times t}$, $\beta \in R^{t\times s}$ and $B, C \in R^{s\times s}$, such that $n = t + cs$, where $c$ is the number of copies of $B$, then we can obtain the spectrum of this matrix as the union of the spectrum of smaller matrices using the following result presented in [12]. In the statement of the theorem, $\sigma^{(k)}(Y)$ indicates the multiset formed by $k$ copies of the spectrum of $Y$, denoted by $\sigma(Y)$.

Now, we consider some special classes of graphs satisfying the hypothesis of Theorem 2.13. Let $S_\omega(a_1, a_2, \dots, a_\omega)$, $a_i \geq 0$, $1 \leq i \leq \omega$, be the family of connected graphs of order $n = \sum_{i=1}^{\omega}(a_i + 1)$ with $m$ edges having $a_i$ pendent vertices attached at the $i^{th}$ vertex of the clique $K_\omega$. For the family of graphs $S_\omega(a_1, a_2, \dots, a_\omega)$, $a_i \geq 1$, $1 \leq i \leq \omega$, if after removing the edges of the maximal complete subgraph $K_\omega$, the resulting graph has exactly $\omega$ non-trivial components. Then, Conjecture 1 always holds by Corollary 2.14. Therefore, we need to consider the cases when some of the $a_i's$ are zero. For the family of graphs $S_\omega(a, 0, \dots, 0)$, $a \geq 1$, we have the following result.

**Theorem 2.15.** [29] For the graph $G \in S_\omega(a, 0, \dots, 0)$, $a \geq 1$, Conjecture 1 holds for all $k$.

Let $S_\omega(a, a, \dots, a, \dots, 0)$, $a \geq 1$ be the family of connected graphs of order $n = \omega + at$ with $m$ edges having $a$ pendent vertices attached to each of the $t$, $1 \leq t \leq \omega - 1$ vertices of the clique $K_\omega$. For the family of graphs $S_\omega(a, a, \dots a, \dots, 0)$, we have the following result.

**Theorem 2.16.** [29] For the graph $G \in S_\omega(a, a, \dots, a, \dots, 0)$, $a \geq 1$, Conjecture 1.1 holds for all $k$.

Let $T_\omega(a_1, a_2, \dots, a_\omega)$, $a_i \geq 0$, $1 \leq i \leq \omega$, be the family of connected graphs of order $n = \sum_{i=1}^{\omega}(a_i + 1)$ with $m$ edges obtained by fusing a pendent vertex of a tree on $a_i + 1$ vertices with the $i^{th}$ vertex of the clique $K_\omega$. For the family of graphs $T_\omega(a_1, a_2, \dots, a_\omega)$, $a_i \geq 1$, $1 \leq i \leq \omega$, if we remove the edges of the maximal complete subgraph $K_\omega$, the resulting graph has exactly $\omega$ non-trivial components. So, by Corollary 2.14, it follows that Conjecture 1.1 always holds. Therefore, we need to consider the cases when some of the $a_i$ are zero.

The following observation can be found in [32].

**Lemma 2.17.** *If $G$ is the graph obtained by joining a vertex of graph $G_1$ of order $n_1 \geq 2$ with a vertex of a graph $G_2$ of order $n_2 \geq 2$, then Conjecture 1.1 holds for $G$, if it holds for both $G_1$ and $G_2$.*

Since the conjecture is true for trees and complete graphs, we have the following observation.

**Theorem 2.18.** [29] For the graph $G \in T_\omega(a_1, a_2, \ldots, a_t \ldots, 0),\ \ a_i \geq 1,\ \ t \geq 1$, conjecture holds for all $k$.

Thus combining all the above observations, we arrive at the following theorem.

**Theorem 2.19.** [29] For the graph $G \in T_\omega(a_1, a_2, \ldots, a_n),\ \ a_i \geq 0$, Conjecture 1.1 holds for all $k$.

All these observations indicates that Conjecture 1 holds for almost all graphs which belong to the families of graphs satisfying the hypothesis of Theorem 2.13.

**Theorem 2.20.** [29] Let $G$ be a connected graph of order $n \geq 4$ having $m$ edges. Let $K_{s,s}$, $s \geq 2$ be the maximal complete bipartite subgraph of graph $G$. If $H = G \setminus K_{s,s}$ is a forest having $r \geq 1$ non-trivial components, then

$$S_k^+(G) \leq m + \frac{k(k+1)}{2},$$

holds for all $k$, if $s \geq 5$ and holds for all $k$, $\ k \neq 2,3,4,5,6,7$, if $2 \leq s \leq 4$.

Theorem 2.20, can be generalized as follows.

**Theorem 2.21.** [29] Let $G$ be a connected graph of order $n \geq 4$ having $m$ edges. Let $K_{s_1,s_2}$, $s_2 = s_1 - (t-2),\ \ t \geq 2$ be the maximal complete bipartite subgraph of graph $G$. If $H = G \setminus K_{s_1,s_2}$ is a forest having $r \geq 1$ non-trivial components, then

$$S_k^+(G) \leq m + \frac{k(k+1)}{2},$$

holds for all $k$, if $s_1 \geq 2t+1$ and holds for all $k$, if $s = 2t$ and $r \geq 3$.

*Remark* 5. *Theorem 2.20 implies that Conjecture 1 holds for all the graphs G having the property that after removing the edges of the maximal complete bipartite subgraph $K_{s,s}$, $s \geq 5$, the resulting graph is a forest. For $s = 4$, Conjecture 1 holds for all the graphs G having the property that after removing the edges of the maximal complete bipartite subgraph $K_{4,4}$, the resulting graph is a forest with at least 3 non-trivial components. For $s = 3$, Conjecture 1 holds for all the graphs G having the property that after removing the edges of the maximal complete bipartite subgraph $K_{3,3}$, the resulting graph is a forest with at least 4 non-trivial components. For $s = 2$, Conjecture 1 holds for all the graphs G having the property that after removing the edges of the maximal complete bipartite subgraph $K_{2,2}$, the resulting graph is a forest with 4 non-trivial components. Thus, we conclude that Conjecture 1 holds for all graphs which belong to the families of graphs satisfying the hypothesis of Theorem 2.20.*

Recently, Helmberg et. al [22], showed that for threshold graphs, the eigenvalues of the signless Laplacian matrix interlace with the degrees of the vertices. As an application, they verified that the Conjecture 1 holds for threshold graphs.

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