Journal of Discrete Mathematics and Its Applications 11 (2) (2026) 99–106

Journal of Discrete Mathematics and Its Applications

Available Online at: http://jdma.sru.ac.ir

Shahid Rajaee Teacher Training University

Research Paper

Complementary distance Seidel equienergetic graphs

Deena C Scaria[1], **Gopalapillai Indulal**[1,*], **Jinu Mary Jameson**[1,2]

[1]Department of Mathematics, St. Aloysius College, Edathua, Alappuzha-689573, Kerala, India

[2]Department of Mathematics, Mar Thoma College, Tiruvalla, Pathanamthitta-689103, India

Academic Editor: Bo Zhou

Abstract. The distance matrix, its spectrum, and the corresponding distance energy of a connected graph have been extensively studied. By contrast, research on the *distance Seidel* matrix is still developing. For a connected graph G, the distance Seidel matrix $DS(G) = J - I - 2D(G)$ yields the distance Seidel eigenvalues $\partial_1^S \geq \partial_2^S \geq \cdots \geq \partial_n^S$. Ivanciuc–Ivanciuc–Balaban introduced the *complementary distance* matrix $CD(G)$ and studied its properties. Motivated by these works, we define the *complementary distance Seidel* matrix

$$CD^S(G) := DS(CD(G)) = J - I - 2CD(G) = (-1-2k)(J-I) + 2D(G),$$

where $k = \text{diam}(G)$. We obtain explicit CD^S-spectra for several classes of graphs with diameter 2, and, in particular, we investigate the CD^S-spectra of complements of line graphs of regular graphs. We derive closed forms for the CD^S-energy of $\overline{L(G)}$ and $\overline{L^2(G)}$, and we construct infinite families of non-CD^S-cospectral yet CD^S-equienergetic graphs arising from complements of iterated line graphs.

Keywords. complementary distance Seidel matrix; complementary distance Seidel spectrum; CD^S-equienergetic graphs.
Mathematics Subject Classification (2020): 05C50, 05C69, 05C76.

1 Preliminaries and notation

Throughout, graphs are finite, simple, undirected, and connected. For terms and notation not defined here we refer to [3,4]. For a graph $G = (V, E)$ of order $n = |V|$ and size $m = |E|$,

*Corresponding author (Email address: indulalgopal@gmail.com)
Received 18 September 2025; Revised 13 November 2025; Accepted 07 December 2025
First Publish Date: 01 June 2026

the degree of a vertex $v \in V$ is $\deg(v)$; the maximum and minimum degrees are denoted by $\Delta(G)$ and $\delta(G)$. The eccentricity of v is $e(v)$, the diameter and radius are $\mathrm{diam}(G)$ and $\mathrm{rad}(G)$, respectively. A graph is r-regular if every vertex has degree r.

The *distance matrix* of G is $D(G) = [d_{ij}]_{i,j=1}^{n}$, where d_{ij} is the graph distance between v_i and v_j ($d_{ii} = 0$). Let I and J denote the $n \times n$ identity and all-ones matrices, respectively.

The *complementary distance* between v_i and v_j is $c_{ij} = 1 + k - d_{ij}$, where $k = \mathrm{diam}(G)$, and the corresponding *complementary distance matrix* is [9]

$$CD(G) = [c_{ij}] = (1 + k)(J - I) - D(G). \tag{1}$$

The *distance Seidel* matrix is $DS(G) = J - I - 2D(G)$ [5]. Motivated by CD and DS, we define the *complementary distance Seidel* matrix as the Seidel transform applied to the complementary distance matrix:

Definition 1.1 (Complementary distance Seidel matrix). *Let G be a graph with diameter k and distance matrix $D(G)$. The complementary distance Seidel matrix is*

$$CD^S(G) := DS(CD(G)) = J - I - 2CD(G) = (-1 - 2k)(J - I) + 2D(G). \tag{2}$$

Its eigenvalues are denoted by $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ and form the CD^S-spectrum of G. Two non-isomorphic graphs having the same CD^S-spectrum are called CD^S-cospectral.

Definition 1.2 (Energy). *The complementary distance Seidel energy of G is*

$$CD^S E(G) = \sum_{i=1}^{n} |\eta_i|. \tag{3}$$

Two graphs G_1, G_2 are CD^S-equienergetic if $CD^S E(G_1) = CD^S E(G_2)$.

