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Research Paper

Tetravalent one-regular graphs of order p^2q^2

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Abstract. A graph is called one-regular if its full automorphism group acts regularly on the set of arcs. In this paper, we classify all connected one-regular graphs of valency 4 of order p^2q^2, where $p > q$ are prime numbers. We also prove that all such graphs are Cayley graphs.

Keywords: one-regular graph, symmetric graph, Cayley graph
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1 Introduction

Gardiner and Praeger in 1994 constructed 4-valent one-regular graphs of prime order, see [6]. Let p and q be two primes. Every tetravalent one-regular graph of order p or pq or p^2 is a circulant graph and all of them have been classified in [15]. Furthermore, in [5, 17] the authors classified tetravalent one-regular graphs of order $2pq$ and $4p^2$. Here, we study the tetravalent one-regular graphs of order p^2q^2 and show all of them have Cayley structure. We prove that in such Cayley graphs either the p-Sylow subgroup of G is cyclic and then the regarded group is abelian or q-Sylow subgroup of G is cyclic. The presentation of a group of order p^2q^2 can be found in [12]. All graphs in this paper are undirected, finite, and connected without loops or multiple edges. For a graph Γ, we use $V(\Gamma)$, $E(\Gamma)$ and $Aut(\Gamma)$ to denote its vertex set, edge set and its full automorphism group, respectively. A graph Γ is said to be vertex-transitive if $Aut(\Gamma)$ acts transitively on $V(\Gamma)$. For a positive integer s, an s-arc of

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Γ is an $(s+1)$-tuple $(v_0, v_1, \cdots, v_s)$ of vertices such that $\{v_{i-1}, v_i\} \in E(\Gamma)$ for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is called an arc for short and a 0-arc is a vertex. If $X \leqslant Aut(\Gamma)$ and X is transitive on s-arcs of Γ, then Γ is called a (X,s)-arc transitive graph. In addition, if X is not transitive on the set of $(s+1)$-arcs of Γ, then Γ is called a (X,s)-transitive graph. If $X = Aut(\Gamma)$, then (X,s)-arc transitive and (X,s)-transitive graphs are called s-arc transitive graphs and s-transitive graphs, respectively. A $(X,1)$-arc transitive graph is called symmetric. A graph is said to be one-regular if its automorphism group acts regularly on the set of its arcs.

Let G be a permutation group on Ω and $\alpha \in \Omega$. The stabilizer G_α is the subgroup of G fixing the point α. The group G is called semi-regular on Ω if $G_\alpha = 1$, for every $\alpha \in \Omega$ and regular if G is transitive and semi-regular. Let G be a finite group and S be a symmeric subset of G $(1 \notin S = S^{-1} = \{g^{-1} | g \in S\})$. The Cayley graph $\Gamma = Cay(G,S)$ has vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{\{g, sg\} \mid g \in G, s \in S\}$. For every element $g \in G$, the map ρ_g given by $x \mapsto xg$, $x \in G$, is a permutation on G and the set of all such permutations is called the right regular representation of G denoted by $\mathcal{R}(G)$. One can see that $\mathcal{R}(G)$ is a regular subgroup of $Aut(\Gamma)$ isomorphic with G. It is a well-known fact that Γ is connected if and only if $G = \langle S \rangle$, that is, S generates G. In general, it is a very difficult task to find the full automorphism group of a graph. Although, we know that a Cayley graph is vertex-transitive, in general it is difficult to determine whether it is edge-transitive or arc-transitive. Suppose that $Aut(G,S) = \{\alpha \in Aut(G), \alpha(S) = S\}$. Obviously, $\mathcal{R}(G) \rtimes Aut(G,S) \leq Aut(\Gamma)$. Let $A = Aut(\Gamma)$, according to [16], we have $N_A(\mathcal{R}(G)) = \mathcal{R}(G) \rtimes Aut(G,S)$. The Cayley graph $\Gamma = Cay(G,S)$ is said to be normal if the right regular representation $\mathcal{R}(G)$ of G is normal in $Aut(\Gamma)$ and in this case, $\mathcal{R}(G) \trianglelefteq Aut(\Gamma)$ or equivalently $Aut(\Gamma) = \mathcal{R}(G) \rtimes Aut(G,S)$. The Cayley graph $\Gamma = Cay(G,S)$ is said to be normal symmetric if $N_A(\mathcal{R}(G))$ acts transitively on the set of arcs.

2 Main results

In this section, we determine all tetravalent one-regular Cayley graphs of order p^2q^2. If $q = 2$, then all tetravalent one-regular graphs of order $4p^2$ have been determined in [5] and we can conclude the following result.

Theorem 2.1. *Let $p \neq 2$ be a prime and $\Gamma = Cay(G,S)$ be tetravalent symmetric Cayley graph on groups of order $4p^2$. Then Γ is 1-transitive. Moreover, if Γ is also one-regular, then Γ is a normal Cayley graph.*

Lemma 2.2. *[14] Every transitive abelian group is regular.*

Lemma 2.3. *[14] Suppose G is a permutation group on Ω and P is a p-Sylow subgroup of G, where p is a prime. Let $w \in \Omega$, if p^m divides the length of the G-orbit containing ω. Then p^m also divides the length of the P-orbit containing w.*

For a finite group G, the product of all nilpotent normal subgroups of G is called the Fitting subgroup of G denoted by $Fit(G)$.

