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Research Paper

Edge deletion and symmetric division degree index of graphs

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Abstract. The symmetric division deg index (or simply *SDD*) was proposed by Vukičević et al. as a remarkable predictor of the total surface area of polychlorobiphenyls. We are interested in how the *SDD* of a graph changes when edges are deleted. The obtained results show that all cases are possible: increased, decreased, and unchanged. In this article, we present some necessary conditions for the occurrence of each of the three different states.

Keywords: Symmetric division degree index, edge deletion, semiregular graph
Mathematics Subject Classification (2010): Primary 05C90; Secondary 92E10.

1 Introduction

Throughout this paper, let $G(V,\ E)$ be a simple graph with vertex set $V(G)$ and edge set $E(G)$. An edge $e \in E(G)$ with end vertices u and v is denoted by uv. In a graph G, the neighborhood $N_G(v)$ or briefly N_v of a vertex v is the set of all vertices adjacent to v. Let $e = uv \in E(G)$, we denote by N_u^v, the set of all vertices adjacent to u, except v, i.e. $N_u^v = N_u \setminus \{v\}$. The degree $d(v_i)$ (or simply d_i) of a vertex v_i is the number of edges incident on v_i. The smallest and the largest degrees of graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. If $\Delta(G) = \delta(G)$, then G is called a regular graph. A graph is (m,n)-semiregular if it is bipartite with a bipartition $\{V_1,\ V_2\}$ in which each vertex of V_1 has degree m and each one of V_2 has degree n.

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Molecular descriptors, which are numerical functions of molecular structure, play an important role in mathematical chemistry. They are used in $QSAR$ and $QSPR$ studies to study and predict the biological or chemical properties of molecules [4]. Topological indices, which are numerical functions of the underlying molecular graph, are an important group of these descriptors. The symmetric division degree index ($SDD(G)$) of a graph G introduced by Vukičević and Gašperov, is one of the 148 so-called Adriatic indices, with a good predictive power for the total surface area of polychlorobiphenyl [23]. It is defined as

$$SDD = SDD(G) = \sum_{v_i v_j \in E(G)} \frac{d_i^2 + d_j^2}{d_i d_j},$$

where d_i is a degree of vertex v_i. Also in [8] Ghorbani and Alidehi-Ravandi defined an eccentric version of the SDD index and gave some properties of it. The importance and application of the SDD, have caused many studies to be carried out in recent years. In some of them, the ability of this index to predict the properties of chemical structures has been investigated [6, 7, 14, 16, 19], and some others, have found graphs with extremal SDD index [1, 11, 13, 15, 18, 20, 22, 24]. Also, some researchers have focused on finding the bound for SDD of graphs [2, 5, 9, 12, 17, 21].

In a different work, Gupta et al. focused on SDD and graph operations such as join, corona product, cartesian product, composition, and symmetric difference of graphs [10].

In this article, we examine the effect of edge removal as another graph operation on the SDD index. In some special graphs, we get the exact value of this change and in some cases, we find a bound for it.

2 Edge deletion

In this section, we will determine the amount of change in the SDD index of graphs when an edge is removed from the graph G. Studies show that this index may increase or decrease or even remain unchanged by removing the edge. For example, consider the graph G as depicted in Figure 1. It is not difficult to see that $SDD(G \setminus e) = SDD(G)$.

According to the definition of SDD, the sum is taken over all edges. Therefore, removing the edge $e = uv$ from graph G, only changes the amount of SDD in the components corresponding to the edges leading to u and v, and the other components of the sum, remain unchanged. It means that if $uw \in E(G)$, the component related to this edge changes from $\frac{d_w}{d_u} + \frac{d_u}{d_w}$ in the $SDD(G)$ to $\frac{d_w}{d_u - 1} + \frac{d_u - 1}{d_w}$ in the $SDD(G \setminus e)$. Therefore, for this edge, the SDD decreases by $\frac{d_u}{d_w} - \frac{d_u - 1}{d_w} = \frac{1}{d_w}$ and increases by $\frac{d_w}{d_u - 1} - \frac{d_w}{d_u} = \frac{d_w}{d_u(d_u - 1)}$. Now, if we calculate these values for all edges leading to u and v, the amount of decrease in $SDD(G \setminus e)$ is equal to

$$\frac{d_u}{d_v} + \frac{d_v}{d_u} + \sum_{v_r \in N_u^v} \frac{1}{d_r} + \sum_{v_s \in N_v^u} \frac{1}{d_s} = \frac{d_u}{d_v} + \frac{d_v}{d_u} + \sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i},$$

and the amount of increase is

$$\sum_{v_r \in N_u^v} \frac{d_r}{d_u(d_u-1)} + \sum_{v_s \in N_v^u} \frac{d_s}{d_v(d_v-1)}.$$