Remark 1.3 (Normalization and a one-parameter family). *One may consider $CD^S_\alpha(G) = (-1 - \alpha k)(J - I) + 2D(G)$ with $\alpha > 0$. For the complete graph K_n (where $k = 1$ and $D = J - I$) we get $CD^S_\alpha(K_n) = (1 - \alpha)(J - I)$. Thus $\alpha = 2$ yields $CD^S_2(K_n) = -(J - I) = DS(K_n)$, recovering the usual distance Seidel in the compact case. This makes $\alpha = 2$ the natural normalization used throughout.*

We record standard spectral facts for regular graphs (see [3]).

Lemma 1.4 (Principal eigenvalue). *If G is r-regular, then r is the largest (simple) adjacency eigenvalue.*

Lemma 1.5 (Complements). *Let G be an r-regular connected graph with $spec(A(G)) = \{r, \lambda_2, \ldots, \lambda_n\}$. Then*

$$spec(A(\overline{G})) = \Big\{ n - r - 1;\ -1 - \lambda_i\ (i = 2, \ldots, n) \Big\}.$$

Lemma 1.6 (Line graphs). *Let G be an r-regular connected graph with $spec(A(G)) = \{r, \lambda_2, \ldots, \lambda_n\}$. Then*

$$spec\big(A(L(G))\big) = \Big\{ 2r - 2;\ \lambda_i + r - 2\ (i = 2, \ldots, n);\ -2\ (mult.\ \tfrac{n(r-2)}{2}) \Big\}.$$

Lemma 1.7 (Second iterated line graphs)**.** *Let G be an r-regular connected graph with $spec(A(G)) = \{r, \lambda_2, \dots, \lambda_n\}$. Then*

$$spec\big(A(L^2(G))\big) = \Big\{ 4r-6;\ \lambda_i + 3r - 6\ (i = 2, \dots, n);\ 2r-6\ (mult.\ \tfrac{n(r-2)}{2});\ -2\ (mult.\ \tfrac{nr(r-2)}{2}) \Big\}.$$

Lemma 1.8 (Orders and degrees of iterated line graphs)**.** *Let G be an r-regular graph on n vertices. If n_k and r_k denote the order and degree of $L^k(G)$, then [1,2]*

$$n_k = \frac{n}{2^k} \prod_{i=0}^{k-1} \big(2^i r - 2^{i+1} + 2\big), \qquad r_k = 2^k r - 2^{k+1} + 2.$$

Lemma 1.9 (Diameters of complements of iterated line graphs)**.** *Let G be an r-regular graph on n vertices. If $r \le \frac{n-1}{2}$, then for all $k \ge 1$, [7]*

$$\operatorname{diam}\big(\overline{L^k(G)}\big) = 2.$$

Graphs of diameter 2

If $\operatorname{diam}(G) = 2$ and A (resp. $\overline{A}$) is the adjacency matrix of G (resp. its complement), then

$$D(G) = A + 2\overline{A} = 2(J - I) - A. \tag{4}$$

Hence, by Definition 1.1, for graphs of diameter 2,

$$CD^S(G) = (-1 - 2 \cdot 2)(J - I) + 2D(G) = I - J - 2A. \tag{5}$$

Moreover (see [6]):

Theorem 1.10 (Distance spectrum for diameter 2)**.** *[6] Let G be an r-regular graph with* $\operatorname{diam}(G) = 2$ *and adjacency spectrum $\{r, \lambda_2, \dots, \lambda_n\}$. Then*

$$spec\big(D(G)\big) = \{\, 2n - r - 2;\ -(\lambda_i + 2)\ (i = 2, \dots, n) \,\}.$$

2 CD^S-spectrum for diameter 2 graphs

The identity (5) yields the following transparent diagonalization.

Theorem 2.1. *Let G be an r-regular graph on n vertices with* $\operatorname{diam}(G) = 2$ *and adjacency eigenvalues $\{r, \lambda_2, \dots, \lambda_n\}$. Then the CD^S-eigenvalues of G are*

1. $-n - 2r + 1$;
2. $1 - 2\lambda_i$ *for* $i = 2, 3, \dots, n$.

Proof. For $\mathrm{diam}(G) = 2$, we have

$$\mathrm{CD}^{\mathrm{S}}(G) = I - J - 2A.$$

Since $J\mathbf{1} = n\mathbf{1}$ and $A\mathbf{1} = r\mathbf{1}$,

$$\mathrm{CD}^{\mathrm{S}}(G)\mathbf{1} = (I - J - 2A)\mathbf{1} = (1 - n - 2r)\mathbf{1},$$

yielding the eigenvalue $-n - 2r + 1$ where $\mathbf{1}$ is the all one column vector.