Theorem 2.4. *[13] If G is solvable group, then $Fit(G) \neq 1$ and $C_G(Fit(G)) \leq Fit(G)$.*

We recall that $O_p(G)$ is the unique largest normal p-subgroup of the finite group G, where p is a prime number and it can be found by taking the intersection of all of the p-Sylow subgroups of G. If a p-Sylow subgroup of a finite group G has a normal p-complement, then G is called p-nilpotent. The set of all p-Sylow subgroups of G is denoted by $Syl_p(G)$.

Theorem 2.5. *[4] Let G be a group acting transitively on a set Ω and $H \lhd G$. Then the group H has at most $|G : H|$ orbits and if the index $|G : H|$ is finite, then the number of orbits of H divides $|G : H|$.*

Theorem 2.6. *(i) [2] A graph Γ is isomorphic to a Cayley graph on a group G if and only if its automorphism group has a subgroup isomorphic to G acting regularly on the vertex set of Γ.*

(ii) [2] A circulant graph is vertex-transitive. A vertex-transitive graph with a prime number of vertices must be a circulant graph.

(iii) [15] Every tetravalent one-regular graph of order p^2 is a circulant graph.

Suppose Γ is a symmetric graph and consider the transitive subgroup X of $Aut(\Gamma)$. Let N be a normal subgroup of X. Then the quotient graph Γ_N is the graph with orbits of N as its vertices and two vertices are adjacent if there is an edge between these two orbits in Γ. If further the valency of Γ_N equals the valency of Γ, then Γ is called a regular cover of Γ_N.

Theorem 2.7. *[6] Let Γ be symmetric graph of valency 4 and $X \leq Aut(\Gamma)$ be arc-transitive. If $N \trianglelefteq X$, then one of the following cases holds,*

1. *N is transitive on $V(\Gamma)$;*
2. *Γ is bipartite and N acts transitively on each part of the bipartition;*
3. *N has $r \geq 3$ orbits on $V(\Gamma)$, the quotient graph Γ_N is a cycle of length r, and X induces the full automorphism group D_{2r} of Γ_N;*
4. *N has $r \geq 5$ orbits on $V(\Gamma)$, N acts semi-regularly on $V(\Gamma)$, the quotient graph Γ_N is a tetravalent connected X/N-symmetric graph and Γ is a regular cover of Γ_N.*

Theorem 2.8. *Let Γ be a one-regular tetravalent graph of an odd order m. Assume $A = Aut(\Gamma)$. Then the following cases holds,*

1. *If A has a subgroup of order m, then Γ is a Cayley graph;*
2. *If $A_v \cong C_4$, then Γ is a normal Cayley graph;*
3. *If $A_v \cong C_2 \times C_2$ and $3 \nmid m$, then Γ is a normal Cayley graph.*

Proof. (1) Let A has a subgroup G of order m. By the orbit-stabilizer theorem for the vertex v, we have $|orb_G(v)| = |G : G_v|$. On the other hand, A acts regularly on the arc set of Γ, hence $|A_v| = 4$. But G_v devides m and so the fact that m is odd implies $G_v \cong \langle 1 \rangle$. Hence, G acts regularly on the vertex set of Γ. Applying Theorem 2.6 yields Γ is a Cayley graph.
(2) By [11, Theorem 7.51], A has a normal subgroup of order m and by the Case 1, Γ is a normal Cayley graph.
(3) Suppose $H \cong C_2 \times C_2 \cong A_v$ is a 2-Sylow subgroup of A. It is not difficult to see that $|Aut(H)| = (2^2 - 1)(2^2 - 2) = 6$ and there is an embedding $N_A(H)/C_A(H) \hookrightarrow Aut(H)$. Obviously, $|N_A(H)/C_A(H)|$ divides $|A|$. Since H is abelian, $H < C_A(H)$ and then $2 \nmid |N_A(H)/C_A(H)|$. On the other hand, $(3, |A|) = 1$, and thus $N_A(H) = C_A(H)$. Hence, by Burnside's Theorem [10, Theorem 6.17] H has a normal complement in A. This means that A has a normal subgroup of order m and by Case 1, Γ is a normal Cayley graph. □

Theorem 2.9. *Let Γ be a one-regular tetravalent graph of order p^2q^2, where $p > q \neq 2$ are prime. Assume $A = Aut(\Gamma)$, then the following cases hold,*

(1) *If A has a subgroup of order p^2q^2, then Γ is a Cayley graph;*

(2) *If $A_v \cong C_4$, then Γ is a normal Cayley graph;*

(3) *If $A_v \cong C_2 \times C_2$ and $q \neq 3$, then Γ is a normal Cayley graph;*