In the following, for convenience, we set $m = d_u$ and $n = d_v$. So we have

$$SDD(G \setminus e) = SDD(G) + \alpha, \tag{1}$$

where

$$\alpha = \sum_{v_r \in N_u^v} \frac{d_r}{m(m-1)} + \sum_{v_s \in N_v^u} \frac{d_s}{n(n-1)} - \frac{m}{n} - \frac{n}{m} - \sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i}.$$

It is clear that $\alpha = 0$, $\alpha > 0$, and $\alpha < 0$ indicate no change, increase, and decrease of SDD, after edge removal, respectively. In the following, we obtain bounds for α and find its exact value for some special categories of graphs.

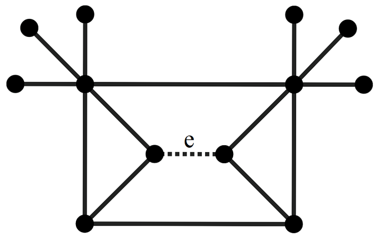

Figure 1. A graph G with $SDD(G \setminus e) = SDD(G)$.

Theorem 2.1. *Let G be a graph and $e = uv \in E(G)$ such that the sum of degrees of all vertices connected to u is less than or equal to $m(m-1)$, and also the sum of degrees of all vertices connected to v is less than or equal to $n(n-1)$. Then*

$$SDD(G \setminus e) < SDD(G).$$

Proof. If $\sum_{v_r \in N_u^v} d_r \leq m(m-1)$ and $\sum_{v_s \in N_v^u} d_s \leq n(n-1)$, then

$$\sum_{v_r \in N_u^v} \frac{d_r}{m(m-1)} \leq 1 \text{ and } \sum_{v_s \in N_v^u} \frac{d_s}{m(m-1)} \leq 1.$$

Since $\frac{m}{n} + \frac{n}{m} \geq 2$, for all real numbers m and n we have

$$\alpha = \sum_{v_r \in N_u^v} \frac{d_r}{m(m-1)} + \sum_{v_s \in N_v^u} \frac{d_s}{n(n-1)} - \frac{m}{n} - \frac{n}{m} - \sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i}$$
$$\leq - \sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i}.$$

Therefore, from Equality (1) we obtain that $SDD(G \setminus e) \leq SDD(G)$. □

Corollary 2.2. *Let G be a graph and $e = uv \in E(G)$ such that the degree of any vertex connected to u is less than or equal to d_u, and also the degree of any vertex connected to v is less than or equal to d_v. Then*

$$SDD(G \setminus e) < SDD(G).$$

Theorem 2.3. *Let G be a graph and $e = uv \in E(G)$, with $d_u = d_v = m$.*
(i) If the degree of all vertices connected to u and v is less than or equal to m, then

$$SDD(G \setminus e) < SDD(G).$$

(ii) If the degree of all vertices connected to u and v is greater than or equal to $2m$, then

$$SDD(G \setminus e) > SDD(G).$$

(iii) If the summation of degrees of all vertices connected to u and v is less than or equal to $2m(m-1)$, then

$$SDD(G \setminus e) < SDD(G).$$

Proof. If $d_u = d_v = m$, then Equality (1) yields that

$$\alpha = \sum_{v_r \in N_u^v} \frac{d_r}{m(m-1)} + \sum_{v_s \in N_v^u} \frac{d_s}{m(m-1)} - 2 - \sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i}. \tag{2}$$

(i) If $d_w \leq m$ for all $w \in N(u) \cup N(v)$, then

$$\sum_{v_r \in N_u^v} \frac{d_r}{m(m-1)} \leq 1 \text{ and } \sum_{v_s \in N_v^u} \frac{d_s}{m(m-1)} \leq 1.$$