If $x \perp \mathbf{1}$ and $Ax = \lambda x$, then $Jx = 0$, and hence

$$\mathrm{CD}^{\mathrm{S}}(G)x = (I - J - 2A)x = (1 - 2\lambda)x.$$

Thus, the remaining CD^S-eigenvalues are $1 - 2\lambda_i$, where λ_i are the nonprincipal adjacency eigenvalues. □

Corollary 2.2. *Let G be r-regular on n vertices with* $\mathrm{diam}(G) = 2$*. Then the CD^S-spectrum of $L(G)$ is*

1. $\dfrac{-nr - 8r + 10}{2}$;
2. $-2\lambda_i - 2r + 5$ *for* $i = 2, 3, \ldots, n$;
3. 5 *with multiplicity* $\dfrac{n(r-2)}{2}$.

Proof. Apply Lemma 1.6 to obtain $spec(A(L(G)))$ and then Theorem 2.1. □

Corollary 2.3. *Let G be r-regular on n vertices with $r \leq \frac{n-1}{2}$. Then the CD^S-spectrum of $\overline{L(G)}$ is*

1. $\dfrac{-3nr}{2} + 4r - 1$;
2. $2(\lambda_i + r) - 1$ *for* $i = 2, 3, \ldots, n$;
3. -1 *with multiplicity* $\dfrac{n(r-2)}{2}$.

Proof. By Lemma 1.6 and Lemma 1.5, $\overline{L(G)}$ is regular of order $nr/2$ and degree $(nr/2) - 2r + 1$. By Lemma 1.9, $\mathrm{diam}(\overline{L(G)}) = 2$. Apply Theorem 2.1 to $\overline{L(G)}$ using the adjacency spectrum from Lemma 1.5. □

Corollary 2.4. *Let G be r-regular on n vertices with $r \leq \frac{n-1}{2}$ and* $\mathrm{diam}(G) = 2$*. Then the CD^S-spectrum of $L^2(G)$ is*

1. $\dfrac{-nr^2}{2} + \dfrac{nr}{2} - 8r + 13$;
2. $-2\lambda_i - 6r + 13$ *for* $i = 2, 3, \ldots, n$;

3. $-4r+13$ *with multiplicity* $\dfrac{n(r-2)}{2}$;

4. 5 *with multiplicity* $\dfrac{nr(r-2)}{2}$.

Proof. Combine Lemma 1.7 with Theorem 2.1. □

Corollary 2.5. *Let G be r-regular on n vertices with* $r \le \frac{n-1}{2}$. *Then the* CD^S*-spectrum of* $\overline{L^2(G)}$ *is*

1. $\dfrac{-3nr^2}{2} + \dfrac{3nr}{2} + 8r - 9$;

2. $2\lambda_i + 6r - 9$ *for* $i = 2,3,\ldots,n$;

3. $4r-9$ *with multiplicity* $\dfrac{n(r-2)}{2}$;

4. 1 *with multiplicity* $\dfrac{nr(r-2)}{2}$.

Proof. Apply Lemma 1.5 to the spectrum in Lemma 1.7, then Theorem 2.1 to $\overline{L^2(G)}$. □

3 CD^S-energy

Theorem 3.1. *Let G be an r-regular graph on n vertices with* $r \le \frac{n-1}{2}$. *Then, for* $r \ge 3$,

$$CD^SE\big(\overline{L(G)}\big) = n(r-2).$$

Proof. By Lemma 1.6 and Lemma 1.5, the adjacency eigenvalues of $\overline{L(G)}$ are

$$\Big\{ \tfrac{nr}{2} - 2r + 1;\ -\lambda_i - r + 1\ (i = 2,\ldots,n);\ 1\ (\text{mult. } \tfrac{n(r-2)}{2}) \Big\}.$$

Since $\operatorname{diam}(\overline{L(G)}) = 2$ (Lemma 1.9), Corollary 2.3 gives the CD^S-eigenvalues. Using $-r \le \lambda_i \le r$ for an r-regular graph (see [4]), we have $\lambda_i + r \ge 0$. Therefore, summing absolute values in Corollary 2.3,

$$\begin{aligned} CD^SE\big(\overline{L(G)}\big) &= \Big|\tfrac{-3nr}{2} + 4r - 1\Big| + \sum_{i=2}^{n} \big(2\lambda_i + 2r - 1\big) + \tfrac{n(r-2)}{2} \cdot |-1| \\ &= n(r-2). \end{aligned}$$