(4) *If $A_v \cong C_2 \times C_2$, $q = 3$, then Γ is a Cayley graph.*

Proof. The Cases 1-3 have been discussed in Theorem 2.8. For the Case 4, let P be a p-Sylow subgroup of A. Then $n_p = 1 + kp \mid 36$, and if $p \neq 5,\ 11,\ 17$, we can conclude that $n_p = 1$; therefore, $P \lhd A$. Now, let Q be a q-Sylow subgroup of A. Hence, $PQ \leq A$ and $|PQ| = p^2q^2$ and the proof is similar to Case 1. For $p = 11$ or $p = 17$, by [9, Theorem 1.37] and [9, Corollaries 1.39, 1.40], we have $|O_p| = p$ *or* p^2. If $|O_p| = p^2$, then Γ is a Cayley graph. If $|O_p| = p$, then the embedding $N_A(O_p)/C_A(O_p) \cong A/C_A(O_p) \hookrightarrow Aut(O_p) \cong C_{p-1}$ yields that $|C_A(O_p)| \geq p^2q^2$ and $C_A(O_p)$ has a subgroup of order p^2q^2. This means that Γ is a Cayley graph. For $p = 5$, let $PQ \nleq A$. Then P and Q are not normal subgroups of A. By using a Gap program, there is only one group of the order $9 \times 25 \times 4 = 900$ with the above conditions which is isomorphic with $A \cong A_5 \times C_{15}$, a contradiction. So, all one-regular graphs of order 225 have Cayley structures. □

Theorem 2.10. *Let G be a finite group of order p^2q^2, where $p > q \neq 2$ are prime numbers and $\Gamma = Cay(G, S)$ be a Cayley graph of valency 4. If Γ is an $(X, 1)$-arc transitive, where $G \leq X \leq Aut(\Gamma)$, then one of the following cases holds:*

1. *G is normal in X, $X_1 \leq D_8$ and $|X_1| \geq 4$;*
2. *There is a subgroup $P < X$ such that $P \lhd G$ and Γ is a cover of Γ_P;*
3. *X has a unique minimal normal subgroup $N \cong C_p^2$ such that*

(a) $G = N \rtimes R \cong C_p^2 \rtimes C_9$;

(b) $X = N \rtimes ((H \rtimes R).C_2) \cong C_p^2 \rtimes ((C_2^2 \rtimes C_9).C_2)$ *and* $X_1 = H.C_2 \cong (C_2 \times C_2).C_2$;

(c) $NH \cong D_{2p} \times D_{2p}$;

(d) $H \rtimes R = C_2^2 \rtimes C_9 = \langle a, b, c \mid a^2 = b^2 = c^9 = 1, ab = ba, c^{-1}ac = ab, c^{-1}bc = a \rangle$;

(e) $X/(NH) \cong D_{18}$.

Proof. By [8, Theorem 1.1] the proof is straightforward. □

Theorem 2.11. *Let* G *be a finite group of order* p^2q^2*, where* $p > q \neq 2$ *are prime numbers and* $\Gamma = Cay(G, S)$ *be a Cayley graph of valency* 4.

(i) *If* Γ *is one-regular,* $A = Aut(\Gamma)$ *and* $P \in Syl_p(G)$*, then the following cases hold,*

(a) $A \cong GA_1$ *and* $A_1 \cong C_4$ *or* $C_2 \times C_2$,

(b) $P \lhd A$ *and* A *is solvable,*

(c) *If* $A_1 \cong C_4$ *then* $G \lhd A$,

(d) *If* $A_1 \cong C_2 \times C_2$ *and* $q \neq 3$ *then* $G \lhd A$.

(ii) *If* Γ *is one-regular,* $A = Aut(\Gamma)$ *and* $P \cong C_{p^2}$ *is a p-Sylow subgroup of* G*. Then* G *is an abelian group.*

Proof. (i) Since Γ is a Cayley graph, the proof of part (a) is clear. For the next one, we know that $P \lhd G$, hence $G \subseteq N_A(P)$ and $|A : N_A(G)| = 1 + kp \mid 4$, $(p > 3)$ which implies that $N_A(P) = A$. It is clear P and A/P are solvable, hence A is solvable and the proof of part (c) is a result of [11, Theorem 7.51]. There is a similar proof for the part (d), as we have done in the Case (3) of Theorem 2.8.

(ii) We know that $P \cong C_{p^2} \leq Fit(A)$, hence $Fit(A) \neq \langle 1 \rangle$. We prove that $Fit(A) = G$ which yields that G is abelian. Suppose $Fit(A) \neq G$, then only one of the following possibilities holds:
$Fit(A) = C_{p^2}$, $Fit(A) = C_{qp^2}$, $Fit(A) = C_{2qp^2}$, $Fit(A) = C_{2p^2}$, $|Fit(A)| = 4p^2$, $|Fit(A)| = 4qp^2$. We prove that all of them are impossible. By Theorem 2.4,

$$N_A(Fit(A))/C_A(Fit(A)) = A/Fit(A).$$

Hence if $Fit(A) = C_{p^2}$, then

$$N_A(Fit(A))/C_A(Fit(A)) \cong A/Fit(A) \to Aut(Fit(A)) \cong C_{p(p-1)}.$$

Therefore, $A/Fit(G)$ is abelian. On the other hand, Γ is a Cayley graph, hence two Cases 1, 2 in Theorem 2.7 for $N = Fit(G)$ are impossible. Let Γ_N be the quotient graph of Γ relative to the orbits of N and K be the kernel of A acting on $V(\Gamma_N)$. By Lemma 2.3, the orbits of N are of length p^2. Thus $|V(\Gamma_N)| = q^2$, $N \leq K$, and A/K acts transitively on arcs of Γ_N. For the Case 3 in Theorem 2.7, we have Γ_N is a cycle of length q^2 and hence $A/K \cong D_{2q^2}$, which yields