So we have that $\alpha \leq -\sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i}$, which implies that $SDD(G \setminus e) \leq SDD(G)$.
(ii) If $d_w \geq 2m$ for all $w \in N_u^v \cup N_v^u$, then

$$\sum_{v_r \in N_u^v} \frac{d_r}{m(m-1)} \geq 2 \text{ and } \sum_{v_s \in N_v^u} \frac{d_s}{m(m-1)} \geq 2.$$

So from Equality (2) we have that

$$\alpha \geq 2 - \sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i}.$$

On the other hand

$$\sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i} \leq \frac{2m-2}{2m} \leq 1,$$

which gives that $\alpha \geq 1$, and so $SDD(G \setminus e) > SDD(G)$.
(iii) If $\sum_{v_i \in N_u^v \cup N_v^u} d_i \leq 2m(m-1)$, then we have

$$\sum_{v_r \in N_u^v} \frac{d_r}{m(m-1)} + \sum_{v_s \in N_v^u} \frac{d_s}{m(m-1)} \geq 2.$$

And Equality (2) yields that

$$SDD(G \setminus e) \leq SDD(G) - \sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i},$$

which gives the desired result. □

From Theorem 2.3 (i), we have the following corollary.

Corollary 2.4. *Let G be a regular graph, for all $e \in E(G)$. Then*

$$SDD(G \setminus e) = SDD(G) - \frac{2(m-1)}{m}.$$

Corollary 2.5. *Let G be a graph, and $e = uv \in E(G)$ with $d_u = d_v = \Delta$. Then*

$$SDD(G \setminus e) \leq SDD(G).$$

Recall that a semiregular graph, also known as a biregular graph, is a bipartite graph G in which every two vertices on the same side of the bipartition have the same degree. Specifically, if the degree of the vertices in one partition is m and the degree of the vertices in the other partition is n, we say that the graph is (m,n)-semiregular [3].

Theorem 2.6. *Let G be an (m,n)-semiregular graph, for all $e \in E(G)$. Then*

$$SDD(G \setminus e) = SDD(G) - \frac{n-1}{m} - \frac{m-1}{n}.$$

Proof. For (m,n)-semiregular graph, we have that

$$\sum_{v_r \in N_u^v} \frac{d_r}{m(m-1)} = \frac{n}{m} \quad \text{and} \quad \sum_{v_s \in N_v^u} \frac{d_s}{n(n-1)} = \frac{m}{n}.$$

So from Equality (1) we have

$$\alpha = -\sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i} = -\frac{n-1}{m} - \frac{m-1}{n}.$$

□

Theorem 2.7. *Let G be a graph and $e = uv \in E(G)$, with $d_u = d_v = m$ and the degree of all vertices connected to u and v is equal to k.*

(i) *If $k = \frac{m+\sqrt{m(5m-4)}}{2}$, then*

$$SDD(G \setminus e) = SDD(G).$$

(ii) *If $k < \frac{m+\sqrt{m(5m-4)}}{2}$, then*

$$SDD(G \setminus e) < SDD(G).$$

(iii) *If $k > \frac{m+\sqrt{m(5m-4)}}{2}$, then*

$$SDD(G \setminus e) > SDD(G).$$

Proof. Let G be a graph and $e = uv \in E(G)$, with $d_u = d_v = m$ and the degree of all vertices connected to u and v is equal to k. Consider

$$\alpha = \frac{2k}{m} - 2 - \frac{2(m-1)}{k} = \frac{2k^2 - 2mk - 2m(m-1)}{mk}.$$

(i) If $k = \frac{m+\sqrt{m(5m-4)}}{2}$, then $\alpha = \frac{2k^2-2mk-2m(m-1)}{mk} = 0$ and we have that $SDD(G \setminus e) = SDD(G)$.
(ii) If $k < \frac{m+\sqrt{m(5m-4)}}{2}$, then $\alpha = \frac{2k^2-2mk-2m(m-1)}{mk} < 0$, which yields that $SDD(G \setminus e) < SDD(G)$.
(iii) Similarly $k > \frac{m+\sqrt{m(5m-4)}}{2}$, results that $\alpha = \frac{2k^2-2mk-2m(m-1)}{mk} > 0$ and so we have that $SDD(G \setminus e) > SDD(G)$. □