□

Theorem 3.2. *Let G be an r-regular graph on n vertices with* $r \le \frac{n-1}{2}$. *Then, for* $r \ge 3$,

$$CD^SE\big(\overline{L^2(G)}\big) = 4nr^2 - 5nr - 16r + 18.$$

Proof. Using Corollary 2.5 and the same regular-graph bound, we obtain

$$\begin{aligned} CD^SE\big(\overline{L^2(G)}\big) &= \left|\tfrac{3nr^2}{2} - \tfrac{3nr}{2} - 8r + 9\right| + \sum_{i=2}^{n} \big(2\lambda_i + 6r - 9\big) \\ &\quad + \tfrac{n(r-2)}{2} \cdot |4r - 9| + \tfrac{nr(r-2)}{2} \cdot |1| \\ &= 4nr^2 - 5nr - 16r + 18. \end{aligned}$$

□

Corollary 3.3. *Let G be an r_0-regular graph of order n_0. If n_k and r_k are the order and degree of $L^k(G)$ (as in Lemma 1.8), and if $r_0 \leq \frac{n_0-1}{2}$, then for all $k \geq 3$,*

$$CD^SE\big(\overline{L^k(G)}\big) = 4n_{k-2}r_{k-2}^2 - 5n_{k-2}r_{k-2} - 16r_{k-2} + 18.$$

Proof. Apply Theorem 3.2 to $L^{k-2}(G) =: H$ and use Lemma 1.8 to express n_{k-2} and r_{k-2} in terms of n_0, r_0. □

Corollary 3.4. *Under the hypotheses of Corollary 3.3, the energy can be written explicitly as*

$$\begin{aligned} CD^SE\big(\overline{L^k(G)}\big) &= \Big[2^k n_0(r_0-2)^2 + 11n_0(r_0-2) + \frac{24n_0}{2^k}\Big] \prod_{i=0}^{k-3} (2^i r_0 - 2^{i+1} + 2) \\ &\quad + 2^{k+2}(2 - r_0) - 14, \qquad k \geq 3. \end{aligned}$$

Proof. Use Lemma 1.8 to substitute n_{k-2} and r_{k-2} into Corollary 3.3 and simplify. □

4 CD^S-equienergetic graphs

If G_1 and G_2 are regular graphs of the same order and degree, then $L(G_1)$ and $L(G_2)$ (and hence their complements) also have the same order and degree.

Lemma 4.1. *Let G_1 and G_2 be r-regular graphs on n vertices with $r \leq \frac{n-1}{2}$. Then, for $k \geq 1$, the graphs $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are CD^S-cospectral if and only if G_1 and G_2 are adjacency-cospectral.*

Proof. Use Lemmas 1.6, 1.7, 1.5 to express the adjacency spectrum of $\overline{L^k(G)}$ in terms of that of G, then invoke Theorem 2.1. □

Theorem 4.2. *Let G_1 and G_2 be two non-CD^S-cospectral regular graphs of the same order n and degree r with $r \leq \frac{n-1}{2}$. Then for any $k \geq 3$ and $r \geq 3$, the graphs $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ form a pair of non-CD^S-cospectral, CD^S-equienergetic graphs of equal order and equal size.*

Examples

Using [6, Thm. 1], the distance spectra are

$$spec_D(K_{n,n}) = \{\, 3n-2;\ n-2;\ -2\ (\text{mult. } 2n-2) \,\},$$

$$spec_D(\mathrm{CP}(n)) = \{\, 2n;\ -2\ (\text{mult. } n);\ 0\ (\text{mult. } n-1)\,\},$$

where $K_{n,n}$ is complete bipartite and $\mathrm{CP}(n)$ is the *cocktail party* graph (obtained from K_{2n} by deleting a perfect matching). Consequently,

$$spec_{CD^S}(K_{n,n}) = \{\, 2n+1;\ 1-4n;\ 1\ (\text{mult. } 2n-2)\,\},$$

$$spec_{CD^S}(\mathrm{CP}(n)) = \{\, 5-6n;\ 1\ (\text{mult. } n);\ 5\ (\text{mult. } n-1)\,\}.$$

5 Conclusion

We introduced the complementary distance Seidel matrix as $CD^S = DS(CD)$ and developed its spectral and energy theory on complements of line graphs of regular graphs. For this class we provided closed forms for the CD^S-energy and produced infinite families of non-cospectral yet CD^S-equienergetic graphs. Extending these techniques to further graph operations and to irregular graphs is a natural direction for future work.

Acknowledgements

The authors express their sincere gratitude to the anonymous referees for their careful reading of the manuscript and for their insightful comments and suggestions, which greatly improved the clarity and quality of this work.

Funding

This research received no external funding.

Data Availability Statement

Data is contained within the article.

Conflicts of Interests

The authors declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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Citation: D. C Scaria, G. Indulal, J. M. Jameson, Complementary distance Seidel equienergetic graphs, J. Disc. Math. Appl. 11(2) (2026) 99–106.
https://doi.org/10.22061/jdma.2025.12527.1165