$|K| = 2p^2$. Since A/K is a subgroup of A/P, it follows that A/P is a non-abelian group, a contradiction. For the Case 4 of Theorem 2.7, Γ_N is A/N-symmetric graph, hence A/N is transitive on the vertices of Γ_N and also is abelian. Therefore, by Lemma 2.2, A/N acts regularly on the vertices of Γ_N, a contradiction. Therefore, $Fit(G) \ncong C_{p^2}$. Similarly, the other cases are impossible. Suppose $|Fit(A)| = 4p^2$ *or* $4qp^2$. Since $N \leq K$, where K is the kernel of A acting on $V(\Gamma_N)$. Γ_N is a symmetric graph of valency 2 or 4 and by Theorem 2.7, A/K acts transitively on arcs of Γ_N. Then $2 \mid |A/K|$, which is clearly impossible, because $|A| = 4p^2q^2$. Therefore, $|Fit(A)| = p^2q^2$ and so G is an abelian group. □

Theorem 2.12. *Let G be a finite group of order p^2q^2, where $p > q \neq 2$ are prime numbers, and let $\Gamma = Cay(G,S)$ be a connected Cayley graph of valency 4. Assume Γ is one-regular, $A = Aut(\Gamma)$ and $P \cong C_p \times C_p \in Syl_p(G)$. Then $G \cong (C_p \times C_p) \rtimes C_{q^2}$.*

Proof. Since Γ is a Cayley graph, two Cases 1, 2 in Theorem 2.7 for $N \cong C_p \times C_p \cong P$, are impossible. By Theorem 2.5, the number of orbits of N on G are q^2. Let Γ_N be the quotient graph of Γ relative to the orbits of N and K be the kernel of A acting on $V(\Gamma_N)$. Thus $|V(\Gamma_N)| = q^2$, $N \leq K$ and A/K acts transitively on the arcs of Γ_N. For the Case 3 in Theorem 2.7, Γ_N is a cycle of length q^2 and hence $A/K \cong D_{2q^2}$, which yields that $|K| = 2p^2$. Since $C_{q^2} \leq A/K$, and A/K is a subgroup of A/P, it follows that the q-Sylow subgroup of A (and G) is cyclic. Now, for the Case 4, let Γ_P be the quotient graph of Γ relative to the orbits of P. By Lemma 2.3, the orbits of N are of length p^2. Thus $|V(\Gamma_P)| = q^2$ and A/P acts transitively on the arcs of Γ_P. Now, by Theorems 2.6(ii) and 2.6(iii), Γ_P is a circulant graph and so it is a Cayley graph on an abelian group. Hence the q-Sylow subgroup of A (and G) is cyclic; therefore, $G \cong (C_p \times C_p) \rtimes C_{q^2}$. □

3 Tetravalent normal symmetric Cayley graphs on group of order p^2q^2

Let G be a group of order $p^2q^2 (p > q)$ with generating set $S = \{a,b,a^{-1},b^{-1}\}$. Suppose $\Gamma = Cay(G,S)$ is a Cayley graph, then an automorphism of $Aut(G,S)$ satisfies in one of the following rules:

$$\alpha : \begin{cases} a \mapsto b^{-1} \\ b \mapsto a \end{cases}, \alpha^2 : \begin{cases} a \mapsto a^{-1} \\ b \mapsto b^{-1} \end{cases}, \alpha^3 : \begin{cases} a \mapsto b \\ b \mapsto a^{-1} \end{cases}, \beta : \begin{cases} a \mapsto b \\ b \mapsto a \end{cases},$$

$$\alpha \circ \beta : \begin{cases} a \mapsto a \\ b \mapsto b^{-1} \end{cases}, \alpha^2 \circ \beta : \begin{cases} a \mapsto b^{-1} \\ b \mapsto a^{-1} \end{cases}, \alpha^3 \circ \beta : \begin{cases} a \mapsto a^{-1} \\ b \mapsto b \end{cases}, i : \begin{cases} a \mapsto a \\ b \mapsto b \end{cases}.$$

It is not difficult to see that $\alpha^4 = \beta^2 = i$, $\beta^{-1} \circ \alpha \circ \beta = \alpha^3$ and so $\langle \alpha, \beta \rangle \cong D_8$. In other words, we can conclude the following theorem.

Theorem 3.1. *Let G be a group of order p^2q^2 with the symmetric generating subset $S = \{a,b,a^{-1},b^{-1}\}$. Then $Aut(G,S) \leq \langle \alpha, \beta \rangle \cong D_8$.*

Theorem 3.2. *Let $\Gamma = Cay(G,S)$ be a normal symmetric Cayley graph of order p^2q^2, where $p > q \neq 2$ are primes and $S = \{a,a^{-1},b,b^{-1}\}, (a \neq b)$. Then $o(a) \neq p, p^2, q^2$.*