Example 2.8. *In Theorem 2.7, when $m = 4$ and $k = 6$, we obtain a family of graphs where $SDD(G \setminus e) = SDD(G)$. This implies that for any graph G containing the graph depicted in Figure 2. as a subgraph (with the degree of the vertices adjacent to the two ends of edge e remaining unchanged), we have $SDD(G \setminus e) = SDD(G)$.*

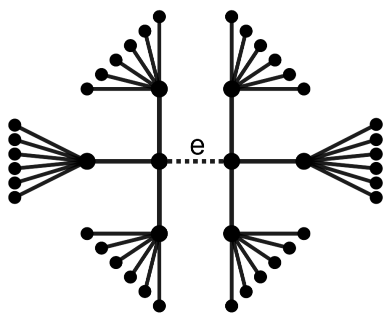

Figure 2. All graphs containing this graph, have $SDD(G \setminus \{e\}) = SDD(G)$.

Theorem 2.9. *Let G be a graph with maximum degree Δ and minimum degree δ and $e = uv \in E(G)$. Then*

$$\frac{\delta - n}{m} + \frac{\delta - m}{n} - \frac{n+m-2}{\delta} \leq SDD(G \setminus \{e\}) - SDD(G) \leq \frac{\Delta - n}{m} + \frac{\Delta - m}{n} - \frac{n+m-2}{\Delta}, \tag{3}$$

and equalities hold if and only if G is a regular graph.

Proof. It is easy to see that

$$\sum_{v_r \in N_u^v} \frac{d_r}{m(m-1)} + \sum_{v_s \in N_v^u} \frac{d_s}{n(n-1)} \geq \frac{\delta}{m} + \frac{\delta}{n},$$

and

$$\sum_{v_i \in N_u^v \cup N_v^u} \frac{1}{d_i} \leq \frac{n+m-2}{\delta}.$$

Therefore

$$\sum_{v_r\in N_u^v}\frac{d_r}{m(m-1)}+\sum_{v_s\in N_v^u}\frac{d_s}{n(n-1)}-\frac{m}{n}-\frac{n}{m}-\sum_{v_i\in N_u^v\cup N_v^u}\frac{1}{d_i}$$
$$\geq\frac{\delta}{m}+\frac{\delta}{n}-\frac{m}{n}-\frac{n}{m}-\frac{n+m-2}{\delta}.$$

Using this and Equality (1) yield the left hand side of Inequality (3).

Also

$$\sum_{v_r\in N_u^v}\frac{d_r}{m(m-1)}+\sum_{v_s\in N_v^u}\frac{d_s}{n(n-1)}\leq\frac{\Delta}{m}+\frac{\Delta}{n},$$

and

$$\sum_{v_i\in N_u^v\cup N_v^u}\frac{1}{d_i}\geq\frac{n+m-2}{\Delta}.$$

So

$$\sum_{v_r\in N_u^v}\frac{d_r}{m(m-1)}+\sum_{v_s\in N_v^u}\frac{d_s}{n(n-1)}-\frac{m}{n}-\frac{n}{m}-\sum_{v_i\in N_u^v\cup N_v^u}\frac{1}{d_i}$$
$$\leq\frac{\Delta}{m}+\frac{\Delta}{n}-\frac{m}{n}-\frac{n}{m}-\frac{n+m-2}{\Delta}.$$

So from Equality (1), we have the right hand side of Inequality (3) and the proof is complete. □

3 Conclusion

In this study, we have examined the impact of edge deletion on the SDD index of graphs. Our primary findings demonstrate that the SDD may increase, decrease, or remain unchanged following the removal of an edge, contingent upon the degree sequences of the vertices incident to the edge. We have presented necessary conditions and explicit formulas for the SDD changes in specific graph types, such as regular and semiregular graphs. Our research prompts several crucial questions for future investigation. Can explicit formulas be derived for the SDD changes in other graph families? Can we establish necessary and sufficient conditions for the SDD to exhibit increases, decreases, or remain unchanged after the deletion of an edge in more general classes of graphs? Furthermore, can we explore the impact of other graph operations, such as edge addition and vertex deletion, on the SDD? In summary, our study contributes significantly to the field of graph theory by advancing our comprehension of the SDD and its responsiveness to alterations in graph structure. We anticipate that our findings will stimulate further research into the behavior of graph invariants under diverse graph operations, fostering new insights and advancements in graph theory.

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