Proof. Suppose $\Gamma = Cay(G,S)$ is a normal symmetric Cayley graph of order $p^2q^2 (p > q)$ where $G = \langle a,b\rangle$ and $S = \{a,a^{-1},b,b^{-1}\},(a \neq b)$. It is a well-known fact that $Aut(G,S)$ is a 2-group. Since $|S| = 4$, we conclude that $|Aut(G,S)| = 2$ *or* 4 *or* 8. On the other hand, Γ is normal symmetric which yields C_4 or $C_2 \times C_2$ is a subgroup of $Aut(G,S)$. First, suppose that $C_4 \cong \langle\alpha\rangle \leq Aut(G,S)$ and necessarily $o(a) = o(b)$. Since $|G| = p^2q^2$, one of the following cases holds:

Case 1. $o(a) = o(b) = p$. Suppose $H = \langle a\rangle$ and $K = \langle b\rangle$, then $H \leq P$ and $K \leq P(P \in Syl_p(G)$ is normal) which implies that $\langle H \cup K\rangle \subseteq P$. This yields $G = \langle a,b\rangle \subseteq P$, a contradiction.

Case 2. $o(a) = o(b) = p^2$ and suppose $H = \langle a\rangle$ and $K = \langle b\rangle$, then $H = P$, $K = P$ and thus $\langle H \cup K\rangle = P = G$, a contradiction.

Case 3. $o(a) = o(b) = q^2$, put $H = \langle a\rangle$ and $K = \langle b\rangle$, then $H,K \in Syl_q(G)$ and then there exists $x \in G$ such that $H = K^x$. Now, according to [12], we have the following subcases:

Subcase 1. $G \cong C_{q^2} \ltimes_\varphi C_{p^2} \cong \langle c,d \mid c^{q^2} = d^{p^2} = 1, c^{-1}dc = d^r\rangle$, which yields (without loss of generality) $a = c,\ b = c^{d^i} = d^{-i}cd^i$. It implies that $\alpha(b) = \alpha(c^{d^i}) = \alpha(c)^{\alpha(d^i)} = a$. Hence $(c^{-1})^{d^i\alpha(d^i)} = (c^{-1})^{d^j} = c$, a contradiction.

Subcase 2.

$$\begin{aligned} G &\cong C_{q^2} \ltimes_\varphi (C_p \times C_p) \\ &\cong \langle c,d,e \mid c^{q^2} = d^p = e^p = 1, c^{-1}dc = d^\lambda, c^{-1}ec = e^\lambda, de = ed\rangle, \end{aligned}$$

which yields $a = c,\ b = c^{d^ie^j} = d^{-i}e^{-j}cd^ie^j$. Hence $\alpha(b) = \alpha(c^{d^ie^j}) = \alpha(c)^{\alpha(d^ie^j)} = a$ and so $(c^{-1})^{d^ie^j\alpha(d^ie^j)} = (c^{-1})^{d^ne^m} = c$, a contradiction.

Subcase 3.

$$\begin{aligned} G &\cong C_{q^2} \ltimes_\varphi (C_p \times C_p) \\ &\cong \langle c,d,e \mid c^{q^2} = d^p = e^p = 1, c^{-1}dc = d, c^{-1}ec = e^\lambda, de = ed\rangle, \end{aligned}$$

which implies that $a = c,\ b = c^{e^j} = e^{-j}ce^j$. In other words, $\alpha(b) = \alpha(c^{e^j}) = \alpha(c)^{\alpha(e^j)} = a$. Hence $(c^{-1})^{e^j\alpha(e^j)} = (c^{-1})^{e^m} = c$, a contradiction.

Subcase 4.

$$\begin{aligned} G &\cong C_{q^2} \ltimes_\varphi (C_p \times C_p) \\ &\cong \langle c,d,e \mid c^{q^2} = d^p = e^p = 1, c^{-1}dc = d^\lambda, c^{-1}ec = e^{\lambda^l}, de = ed\rangle \end{aligned}$$

and we can verify that $a = c,\ b = c^{d^ie^j} = d^{-i}e^{-j}cd^ie^j$. Similarly, we have $\alpha(b) = \alpha(c^{d^ie^j}) = \alpha(c)^{\alpha(d^ie^j)} = a$ and so $(c^{-1})^{d^ie^j\alpha(d^ie^j)} = (c^{-1})^{d^ne^m} = c$, a contradiction.

Subcase 5. $G \cong C_{q^2} \ltimes_\varphi (C_p \times C_p)$ $\cong \langle c,d,e \mid c^{q^2} = d^p = e^p = 1, c^{-1}dc = d^\lambda e^{\gamma N}, c^{-1}ec = d^\gamma e^\lambda, de = ed, \lambda^2 - \gamma^2N \neq 0, N \neq n^2, \lambda + \gamma\sqrt{N} \neq 1\rangle$. Again, we can verify that $a = c,\ b = c^{d^ie^j} = d^{-i}e^{-j}cd^ie^j$ and thus $\alpha(b) = \alpha(c^{d^ie^j}) = \alpha(c)^{\alpha(d^ie^j)} = a$. Consequently, $(c^{-1})^{d^ie^j\alpha(d^ie^j)} = (c^{-1})^{d^ne^m} = c$, a contradiction.

Now, suppose $C_2 \times C_2 \cong \langle \alpha^2, \beta \rangle \subseteq Aut(G,S)$, then $Aut(G,S)$ acts transitivily on S. Hence, in this case, the Cayley graph Γ is normal symmetric. Again, we can consider the following cases:
Case 1. $o(a) = o(b) = p$.
Case 2. $o(a) = o(b) = p^2$. For both of them the proof is similar to that of in Subcase 4.
Case 3. $o(a) = o(b) = q^2$, put $H = \langle a \rangle$ and $K = \langle b \rangle$ then $H, K \in Syl_q(G)$ and hence $H = K^x$ for some $x \in G$. Now, according to [12], we have the following subcases:
Subcase 1. $G \cong C_{q^2} \ltimes_\varphi C_{p^2} = \langle c, d \mid c^{q^2} = d^{p^2} = 1, c^{-1}dc = d^r \rangle$, where $a = c$, $b = c^{d^i} = d^{-i}cd^i$. It implies that $\beta(d) = d^{-1}$, $\alpha^2 o \beta(d) = d^{-1}$, $\alpha^2(d) = d$. Hence $\alpha^2(c^{-1}dc) = \alpha^2(d^r)$, so $c^2 d = dc^2$, a contradiction.
Subcase 2. $G \cong C_{q^2} \ltimes_\varphi (C_p \times C_p)$
$= \langle c, d, e \mid c^{q^2} = d^p = e^p = 1, c^{-1}dc = d^\lambda, c^{-1}ec = e^\lambda, de = ed, \lambda^q = 1 \rangle$. Hence $Z(G) = \langle c^q \rangle \cong C_q$, where $a = c$, $b = c^{d^i e^j} = d^{-i}e^{-j}cd^i e^j$. This implies that $\beta(d^i e^j) = (d^i e^j)^{-1}$, $\alpha^2 o \beta(d^i e^j) = (d^i e^j)^{-1}$, $\alpha^2(d^i e^j) = d^i e^j$. Hence $\alpha^2(c^{-1}d^i e^j c) = \alpha^2((d^i e^j)^\lambda)$, so $c^2(d^i e^j) = (d^i e^j)c^2$ and $a^2 = b^2$, a contradiction.
Subcase 3. $G \cong C_{q^2} \ltimes_\varphi (C_p \times C_p)$
$= \langle c, d, e \mid c^{q^2} = d^p = e^p = 1, c^{-1}dc = d, c^{-1}ec = e^\lambda, de = ed, \lambda^q = 1 \rangle$. Hence $Z(G) = \langle c^q, d \rangle \cong C_{pq}$, where $a = c$, $b = c^{e^j} = e^{-j}ce^j$. In other words, $\beta(e^j) = (e^j)^{-1}$, $\alpha^2 o \beta(e^j) = (e^j)^{-1}$, $\alpha^2(e^j) = e^j$. Hence $\alpha^2(c^{-1}e^j c) = \alpha^2((e^j)^\lambda)$, so $c^2(e^j) = (e^j)c^2$ and $a^2 = b^2$, a contradiction.
Subcase 4. $G \cong C_{q^2} \ltimes_\varphi (C_p \times C_p)$
$= \langle c, d, e \mid c^{q^2} = d^p = e^p = 1, c^{-1}dc = d^\lambda, c^{-1}ec = e^{\lambda^t}, de = ed, \lambda^q = 1 \rangle$. Hence $Z(G) = \langle c^q \rangle \cong C_q$, where $a = c$, $b = c^{d^i e^j} = d^{-i}e^{-j}cd^i e^j$. Consequently, $\beta(d^i e^j) = (d^i e^j)^{-1}$, $(\alpha^2 \circ \beta)(d^i e^j) = (d^i e^j)^{-1}$, $\alpha^2(d^i e^j) = d^i e^j$. Hence $\alpha^2(c^{-1}d^i e^j c) = \alpha^2((d^i)^\lambda (e^j)^{\lambda^t})$, $(\alpha^2 \circ \beta)(c^{-1}d^i e^j c) = (\alpha^2 \circ \beta)((d^i)^\lambda (e^j)^{\lambda^t})$, $\beta(c^{-1}d^i e^j c) = \beta((d^i)^\lambda (e^j)^{\lambda^t})$, so $\beta(d) = (\alpha^2 \circ \beta)(d) = d^{-1}$, $\alpha^2(d) = d$, $\beta(e) = (\alpha^2 \circ \beta)(e) = e^{-1}$, $\alpha^2(e) = e$, so $c^2 d = dc^2$ and $c^2 e = ec^2$, a contradiction.
Subcase 5. $G \cong C_{q^2} \ltimes_\varphi (C_p \times C_p)$
$= \langle c, d, e \mid c^{q^2} = d^p = e^p = 1, c^{-1}dc = d^\lambda e^{\gamma N}, c^{-1}ec = d^\gamma e^\lambda, de = ed, \lambda^2 - \gamma^2 N \neq 0, N \neq n^2, \lambda + \gamma\sqrt{N} \neq 1 \rangle$, where $a = c, b = c^{d^i e^j} = d^{-i}e^{-j}cd^i e^j$ and so $\beta(d^i e^j) = (d^i e^j)^{-1}$, $(\alpha^2 \circ \beta)(d^i e^j) = (d^i e^j)^{-1}$, $\alpha^2(d^i e^j) = d^i e^j$. Thus $\alpha^2(c^{-1}d^i e^j c) = \alpha^2(d^{i\lambda + j\gamma} e^{j\lambda + i\gamma N})$, $(\alpha^2 \circ \beta)(c^{-1}d^i e^j c) = (\alpha^2 \circ \beta)(d^{i\lambda + j\gamma} e^{j\lambda + i\gamma N})$, $\beta(c^{-1}d^i e^j c) = \beta(d^{i\lambda + j\gamma} e^{j\lambda + i\gamma N})$, so $\beta(d) = (\alpha^2 \circ \beta)(d) = d^{-1}$, $\alpha^2(d) = d$, $\beta(e) = (\alpha^2 \circ \beta)(e) = e^{-1}$, $\alpha^2(e) = e$, so $c^2 d = dc^2$ and $c^2 e = ec^2$, a contradiction. □

3.1 symmetric Cayley graphs on abelian groups of order p^2q^2

Here, we determine the full automorphism group of symmetric tetravalent Cayley graphs $Cay(G,S)$, where G is an abelian group of order a square product of two primes. To do this, first notice that there are only four abelian groups of order p^2q^2. In the case that $q = 2$, in [7] all tetravalent symmetric graphs of order $4p^2$ have been determined. In the following, we determine the automorphism group for each graph. Here, in this section, α, β are as given in Theorem 3.1. For solving all congruence equations, we applied [3, Theorem 9.13].

Theorem 3.3. *Let G be an abelian group of order p^2q^2, where $p > q \neq 2$ are primes with the symmetric* *generating subset* $S = \{a, b, a^{-1}, b^{-1}\}$ *and* $\Gamma = Cay(G,S)$ *be a symmetric Cayley graph. Then the following cases holds,*

1. $o(a) \neq p, p^2, q, q^2$,
2. *If* $o(a) = pq$, *then* $G \cong C_{pq} \times C_{pq}$ *and* $Aut(\Gamma) \cong (C_{pq} \times C_{pq}) \rtimes D_8$,
3. *If* $o(a) = p^2q$, *then* $G \cong C_{p^2} \times C_q \times C_q$ *and* $|Aut(G,S)| = 4$,
4. *If* $o(a) = pq^2$, *then* $G \cong C_{q^2} \times C_p \times C_p$ *and* $|Aut(G,S)| = 4$,
5. *If* $o(a) = p^2q^2$, *then* $G \cong C_{p^2q^2}$ *and* $|Aut(G,S)| = 4$.

Proof. By [1, Theorem 1.2], we have $Aut(\Gamma) \cong G \rtimes Aut(G,S)$ and G is an abelian group, so the proof of part 1 is clear. For the second one, we know that $G = \langle a,b \rangle = \langle a \rangle . \langle b \rangle = \langle a \rangle \times \langle b \rangle \cong C_{pq} \times C_{pq}$, then it is not difficult to see that $Aut(G,S) = \langle \alpha, \beta \rangle \cong D_8$. Hence Γ is not an one-regular Cayley graph and $Aut(\Gamma) \cong (C_{pq} \times C_{pq}) \rtimes D_8$.

For the part 3, let $o(a) = o(b) = p^2q$, $H = \langle a \rangle$, and $K = \langle b \rangle$. Then $G = \langle a,b \rangle = \langle a \rangle . \langle b \rangle = HK$ and $\mid G \mid = \mid HK \mid = \frac{|H||K|}{|H \cap K|} = p^2q^2$. Since $a \neq b$, we conclude that $|H \cap K| = p^2$. Suppose that $a = xz, b = yz^i$, where $(i, p^2) = 1$. Hence $G \cong C_q \times C_q \times C_{p^2} \cong \langle x,y,z \mid x^q = y^q = z^{p^2} = 1, xy = yx, xz = zx, yz = zy \rangle = \langle a,b \mid a = xz, b = yz^i, (i,p^2) = 1 \rangle$. Now, by a same discussion in the proof of Theorem 3.2, two following cases hold:

Case 1. Suppose $\langle \alpha \rangle \leq Aut(G,S)$, since $Aut(G) \cong C_{p(p-1)} \times GL(2,q)$, we have $\alpha(a) = b^{-1}$ and $\alpha(b) = a$. This means that $\alpha(xz) = y^{-1}z^{-i}$, $\alpha(yz^i) = xz$, $\alpha(z) = z^{-i}$, $\alpha(z^i) = z$, $\alpha(x) = y^{-1}$ and $\alpha(y) = x$. Consequently, $z^{i^2+1} = 1$ and so $1 + i^2 \equiv 0 \ (mod\ p^2)$ or $p = 4k+1$. Finally, if $o(a) = o(b) = p^2q$, the Cayley graph Γ is symmetric if and only if $a^{iq} = b^q$, $1 + i^2 \equiv 0 (mod\ p^2)$ and $p = 4k+1$. Clearly, $Aut(G,S) \cong C_4$. Since $\beta(a) = b$ and $\beta(b) = a$; it means that $\beta(xz) = yz^i$ and $\beta(yz^i) = xz$. We conclude that $z^{i^2} = z$ and so $i^2 - 1 \equiv 0 (mod\ p^2)$, $(i^2 + 1 \equiv 0 (mod\ p^2))$. Consequently, p^2 divides 2, a contradiction. This means that $\beta \notin Aut(G,S)$ and Γ is one-regular Cayley graph. Hence $Aut(\Gamma) \cong (C_q \times C_q \times C_{p^2}) \rtimes C_4$.

Case 2. Suppose that $\langle \alpha^2, \beta \rangle \leq Aut(G,S)$. In this case, $i^2 \equiv 1 (mod\ p^2)$ and it is not difficult to see that $Aut(G,S) = \langle \alpha^2, \beta \rangle \cong C_2 \times C_2$. Hence $Aut(\Gamma) \cong (C_q \times C_q \times C_{p^2}) \rtimes (C_2 \times C_2)$ and Γ is one-regular graph.

For the part 4, let $o(a) = o(b) = pq^2$, $H = \langle a \rangle$ and $K = \langle b \rangle$. Then $G = \langle a,b \rangle = \langle a \rangle . \langle b \rangle = HK$ and $\mid G \mid = \mid HK \mid = \frac{|H||K|}{|H \cap K|} = p^2q^2$. Since $a \neq b$, we conclude that $H \cap K = q^2$. Suppose that $a = xz$, $b = yz^i$, where $(i, q^2) = 1$. Hence $G \cong C_p \times C_p \times C_{q^2} \cong \langle x,y,z \mid x^p = y^p = z^{q^2} = 1, xy = yx, xz = zx, yz = zy \rangle = \langle a,b \mid a = xz, b = yz^i, (i,q^2) = 1 \rangle$. Again, we consider two cases:

Case 1. Suppose $\langle \alpha \rangle \leq Aut(G,S)$. According to the structure of $Aut(G) \cong C_{q(q-1)} \times GL(2,p)$, we have $\alpha(a) = b^{-1}$ and $\alpha(b) = a$. This means that $\alpha(xz) = y^{-1}z^{-i}$ and $\alpha(yz^i) = xz$. Consequently, $\alpha(z) = z^{-i}$, $\alpha(z^i) = z$, $\alpha(x) = y^{-1}$ and $\alpha(y) = x$. Hence $z^{i^2+1} = 1$ and thus $1 + i^2 \equiv 0 (mod\ q^2)$. Therefore, according to [12, Theorem 3] we have $q = 4k+1$. Finally, if $o(a) = o(b) = pq^2$, the Cayley graph $Cay(G,S)$ is tetravalent normal symmetric if and only if $a^{ip} =$

b^p, $1+i^2 \equiv 0(mod\ q^2)$, where $q = 4k+1$. It is not difficult to prove that $Aut(G,S) \cong C_4$, since $\beta(a) = b$ and $\beta(b) = a$. Consequently, $\beta(xz) = yz^i$ and $\beta(yz^i) = xz$. This means that $\beta(z) = z^i$, $\beta(z^i) = z$, $\beta(x) = y$, and $\beta(y) = x$. Thus $z^{i^2} = z$ and so $i^2 - 1 \equiv 0(mod\ q^2)$, $(i^2+1 \equiv 0(mod\ q^2))$, a contradiction. Hence $\beta \notin Aut(G,S)$ and $Aut(\Gamma) \cong (C_p \times C_p \times C_{q^2}) \rtimes C_4$ and Γ is a one-regular graph.

Case 2. Suppose $\langle \alpha^2, \beta \rangle \leq Aut(G,S)$. In this case, $i^2 \equiv 1(mod\ p^2)$ and it is not difficult to see that $Aut(G,S) = \langle \alpha^2, \beta \rangle \cong C_2 \times C_2$. Hence $Aut(\Gamma) \cong (C_p \times C_p \times C_{q^2}) \rtimes (C_2 \times C_2)$ or Γ is one-regular graph.

For the last part, let $G = C_{p^2q^2} \cong \langle a \rangle$. Assume $a = b^i$, where $(i, p^2q^2) = 1$. Two cases hold:

Case 1. Suppose $\langle \alpha \rangle \leq Aut(G,S)$. So $\alpha(a) = \alpha(b^i)$ which means that $b^{-1} = a^i$. Consequently, $\alpha^2(a) = \alpha^2(b^i)$ and so $a^{-1} = b^{-i}$. This yields $b^{i^2+1} = 1$, hence $1 + i^2 \equiv 0(mod\ p^2q^2)$ and thus $p = 4k+1$, $q = 4k^{'}+1$. In other words, $Aut(G,S) = C_4$, since $\beta \in Aut(G,S)$, then $a = b^i$ and $\beta(a) = \beta(b^i)$. Hence $b = a^i$ yields $a = a^{i^2}$ and so $a^{i^2-1} = 1$. It means that p^2q^2 divides $i^2 - 1$ and $i^2 + 1$, which implies that $p^2q^2 \mid 2$, a contradiction. Therefore, $\beta \notin Aut(G,S)$. Hence $Aut(\Gamma) \cong C_{p^2q^2} \rtimes C_4$ and Γ is one-regular graph.

Case 2. Suppose $\langle \alpha^2, \beta \rangle \leq Aut(G,S)$. In this case, $i^2 \equiv 1(mod\ p^2q^2)$ and thus $p = 4k+1$, $q = 4k^{'}+1$. We can verify that $Aut(G,S) = \langle \alpha^2, \beta \rangle \cong C_2 \times C_2$. Hence $Aut(\Gamma) \cong C_{p^2q^2} \rtimes (C_2 \times C_2)$ and Γ is one-regular graph. $\square$

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